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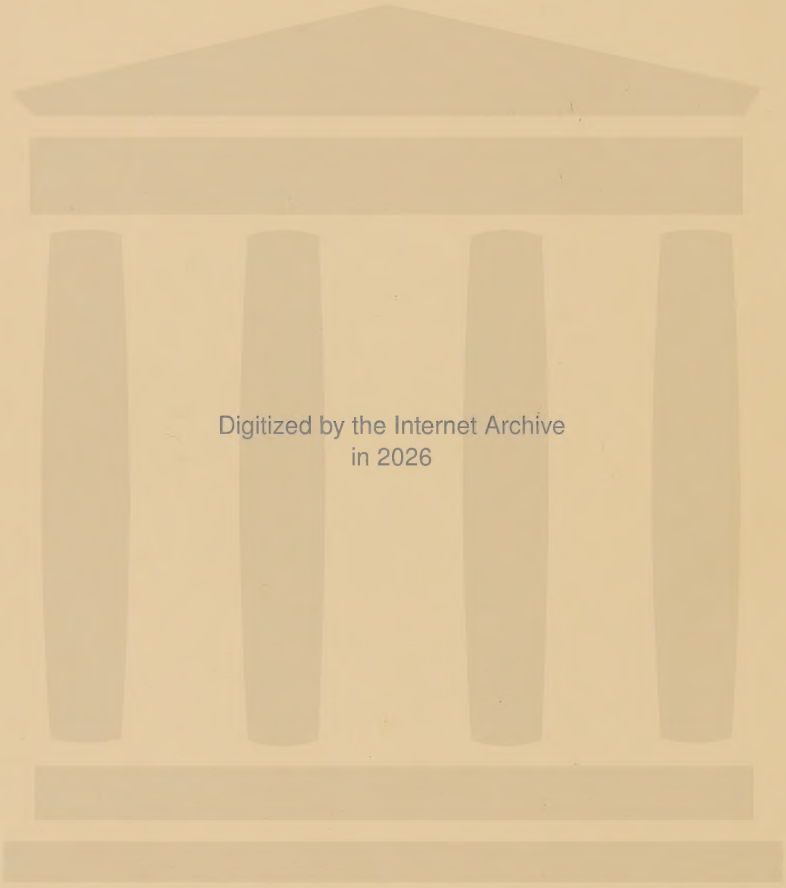

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PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY

SECOND SERIES

VOLUME 30

LONDON:
PRINTED BY C. F. HODGSON & SON, LTD.,
2 NEWTON STREET, KINGSWAY, W.C.2;
AND PUBLISHED FOR THE SOCIETY BY
FRANCIS HODGSON, 15 ST. BRIDE STREET, E.C.4.

1930

BIBL.
LONDIN.
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PAPERS

PUBLISHED IN THE

PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY

A TYPE OF TAUBERIAN THEOREM APPLYING TO FOURIER SERIES

By NORBERT WIENER.

[Received 1 January, 1929.—Read 10 January, 1929.]

The purpose of the present paper is to apply a method already developed by the author* to the proof of a theorem of Hardy and Littlewood† concerning the summability of Fourier series. Restated in an equivalent form for Fourier integrals, it reads: *Let $f(x)$ be a measurable function defined over $(-\infty, \infty)$ and zero outside $(-A, A)$. Let*

$$\phi(y) = \tfrac{1}{2}\{f(x+y)+f(x-y)-2C\}.$$

Then if we write B_m for the proposition

$$\lim_{\lambda\to\infty} \lambda \lim_{\epsilon\to 0} \int_{\epsilon}^{1/\lambda} \phi(y)(1-\lambda y)^m\, dy = 0$$

and C_m for the proposition

$$\lim_{\lambda\to\infty} \lambda \int_0^{\infty} \phi(y)\, dy \int_0^1 (1-z)^m \cos \lambda yz\, dz = 0,$$

the infinite integral being taken in the sense $\lim_{\epsilon\to 0,\ E\to\infty} \int_{\epsilon}^{E}$, the truth of B_m for some m implies the truth of C for some n, and vice versa. C_m asserts

* N. Wiener, "A new method in Tauberian theorems", *Journal Math. and Phys., Mass. Inst. of Tech.*, 7 (1928), 161–184.

† G. H. Hardy and J. E. Littlewood, "Fourier series and power series", *Proc. London Math. Soc.* (2), 21 (1922–23), xxxi–xxxii. Cf. also Hardy and Littlewood, "The equivalence of certain integral means", *ibid.*, 22 (1923–24), xl–xliii; "Solution of the Cesàro summability problem for power series and Fourier series", *Math. Zeitschrift*, 19 (1924), 67–96.

the m-th order Cesàro summability of the Fourier integral for $f(x)$ to the value C.

In proving this theorem, we shall use the following lemma :

LEMMA I. *Let $F(x)$ be a (real or complex) function of the real variable x, defined over $0 \leqq x < \infty$ and summable over every interval $(0, A)$. Let $Q(x)$ be a function defined and measurable over $0 \leqq x < \infty$ and not vanishing for $x > B$. Let $x^n Q(x)$ be of limited total variation over $(0, \infty)$, and let its reciprocal be of limited total variation over (B, ∞), so that $L = \lim_{x \to \infty} x^n Q(x)$ exists and is not zero. Let $\int_B^\infty F(x)\, Q(x)\, dx$ exist as the limit of a Lebesgue integral, and let us write*

$$g(y, \lambda) = \lambda \int_B^y F(x)\, Q(\lambda x)\, dx ;$$

$$h(y) = \int_B^y F(x)\, x^{-n}\, dx.$$

Then $h(\infty)$ and $g(\infty, \lambda)$ exist, and

$$g(\infty, \lambda) = \lambda^{1-n} \left\{ h(\infty)\, L - \int_B^\infty h(x)\, d\left[(\lambda x)^n\, Q(\lambda x)\right] \right\} = O(\lambda^{1-n}).$$

Proof. We have

$$h(N) = \int_B^N F(x)\, Q(x) \frac{dx}{x^n Q(x)} = \frac{g(N, 1)}{N^n Q(N)} - \int_B^N g(x, 1)\, d\left(\frac{1}{x^n Q(x)}\right).$$

Clearly $g(x, 1)$ is bounded. Going to the limit, $h(\infty)$ exists and

$$h(\infty) = \frac{g(\infty, 1)}{L} - \int_B^\infty g(x, 1)\, d\left(\frac{1}{x^n Q(x)}\right).$$

In the same way,

$$\begin{aligned} g(N, \lambda) &= \lambda \int_B^N F(x)\, x^{-n} \left\{ x^n Q(\lambda x) \right\} dx \\ &= \lambda h(N)\, N^n Q(\lambda N) - \lambda \int_B^N h(x)\, d\left\{ x^n Q(\lambda x) \right\} \\ &= \lambda^{1-n} \left\{ h(N)(\lambda N)^n Q(\lambda N) - \int_B^N h(x)\, d\left[(\lambda x)^n Q(\lambda x)\right] \right\}. \end{aligned}$$

Since $h(x)$ is bounded, $g(\infty, \lambda)$ exists, and

$$g(\infty, \lambda) = \lambda^{1-n} \left\{ h(\infty)\, L - \int_B^\infty h(x)\, d\left[(\lambda x)^n\, Q(\lambda x)\right] \right\}.$$

Thus, for $\lambda > 0$,

$$|g(\infty, \lambda)| \leqslant \lambda^{1-n}\left\{|L\,h(\infty)| + \max_{B \leqslant x < \infty} |h(x)| \int_0^\infty |d[x^n Q(x)]|\right\} = O(\lambda^{1-n}).$$

This completes our proof. We are now in a position to prove

LEMMA II. *If F and Q are subject to the restrictions of Lemma I, and if*

$$\int_0^\infty \lambda^{1-n} P(\lambda)\, d\lambda$$

is absolutely convergent, then

$$\int_0^\infty \lambda P(\lambda)\, d\lambda \int_B^\infty F(x)\, Q(\lambda x)\, dx = \int_0^\infty F(x)\, dx \int_0^\infty \lambda P(\lambda)\, Q(\lambda x)\, d\lambda.$$

Proof. The integral

$$\int_0^\infty \lambda P(\lambda)\, d\lambda \int_B^\infty F(x)\, Q(\lambda x)\, dx$$

may be arranged in the form

$$\int_0^\infty P(\lambda)\, g(\infty, \lambda)\, d\lambda$$

$$= \int_0^\infty \lambda^{1-n} P(\lambda)\, d\lambda \left\{h(\infty)\, L - \int_B^\infty h(x)\, d[(\lambda x)^n Q(\lambda x)]\right\}$$

$$= L\,h(\infty) \int_0^\infty \lambda^{1-n} P(\lambda)\, d\lambda - \int_B^\infty h(x)\, d\left[x^n \int_0^\infty \lambda P(\lambda)\, Q(\lambda x)\, d\lambda\right]$$

$$= L\,h(\infty) \int_0^\infty \lambda^{1-n} P(\lambda)\, d\lambda - \lim_{N \to \infty} \left\{h(N)\, N^n \int_0^\infty \lambda P(\lambda)\, Q(\lambda N)\, d\lambda + \int_B^N \left[x^n \int_0^\infty \lambda P(\lambda)\, Q(\lambda x)\, d\lambda\right] dh(x)\right\}.$$

But $$\lim_{N \to \infty} \left(h(N)\, N^n \int_0^\infty \lambda P(\lambda)\, Q(\lambda N)\, d\lambda\right)$$

$$= \lim_{N \to \infty} \left(h(N) \int_0^\infty \lambda^{1-n} P(\lambda)(\lambda N)^n Q(\lambda N)\, d\lambda\right)$$

$$= L\,h(\infty) \int_0^\infty \lambda^{1-n} P(\lambda)\, d\lambda.$$

Hence $$\int_0^\infty P(\lambda)\, g(\infty, \lambda)\, d\lambda = \int_B^\infty F(x)\, dx \int_0^\infty \lambda P(\lambda)\, Q(\lambda x)\, d\lambda.$$

Lemma II is of particular interest when

$$\lim_{\lambda\to\infty} \lambda \int_B^\infty F(x)\, Q(\lambda x)\, dx = M$$

and

$$P(\lambda) = \frac{1}{\mu}\, \Phi\left(\frac{\lambda}{\mu}\right).$$

Then our condition concerning P assumes the form that

$$\int_0^\infty \lambda^{1-n}\, \Phi(\lambda)\, d\lambda$$

is to converge absolutely. Let us put

$$\int_0^\infty \lambda \Phi(\lambda)\, Q(\lambda x)\, d\lambda = K(x).$$

Let us introduce the new assumption that

$$\int_0^\infty \Phi(\lambda)\, d\lambda$$

is to converge absolutely. We then have

$$\begin{aligned} M\int_0^\infty \Phi(\lambda)\, d\lambda &= \lim_{\mu\to 0} \int_0^\infty \Phi(\lambda)\, g(\infty, \lambda\mu)\, d\lambda \\ &= \lim_{\mu\to 0} \int_0^\infty \Phi(\lambda)\, d\lambda \int_B^\infty \lambda\mu F(x)\, Q(\lambda\mu x)\, dx \\ &= \lim_{\mu\to 0} \mu \int_B^\infty F(x)\, dx \int_0^\infty \lambda \Phi(\lambda)\, Q(\lambda\mu x)\, dx \\ &= \lim_{\mu\to 0} \mu \int_B^\infty F(x)\, K(\mu x)\, dx. \end{aligned}$$

All the integrals and limits in this last formula exist, as we may verify by means of Lemma I. The inversion of the order of integration is justified by the fact that it is carried out in an absolutely convergent double Stieltjes integral.

Hence, *if Q and F satisfy the condition which we have already laid down for them, and if the integral equation*

$$\int_0^\infty \lambda \Phi(\lambda)\, Q(\lambda x)\, d\lambda = K(x) \tag{I}$$

has a solution $\Phi(\lambda)$ for which

$$\int_0^\infty |\Phi(\lambda)|\, d\lambda, \quad \int_0^\infty \lambda^{1-n} |\Phi(\lambda)|\, d\lambda$$

are finite, then

$$\lim_{\mu \to 0} \mu \int_B^{\infty} F(x)\, K(\mu x)\, dx = M \int_0^{\infty} \Phi(\lambda)\, d\lambda.$$

This is a theorem of the type termed Tauberian* by Hardy and Littlewood.

Now let us put $\lambda = e^{\xi}$; $x = e^{\eta}$;

$$Q(x) = R(\eta); \quad \Phi(x) = S(\eta); \quad K(x) = T(\eta).$$

Then (I) assumes the form

$$\int_{-\infty}^{\infty} e^{2\xi}\, S(\xi)\, R(\xi+\eta)\, d\xi = T(\eta). \tag{I^1}$$

Now let us suppose that $e^{\xi} R(\xi)$ and $e^{\xi} T(\xi)$ are quadratically summable over $(-\infty, \infty)$. Let

$$r(u) = \underset{A \to \infty}{\text{l.i.m.}} \int_{-A}^{A} e^{\xi} R(\xi)\, e^{i\xi u}\, d\xi = \underset{\epsilon \to 0}{\text{l.i.m.}} \int_{\epsilon}^{1/\epsilon} Q(x)\, x^{iu}\, dx;$$

$$t(u) = \underset{A \to \infty}{\text{l.i.m.}} \int_{-A}^{A} e^{\xi} T(\xi)\, e^{i\xi u}\, d\xi = \underset{\epsilon \to 0}{\text{l.i.m.}} \int_{\epsilon}^{1/\epsilon} K(x)\, x^{iu}\, dx.$$

These limits, in the mean, exist by the Plancherel theory of the Fourier transforms, which we now assume *in toto*†. In case $t(u)/r(u)$ is quadratically summable, it follows from the Parseval-Plancherel theorem as applied to the Fourier transform of the product of two functions that

$$S(\xi) = \underset{A \to \infty}{\text{l.i.m.}}\, e^{-\xi} \int_{-A}^{A} \left(\frac{t(-u)}{r(-u)}\right) e^{-i\xi u}\, du$$

is a possible solution of (I^1). This gives us

$$\Phi(x) = \underset{A \to \infty}{\text{l.i.m.}} \int_{-A}^{A} \frac{t(-u)}{r(-u)}\, x^{-iu-1}\, du;$$

$$\frac{2\pi t(-u)}{r(-u)} = \underset{\epsilon \to 0}{\text{l.i.m.}} \int_{\epsilon}^{1/\epsilon} \Phi(x)\, x^{iu}\, dx.$$

In a particular case, it may happen that

$$\int_0^{\infty} Q(x)\, x^{w}\, dx, \quad \int_0^{\infty} K(x)\, x^{w}\, dx$$

---

* Cf., *e.g.*, G. H. Hardy and J. E. Littlewood, "Tauberian theorems concerning power series and Dirichlet's series whose coefficients are positive", *Proc. London Math. Soc.* (2), 19 (1921), 174–191.

† M. Plancherel, "Contribution à l'étude de la représentation d'une fonction arbitraire par des intégrales définies", *Rend. di Palermo*, 30 (1910), 289–335.

exist and define analytic functions over the range $-B < \Re(w) < C$; $-\infty < \Im(w) < \infty$. Let us call the functions $\rho(w)$ and $\tau(w)$ respectively. Let the function $\tau(w)/\rho(w)$ be analytic over the range $\mu < \Re(w) < \nu$, including $\Re(w) = 0$. Let $\tau(w)/\rho(w)$ be uniformly $o(1)$ over this strip as $|\Im(w)| \to \infty$. By Cauchy's theorem,

$$S(\xi) = e^{-\xi} \int_{-\infty}^{\infty} \frac{\tau(-iu)}{\rho(-iu)} e^{-i\xi u}\, du = \int_{-\infty}^{\infty} \frac{\tau(w_0 - iu)}{\rho(w_0 - iu)} e^{\xi(w_0 - iu - 1)}\, du,$$

for any w_0 of the strip, provided that either integral exists. Indeed, the integrals in question need only exist as limits in the means. Thus, if

$$\tau(w_0 - iu)/\rho(w_0 - iu)$$

is quadratically summable, it is the Fourier transform of $S(\xi)e^{-\xi(w_0-1)}$, which is therefore also quadratically summable.

Let us assume that $\tau(w_0 - iu)/\rho(w_0 - iu)$ is quadratically summable in u for $\mu < \Re(w) < \nu$, and is uniformly $o(1)$ in this strip as $|u| \to \infty$. Since

$$\int_0^A |S(\xi)|\, e^{-\xi(w_0-1)}\, d\xi \leqslant \sqrt{\left\{\int_0^A |S(\xi)\, e^{-\xi(w-1)}|^2\, d\xi \int_0^A e^{-\xi(w_0-w_1)}|^2\, d\xi\right\}},$$

where w_1 is a number of the strip less than w_0, we see that

$$\int_0^{\infty} |S(\xi)|\, e^{-\xi(w_0-1)}\, d\xi$$

exists for every w_0 in our strip. Similarly,

$$\int_{-\infty}^{0} |S(\xi)|\, e^{-\xi(w_0-1)}\, d\xi$$

exists, and hence

$$\int_0^{\infty} |\Phi(x)|\, x^{-w_0}\, dx = \int_{-\infty}^{\infty} |S(\xi)|\, e^{-\xi(w_0-1)}\, d\xi$$

exists. Hence, if 0 and $1-n$ lie within the range $(-\nu, -\mu)$, the function Φ satisfies the condition laid down in the hypothesis of our Tauberian theorem. Assuming that $n > 1$, we may write the condition that 0 and $1-n$ are to lie within this range in the form $\mu < 0 < n-1 \leqslant \nu$. Thus μ may have any value less than 0, and ν any value greater than $n-1$. Combining these results, we obtain

LEMMA III. *Let $F(x)$ be a (real or complex) function, bounded for $0 \leqslant x \leqslant A$ for all A, and defined over $(0, \infty)$. Let $Q(x)$ be a function defined over $(0, \infty)$ and not vanishing for $x > B$. Let $x^n Q(x)$ be of*

limited total variation over $(0, \infty)$, *and let its reciprocal be of limited total variation over* (B, ∞), *for a given* $n > 1$. *Let* $K(x)$ *be a measurable function defined over* $(0, \infty)$, *and let*

$$\int_0^\infty Q(x)\,dx = \int_0^\infty K(x)\,dx$$

(as will be the case when $\int_0^\infty \Phi(\lambda)\,d\lambda = 1$, *according to our earlier definitions). Let*

$$\rho(w) = \int_0^\infty Q(x)\,x^w\,dx \quad \textit{and} \quad \tau(w) = \int_0^\infty K(x)\,x^w\,dx$$

exist as functions such that $\tau(w)/\rho(w)$ *is analytic over*

$$\{-\eta < \Re(w) < n-1+\eta\},$$

where $\eta > 0$. *Let* $\tau(w)/\rho(w)$ *be* $o(1)$ *uniformly in* $\Im(w)$ *at* $+\infty$ *for* $\{-\eta < \Re(w) < n-1+\eta\}$, *and let it be quadratically summable over every ordinate in that interval. Let*

$$\lim_{\lambda \to 0} \int_A^\infty F(x)\,Q(\lambda x)\,dx = M.$$

Then

$$\lim_{\lambda \to 0} \int_A^\infty F(x)\,K(\lambda x)\,dx = M.$$

By the transformation $y = 1/x$, we get

LEMMA III[1]. *Let* $F(x)$ *be a (real or complex) function, bounded over* (ϵ, ∞) *for all* ϵ, *and defined over* $(0, \infty)$. *Let* $Q(y)$ *be a function defined over* $(0, \infty)$ *and not vanishing for* $x < \epsilon$. *Let* $y^{-m}\,Q(y)$ *be of limited total variation over* $(0, \infty)$, *and let its reciprocal be of limited total variation over* $(0, \epsilon)$, *for a given* $m > -1$. *Let* $K(y)$ *be a measurable function defined over* $(0, \infty)$, *and let*

$$\int_0^\infty Q(y)\,dy = \int_0^\infty K(y)\,dy.$$

Let

$$\rho'(w) = \int_0^\infty Q(y)\,y^w\,dy \quad \textit{and} \quad \tau'(w) = \int_0^\infty K(y)\,y^w\,dy$$

exist as functions such that $\tau'(w)/\rho'(w)$ *is analytic over*

$$\{-m-1-\eta < \Re(w) < \eta\}.$$

Let $\tau'(w)/\rho'(w)$ *be* $o(1)$ *uniformly in* $\Im(w)$ *at* $\pm\infty$ *for*

$$\{-m-1-\eta < \Re(w) < \eta\},$$

and let it be quadratically summable over every ordinate in that interval. Let

$$\lim_{\lambda\to\infty} \lambda \int_0^\epsilon F(y)\, Q(\lambda y)\, dy = M.$$

Then

$$\lim_{\lambda\to\infty} \lambda \int_0^\epsilon F(y)\, K(\lambda y)\, dy = M.$$

We now proceed to our main theorem. As $Q(y)$ or $K(y)$ we consider the functions

$$K_\mu(y) = \begin{cases} (\mu+1)(1-y)^\mu & [0 < y < 1], \\ 0 & [y > 1]; \end{cases}$$

$$K^1_\mu(y) = \frac{2}{\pi} \int_0^1 (1-z)^\mu \cos yz\, dz.$$

The quantity m is obviously 0^*. If we write

$$\tau_\mu(w) = \int_0^\infty K_\mu(y)\, y^w\, dy, \quad \sigma_\mu(w) = \int_0^\infty K'_\mu(y)\, y^w\, dy,$$

we have

$$\tau_\mu(w) = \frac{\Gamma(\mu+2)\,\Gamma(w+1)}{\Gamma(w+\mu+2)}, \quad \sigma_\mu(w) = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1-w)\cos\frac{1}{2}w\pi}.$$

For any positive μ, these functions are free from zeros and singularities for all w, such that $\Re(w)$ lies in $(-1-\eta,\ \eta)$ for sufficiently small η, with the exception in all cases of a simple pole at -1. Thus $\sigma_\mu(w)/\tau_\nu(w)$ and $\tau_\mu(w)/\sigma_\nu(w)$ are alike free from singularities over $(-1-\eta,\ \eta)$. It is easy to see from the known asymptotic properties of the gamma function that

$$\left|\frac{\sigma_\mu(w)}{\tau_\nu(w)}\right| \sim \frac{2\Gamma(\mu+1)}{e\sqrt{2\pi}\,\Gamma(\nu+2)}\, \Im(w)^{\nu-\mu+\Re(w)+\frac{1}{2}}.$$

Over $-1-\eta < \Re(w) < \eta$ the exponent of $\Im(w)$ varies between $\nu-\mu-\frac{1}{2}-\eta$ and $\nu-\mu+\frac{1}{2}+\eta$. If both these numbers are greater than $\frac{1}{2}$, K_μ can serve for Q and K^1_ν for K. In this case $\mu > \nu+1$. If both these numbers are greater than $\frac{1}{2}$, K^1_μ can serve for Q and K_ν for K. In this case $\nu > \mu+1$. Thus, in the notation of the main theorem of our paper, B_m implies $C_{m+1+\eta}$ and C_m implies $B_{m+1+\eta}$. This is, however, sufficient to establish our main theorem.

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* Cf. M. Jacob, "Über die Summierbarkeit von Fourierschen Reihen und Integralen", *Math. Zeitschrift*, 29 (1928), 21.

ON THE INTEGRATION OF PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER WITH ONE DEPENDENT AND TWO INDEPENDENT VARIABLES

By M. J. M. Hill.

[Received 8 June, 1927.—Read 9 June, 1927.—Revised 13 October, 1928.]

1. The object of this paper is to point out the relation between the nature of an integral of a partial differential equation and the way in which the condition of integrability is satisfied.

It is convenient to begin by writing down the condition that the values of p and q, given by the equations

$$F(x, y, z, p, q) = a, \tag{1}$$

$$G(x, y, z, p, q) = b, \tag{2}$$

make $pdx+qdy$ a complete differential dz. It is supposed that F and G are functions of the variables and of fixed constants, while a and b are constants whose character will be determined later.

The condition is that the values of p and q, given by (1) and (2), are such that

$$\left(\frac{DF}{Dx}+p\frac{DF}{Dz}\right)\frac{DG}{Dp}+\left(\frac{DF}{Dy}+q\frac{DF}{Dz}\right)\frac{DG}{Dq} - \left(\frac{DG}{Dx}+p\frac{DG}{Dz}\right)\frac{DF}{Dp}-\left(\frac{DG}{Dy}+q\frac{DG}{Dz}\right)\frac{DF}{Dq}=0. \tag{3}$$

This will be called the condition of integrability. The expression on the left is denoted by

$$[F, G].$$

It is a function of the variables, but it does not contain a or b. Its form is completely determined when the functions F and G are given. It can vanish in three ways.

I. When, after the values of F and G are substituted, all the terms cancel out, and nothing is left.

I think that it is in conformity with ordinary practice to say that $[F, G]$ then vanishes identically, but I shall subsequently say merely that the condition of integrability is satisfied in the first way. In this case, (3) imposes no restriction on a and b, both of which are therefore arbitrary. When the equation $dz = pdx + qdy$ is integrated, z will be a function of the variables, of the arbitrary constants a and b, and of a third arbitrary constant c.

II. It may be that $[F, G]$ vanishes as a consequence of (1). Since $[F, G]$ contains no arbitrary constant, it is necessary in this case that a should be a fixed constant; but there is no restriction on the value of b, which is therefore arbitrary. In this case, z is a function of the variables, of the fixed constant a, and of the arbitrary constants b and c. And now $[F, G]$ is an algebraic multiple of $F-a$.

III. It may be that $[F, G]$ vanishes as a consequence of both (1) and (2). Since $[F, G]$ contains no arbitrary constant, it is necessary in this case that both a and b should be fixed. In this case z is a function of the variables, of the fixed constants a and b, and of the arbitrary constant c. And now $[F, G]$ is expressible as the sum of algebraic multiples of $F-a$ and $G-b$.

The three cases correspond to natural divisions of the subject :—

I. a and b both arbitrary.

II. a fixed and b arbitrary.

III. a and b both fixed.

2. We now discuss the solution of the differential equation

$$F(x, y, z, p, q) = 0 \tag{4}$$

in the light of the previous remarks. The constant a of (1) is now zero.

The equation

$$G(x, y, z, p, q) = b \tag{5}$$

is now obtained as an integral of Charpit's equations associated with (4). Hence, whether $[F, G]$ vanishes in the first or in the second way, the value found for z contains two arbitrary constants b and c; and so a "complete integral" is obtained.

In the third case, however, when $[F, G]$ vanishes only when $a = 0$ and b is a fixed constant, we have an integral containing one arbitrary

constant c; and there may appear, in the process of integration, an integral which contains *no* arbitrary constant.

Whichever of these possibilities occurs, the integral has some kind of tangential relation to complete integrals, but the nature of the relation varies in different cases. It is to be noted that in particular cases* $[F, G]$ may vanish in consequence of equation (5) only; but, in general, the vanishing of $[F, G]$ is a consequence of both (4) and (5).

The integral which contains a single arbitrary constant may be a particular case of a general integral, or of what Forsyth calls a "special integral", or it may be a particular case of a complete integral.

The integral which contains no arbitrary constant is said to be singular.

I now give various examples, with the object of working out the form of the condition of integrability in several cases and of illustrating the various possibilities.

3. *Example* 1. In this example $[F, G]$ vanishes in the first way. Let

$$F \equiv x+p(1+p^2+q^2)^{-\frac{1}{2}} = a,$$

$$G \equiv y+q(1+p^2+q^2)^{-\frac{1}{2}} = b.$$

When we calculate $[F, G]$, we find that all the terms cancel, so that a and b are arbitrary; it will be found that

$$z = c+[1-(x-a)^2-(y-b)^2]^{\frac{1}{2}}.$$

This example is a particular case of the following :—

Let $\Phi(x, y, a, b)$ be any function of x, y, a, b. Form the equations

$$p = \frac{\partial}{\partial x}\Phi(x, y, a, b),$$

$$q = \frac{\partial}{\partial y}\Phi(x, y, a, b);$$

solve these equations for a and b, and let the expressions obtained for a and b in terms of x, y, p, q be called A and B. We find, on differentiating out, that A and B satisfy the condition of integrability $[A, B] = 0$, all the terms which occur on the left cancelling. Hence the equations $A = a$ and $B = b$ form an integrable combination; these equations are

* It is convenient to regard these as examples of the third way of $[F, G]$ vanishing, not of the second way.

obviously equivalent to those written above for p and q, and their complete integral is

$$z = c+\Phi(x, y, a, b).$$

The particular example was obtained by writing

$$\Phi(x, y, a, b) \equiv [1-(x-a)^2-(y-b)^2]^{\frac{1}{2}}.$$

Example 2. This is the differential equation associated with the double family of spheres whose radii are all equal to l, and whose centres lie on the sphere of radius k with centre at the origin.

The equation of the family of spheres is

$$(x-a)^2+(y-b)^2+(z-c)^2 = l^2,$$

where

$$a^2+b^2+c^2 = k^2.$$

The differential equation is

$$F \equiv (1+p^2+q^2)(x^2+y^2+z^2+l^2-k^2)^2-4l^2(xp+yq-z)^2 = 0.$$

Write, for brevity,

$$x^2+y^2+z^2+l^2-k^2 \equiv T,$$

$$xp+yq-z = S;$$

then the differential equation becomes

$$F \equiv (1+p^2+q^2)\,T^2-4l^2S^2 = 0.$$

Three solutions of the condition of integrability, each containing one arbitrary constant, are

$$G \equiv z+\frac{T}{2S} = c,$$

$$H \equiv x-\frac{pT}{2S} = a,$$

$$K \equiv y-\frac{qT}{2S} = b.$$

A solution which contains no arbitrary constant is

$$L \equiv yp-xq = 0;$$

this is obtainable from the equations

$$F = 0, \quad \frac{DF}{Dp} = 0, \quad \frac{DF}{Dq} = 0,$$

and it may be expected to lead to a singular integral.

When we combine $F=0$ with $G=c$, we find that

$$[F, G] = -\frac{2}{S^2}\{x^2+y^2+(xp+yq)^2\}F,$$

and so $[F, G]$ vanishes in consequence of F vanishing; this is an example of the second mode of vanishing, and it leads to the complete integral from which we started.

When we combine $F=0$ with $L=0$, we find that

$$[F, L] = \{4Tz(1+p^2+q^2)+8l^2S\}L;$$

and so $[F, L]$ vanishes in consequence of L vanishing; this is an example of the third mode of vanishing*.

Since the general integral of the equation $py-qx=0$ represents a surface of revolution, z is a function of $\sqrt{(x^2+y^2)}$. If we take z to be a function of ρ only, where $\rho=\sqrt{(x^2+y^2)}$, in the equation

$$(1+p^2+q^2)T^2-4l^2S^2=0,$$

the equation assumes the form

$$\left\{1+\left(\frac{dz}{d\rho}\right)^2\right\}(\rho^2+z^2+l^2-k^2)^2-4l^2\left(\rho\frac{dz}{d\rho}-z\right)^2;$$

and the integral of this equation is

$$(z-c)^2+\{\rho-\sqrt{(k^2-c^2)}\}^2=l^2.$$

This represents a family of anchor rings. It is obtainable from the complete integral by taking c to be fixed, and constructing the special case of the general integral in which a and b vary in such a way that

$$a^2+l^2=k^2-c^2.$$

The singular integral of the last differential equation is

$$(z^2+\rho^2+l^2-k^2)^2=4l^2(z^2+\rho^2),$$

which reduces to the two spheres

$$x^2+y^2+z^2=(k\pm l)^2.$$

Another solution of the condition of integrability (obtained by combining particular cases of G and H) is

$$M\equiv x+zp=0;$$

* As mentioned in § 2, examples can be constructed in which $[F, M]$ is of the form $AF+BM$; these are more general cases of the third way of vanishing; one such example will be given immediately.

this gives $[F, M] = -2pF+[4Tz(1+p^2+q^2)+8l^2S]M,$

and so $[F, M]$ vanishes in consequence of both F and M vanishing.

If we substitute the value of p given by $M=0$ in $F=0$ and multiply through by z^2, we get

$$(x^2+z^2+z^2q^2)(x^2+y^2+z^2+l^2-k^2)^2 = 4l^2(-x^2-z^2+yzq)^2;$$

putting $z^2=v^2-x^2$, we may write this in the form

$$\left\{1+\left(\frac{\partial v}{\partial y}\right)^2\right\}(v^2+y^2+l^2-k^2)^2 = 4l^2\left(y\frac{\partial v}{\partial y}-v\right)^2.$$

The integral of this equation is

$$(v-c)^2+(y-b)^2 = l^2,$$

where $b^2+c^2=k^2$; this also represents a family of anchor rings.

Finally, it may be shown that $[H, K]$ vanishes identically; the integral of the equations $H=a$, $K=b$ is

$$(x-a)^2+(y-b)^2+(2-c)^2 = l^2+a^2+b^2+\mu^2-k^2,$$

where μ is an arbitrary constant.

This represents a triple family of spheres; it is not an integral of $G=0$, in general. It reduces to the double family when $\mu=c$, where $a^2+b^2+c^2=k^2$. This illustrates the first way in which the condition of integrability can be satisfied.

Example 3. The equation

$$F \equiv (z-\tfrac{1}{2}p^2)^2-\tfrac{1}{9}q^3 = 0$$

has been considered by Darboux*. He regards it as possessing an infinite number of singular integrals given by

$$z = \tfrac{1}{2}(x-a)^2, \tag{6}$$

because this equation satisfies all of the five equations which a singular integral must satisfy. His complete integral is

$$z = \tfrac{1}{2}(x-A)^2+\tfrac{1}{3}(y-B)^3, \tag{7}$$

and he deduced his singular integral by fixing A and varying B, so that his singular integral is really a particular case of a general integral.

The surface represented by (6) has contact of the second order with (7) along their curves of intersection.

* "Singulières intégrales des équations aux derivées partielles du premier ordre", 119, 203. *Mémoires des savants étrangers* (*Paris*), 27 (1883), No. 2.

The integral (6) can be obtained by combining F and G, where

$$G \equiv z - \tfrac{1}{2}p^2 = 0.$$

With this value of G, we have $[F, G] = -3F + 3G^2$, and so $[F, G]$ vanishes in the third way.

The integral given by Darboux is also obtainable by combining F and H where

$$H \equiv q/p = b.$$

The complete integral so obtained is

$$\{z - \tfrac{1}{12}b^6 - \tfrac{1}{2}b^3(x+by-a) - \tfrac{1}{2}(x+by-a)\}^2 = \tfrac{4}{9}b^3\{\tfrac{1}{4}b^3 + (x+by-a)\}^3,$$

which reduces to (6) when $b = 0$. It can be shown that $[F, H] \equiv 0$, so the condition of integrability is satisfied in the first way, and (6) is now exhibited as a special case of a complete integral.

Example 4. The equation

$$F \equiv (xp+yq-z)^2 + \frac{z^2}{x^2+y^2-1} - p^2 - q^2 = 0$$

has been discussed by Forsyth in his Presidential Address to this Society*. It is of importance in his theory of special integrals†.

The equations which must be satisfied by singular integrals reduce in this case to two, namely‡

$$p = \frac{xz}{x^2+y^2-1}, \quad q = \frac{yz}{x^2+y^2-1}.$$

These give the special integral

$$z^2 = c(x^2+y^2-1),$$

which is a hyperboloid of one sheet when c is positive.

If $$G \equiv p(x^2+y^2-1) - xz,$$

it is found that

$$[F, G] = -2xF + 4\left\{xp+yq-z - \frac{z}{x^2+y^2-1}\right\} G,$$

so the condition of integrability is satisfied in the third way.

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* *Proc. London Math. Soc.* (2), 4 (1906), 431–460 (435).

† *Theory of differential equations*, 5 (1906), 112, 188, and 280.

‡ In view of what follows it is interesting to note that p and q are *one-valued* functions of x, y, and z; this has not been the case in the previous examples.

The differential equation is resoluble into two equations, which are linear in p and q, with irrational coefficients, namely

(i) $$p(x^2-1)+q(xy-w)-z\left(x-\frac{y}{w}\right)=0,$$

(ii) $$p(x^2-1)+q(xy+w)-z\left(x+\frac{y}{w}\right)=0,$$

where $$w=\surd(x^2+y^2-1).$$

The general integral of (i) is

$$\frac{z}{w}=\phi\left(\frac{xy-w}{x^2-1}\right),$$

so that $$\frac{z}{w}=a\left(\frac{xy-w}{x^2-1}\right)+b$$

is a complete integral. When rationalised, this equation represents a double family of ruled quartic surfaces, which have no proper envelope. The special integral is the particular set obtained by taking $a=0$ and writing c for b^2.

Each of the special integrals meets each complete integral in four straight lines and a circle of contact. At any point of intersection of a complete integral and a special integral, which is not on the circle of contact, the surfaces have no common tangent plane, but they have two common tangent lines which are those generators of the special integral which pass through the point.

Example 5. We lastly consider the equation

$$F\equiv z-px-qy+p^nq^n=0,$$

and associate with it $$G\equiv px-qy=0.$$

We have $[F, G]=G$, so the condition of integrability is satisfied in the third way.

Using $G=0$ to eliminate q from $F=0$ and from $dz=pdx+qdy$, we get

$$z=2px-p^{2n}\frac{x^n}{y^n},\quad dz=pdx+\frac{px}{y}dy.$$

Substituting in the second of these the value of z given by the first, we get, on reduction,

$$\left(1-np^{2n-1}\frac{x^{n-1}}{y^n}\right)\left(2\frac{dp}{p}+\frac{dx}{x}-\frac{dy}{y}\right)xp=0.$$

When the first factor (which contains no differentials) is taken with $F=0$ and $G=0$, the result of eliminating p and q is the singular integral of $F=0$, namely

$$\cdot\left(\frac{z}{2n-1}\right)^{2n-1}=\left(\frac{x}{n}\right)^{n}\left(\frac{y}{n}\right)^{n}.$$

Taking the second factor,

$$2\frac{dp}{p}+\frac{dx}{x}-\frac{dy}{y}=0,$$

we get, on integration, $xp^2=cy$;

the result of eliminating p and q from this equation, combined with $F=0$ and $G=0$, is

$$z=2\sqrt{(cxy)}-c^n.$$

This is a special case of the general integral of $F=0$ which is derived from the complete integral

$$z=ax+by-a^n b^n$$

by taking $b=c/a$ and varying a, keeping c constant.

A NOTE ON HANKEL'S THEOREM

By C. Fox.

[Received 17 October, 1928.—Read 8 November, 1928.]

Hankel's theorem is as follows :—

If

(i) $\int_0^\infty |F(R)|\sqrt{R}\,dR$ *exists,*

(ii) $\nu \geqslant -\frac{1}{2}$,

(iii) $r\ (>0)$ *lies inside an interval in which* $F(R)$ *is of bounded variation, then*

$$(1) \qquad \int_0^\infty u\,du \int_0^\infty F(R)\,J_\nu(uR)\,J_\nu(ur)\,R\,dR = \tfrac{1}{2}\{F(r+0)+F(r-0)\}.$$

In this note I show that (1) is also true under the following conditions :—

(iA) *A positive number* $R_0\ (>0)$ *can be found such that* $F(R_0)\sqrt{R_0}$ *is finite,* $\int_0^{R_0} |F(R)|\sqrt{R}\,dR$ *exists, and* $F(R)\sqrt{R}$ *tends steadily to zero as* R *tends to infinity from* R_0 *onwards.*

(iiA) $\nu > -\frac{1}{2}$.

(iiiA) *The same as* (iii) *above.*

I am indebted to the referee for pointing out that (iA) is equivalent to the following condition :—

(iB) *A positive number* $R_0\ (>0)$ *can be found such that* $F(R_0)\sqrt{R_0}$ *is finite,* $\int_0^{R_0} |F(R)|\sqrt{R}\,dR$ *exists,* $F(R)\sqrt{R}$ *is of bounded variation in the infinite interval* (R_0, ∞) *and* $\lim_{R\to\infty} F(R)\sqrt{R} = 0$.

Watson's* proof of Hankel's theorem is based on four lemmas. The third lemma is deduced from the first two, while the fourth is concerned with properties of Bessel functions only. Hence it is sufficient to show that the first two lemmas of Watson's proof still remain true under conditions (iA), (iiA), and (iiiA) above. The lemmas in question are :—

1. *As* $\lambda \to \infty$,

$$\int_a^b F(R)\, J_\nu(\lambda R)\, R\, dR = o(\lambda^{-\frac{1}{2}}),$$

where $a \geqslant 0$ *and* b *may be infinite.*

2.
$$\int_0^\infty u\, du \int_0^\infty F(R)\, J_\nu(uR)\, J_\nu(ur)\, R\, dR$$
$$= \lim_{\lambda \to \infty} \int_0^\infty F(R) \left\{ \int_0^\lambda J_\nu(uR)\, J_\nu(ur)\, u\, du \right\} R\, dR,$$

provided that the limit on the right exists.

These we now proceed to prove.

Proof of Lemma 1. Following Watson, we divide the proof into three parts.

Part I. $F(R)\sqrt{R}$ bounded and $b = R_0$.

Part II. $F(R)\sqrt{R}$ bounded and b infinite.

Part III. $F(R)\sqrt{R}$ unbounded.

Part I. The conditions, and so the arguments, are the same as those used by Watson† in Part I of his proof of this lemma. His arguments establish the following result :—

From the asymptotic expansion of the Bessel function, both

$$|x^{\frac{1}{2}} J_\nu(x)| \quad \text{and} \quad \left| \int_0^x t^{\frac{1}{2}} J_\nu(t)\, dt \right|$$

are bounded whenever $\nu \geqslant -\frac{1}{2}$ and $x \geqslant 0$, their upper bounds being A and B, say. Let K be the upper bound of $|F(R)|\sqrt{R}$. *Then*

(2)
$$\left| \int_a^{R_0} F(R)\, J_\nu(\lambda R)\, R\, dR \right| \leqslant \frac{2BnK}{\lambda^{\frac{3}{2}}} + \frac{A\epsilon}{\lambda^{\frac{1}{2}}}$$

for every positive ϵ, *where* n *is a number depending upon* ϵ, a, *and* R_0, *but independent of* λ.

* *Theory of Bessel functions* (1922), 457–464. I adopt Watson's notation.

† *Loc. cit.*, 457.

The actual definition of n is as follows. Divide the range of integration (a, R_0) into n equal intervals by the points $x_1, x_2, \ldots, x_{n-1}$ $(x_0 = a, x_1 = R_0)$, and choose n so large that

$$\sum_{m=1}^{n} (U_m - L_m)(x_m - x_{m-1}) < \epsilon,$$

where ϵ is the number used in (2) above, and U_m and L_m are the upper and lower bounds of $F(R)\sqrt{R}$ in the m-th interval.

We shall take (2) as proved.

Part II. We first show that

$$\left|\int_{R_0}^{\infty} F(R)\, J_\nu(\lambda R)\, R\, dR\right| \leqslant 2\,|F(R_0)|\, R_0^{\frac{1}{2}}.B.\lambda^{-\frac{3}{2}}. \tag{3}$$

From the second mean value theorem

$$\int_{R_0}^{\infty} F(R)\, J_\nu(\lambda R)\, R\, dR = F(R_0)\, R_0^{\frac{1}{2}} \int_{R_0}^{R_1} J_\nu(\lambda R)\, R^{\frac{1}{2}}\, dR, \tag{4}$$

where $R_1 \geqslant R_0$;

$$\left|\int_{R_0}^{R_1} J_\nu(\lambda R)\, R^{\frac{1}{2}}\, dR\right| = \lambda^{-\frac{3}{2}} \left|\int_{\lambda R_0}^{\lambda R_1} J_\nu(t)\, t^{\frac{1}{2}}\, dt\right| \leqslant 2B\lambda^{-\frac{3}{2}}.$$

If we apply this to (4) the truth of (3) becomes evident.

On using the inequality

$$\left|\int_a^{\infty} F(R)\, J_\nu(\lambda R)\, R\, dR\right| \leqslant \left|\int_a^{R_0} F(R)\, J_\nu(\lambda R)\, R\, dR\right| + \left|\int_{R_0}^{\infty} F(R)\, J_\nu(\lambda R)\, R\, dR\right|$$

we see from Part I that

$$\left|\int_0^{\infty} F(R)\, J_\nu(\lambda R)\, R\, dR\right| < \frac{2BnK}{\lambda^{\frac{3}{2}}} + \frac{A\epsilon}{\lambda^{\frac{1}{2}}} + \frac{2BF(R_0)\, R_0^{\frac{1}{2}}}{\lambda^{\frac{3}{2}}}.$$

Since A, B, n, K, and $F(R_0)R_0^{\frac{1}{2}}$ are independent of λ, it follows that the integral with upper limit infinite is $o(\lambda^{-\frac{1}{2}})$.

Part III. If $F(R)\sqrt{R}$ satisfies condition (iA), it can be unbounded only for values of R lying in the range $(0, R_0)$. Hence we may enclose the points at which it is unbounded in a number of intervals δ such that

$$\sum_{\delta} \int_{\delta} |F(R)|\, \sqrt{R}\, dR < \epsilon.$$

But $|J_\nu(\lambda R)\sqrt{R}| < A\lambda^{-\frac{1}{2}}$ and so

$$\left|\sum_{\delta} \int_{\delta} F(R)\, J_\nu(\lambda R)\, R\, dR\right| \leqslant A\epsilon\lambda^{-\frac{1}{2}}.$$

Now apply the arguments of I and II to the $p+1$ parts of (a, b) outside these intervals and we get

$$\left|\int_a^b F(R)\, J_\nu(\lambda R)\, R\, dR\right| < 2Bn(p+1)K\lambda^{-\frac{3}{2}} + 2A\epsilon\lambda^{-\frac{1}{2}} + 2BF(R_0)\, R_0^{\frac{1}{2}}\lambda^{-\frac{3}{2}},$$

where K is the upper bound of $|F(R)|\sqrt{R}$ outside the intervals δ. Both K and n depend upon ϵ through the choice of the intervals δ, but they are independent of λ. Hence the integral on the left must be $o(\lambda^{-1})$ as $\lambda \to \infty$.

Proof of Lemma 2. There is little difficulty in showing that, for any given λ and β,

$$\int_0^\lambda J_\nu(ur)\, u \left\{\int_0^\beta F(R)\, J_\nu(uR)\, R\, dR\right\} du = \int_0^\beta F(R)\, R \left\{\int_0^\lambda J_\nu(uR)\, J_\nu(ur)\, u\, du\right\} dR,$$

and so, if we write $\Phi(u, R) = F(R)\, J_\nu(uR)\, J_\nu(ur)\, Ru$, we have

$$\left|\int_a^\lambda\int_0^\infty \Phi(u, R)\, dR\, du - \int_0^\infty\int_0^\lambda \Phi(u, R)\, du\, dR\right|$$

$$= \left|\int_0^\lambda\int_\beta^\infty \Phi(u, R)\, dR\, du - \int_\beta^\infty\int_0^\lambda \Phi(u, R)\, du\, dR\right|$$

$$\leqslant \left|\int_0^\lambda\int_\beta^\infty \Phi(u, R)\, dR\, du\right| + \left|\int_\beta^\infty\int_0^\lambda \Phi(u, R)\, du\, dR\right|.$$

But if $\nu > -\frac{1}{2}$, then $f(\lambda) = \int_0^\lambda |J_\nu(t)|\, t^{-\frac{1}{2}}\, dt$ exists* for every given λ. Hence, given λ and any arbitrary positive ϵ (>0), we may choose β_0 $(> R_0)$ so that $|F(\beta)|\, \beta^{\frac{1}{2}} < \epsilon / f(r\lambda)$ whenever $\beta > \beta_0$. Now from the second mean value theorem we have

$$\int_0^\lambda\int_\beta^\infty \Phi(u, R)\, dR\, du = F(\beta)\, \beta^{\frac{1}{2}} \int_0^\lambda J_\nu(ur)\, u \left\{\int_\beta^{\beta'} J_\nu(uR)\, R^{\frac{1}{2}}\, dR\right\} du,$$

where $\beta' \geqslant \beta$. Hence, using A and B as defined in Lemma 1, Part I, we see that

$$\left|\int_0^\lambda\int_\beta^\infty \Phi(u, R)\, dR\, du\right| < \frac{\epsilon}{f(r\lambda)} \cdot 2B \cdot \int_0^\lambda |J_\nu(ur)|\, u^{-\frac{1}{2}}\, du < 2Br^{-\frac{1}{2}}\epsilon.$$

Again†

$$\int_\beta^\infty\int_0^\lambda \Phi(u, R)\, du\, dR = \int_\beta^\infty F(R)\, \frac{RJ_{\nu+1}(\lambda R)\, J_\nu(\lambda r) - rJ_{\nu+1}(\lambda r)\, J_\nu(\lambda R)}{R^2 - r^2}\, \lambda R\, dR.$$

* $f(\lambda) > 0$ whenever $\lambda > 0$, $f(\lambda) \to 0$ as $\lambda \to 0$ and $f(\lambda) = O(\log \lambda)$ as $\lambda \to \infty$.

† We may evidently choose β greater than r.

But since $F(R)\sqrt{R}$ diminishes steadily as $R\ (>\beta>r)$ increases, so do $F(R)R^{\frac{3}{2}}/(R^2-r^2)$ and $F(R)R^{\frac{1}{2}}/(R^2-r^2)$. From Lemma I, Part II, equation (3), we then have

$$\left|\int_\beta^\infty F(R)\frac{R^2 J_{\nu+1}(\lambda R)\,J_\nu(\lambda r)}{R^2-r^2}\lambda dR\right| \leqslant 2\lambda\,|J_\nu(\lambda r)|\,|F(\beta)|\,\beta^{\frac{3}{2}}(\beta^2-r^2).B.\lambda^{-\frac{3}{2}}$$

$$< 2AB r^{-\frac{1}{2}}\beta\epsilon\left[f(r\lambda).(\beta^2-r^2).\lambda\right]^{-1}$$

and

$$\left|\int_\beta^\infty F(R)\frac{RJ_\nu(\lambda R)}{R^2-r^2}\lambda r J_{\nu+1}(\lambda r)\,dR\right| \leqslant 2\lambda r\,|J_{\nu+1}(\lambda r)|\,|F(\beta)|\,\beta^{\frac{1}{2}}(\beta^2-r^2)^{-1}.B.\lambda^{-\frac{3}{2}}$$

$$< 2AB r^{\frac{1}{2}}\epsilon\left[f(r\lambda)(\beta^2-r^2)\,\lambda\right]^{-1},$$

from which

$$\left|\int_\beta^\infty\int_0^\lambda \Phi(u,R)\,du\,dR\right| < 2AB r^{-\frac{1}{2}}\epsilon\left[f(r\lambda)(\beta-r)\,\lambda\right]^{-1}.$$

It follows now that

$$(5)\quad \left|\int_0^\lambda\int_0^\infty \Phi(u,R)\,dR\,du-\int_0^\infty\int_0^\lambda \Phi(u,R)\,du\,dR\right|$$

$$< 2Br^{-\frac{1}{2}}\epsilon\left\{1+A\left[f(r\lambda)(\beta-r)\,\lambda\right]^{-1}\right\}$$

whenever $\beta>\beta_0$, where ϵ is arbitrary, β_0 and $f(r\lambda)$ are functions of λ only, and $f(r\lambda)>0$ whenever $\lambda>0$. Hence, for any given λ, we can make the left-hand side of (5) less than any arbitrary positive number, however small. We must therefore have

$$\int_0^\lambda\int_0^\infty \Phi(u,R)\,dR\,du=\int_0^\infty\int_0^\lambda \Phi(u,R)\,du\,dR.$$

The proof of the lemma is then completed by making $\lambda\to\infty$.

NOTES ON THE THEORY OF SERIES (XI): ON TAUBERIAN THEOREMS

By G. H. HARDY *and* J. E. LITTLEWOOD.

[Received 10 November, 1928.—Read 13 December, 1928.]

1. This note originated from our desire to prove a theorem for *moment constants* corresponding to our "Tauberian" theorem concerning power series with positive coefficients*. We quote here, for reasons which will appear in the sequel, not this theorem, but the corresponding theorem for integrals (from which it is easily deduced).

THEOREM 1. *If $f(t)$ is positive† and integrable‡ over every finite range $(0, T)$ and $e^{-xt}f(t)$ is integrable over $(0, \infty)$ for every $x>0$; and if*

$$g(x) = \int_0^\infty e^{-xt} f(t)\, dt \sim Hx^{-\alpha}, \tag{1.1}$$

where $\alpha>0$, $H>0$, when $x\to 0$; then

$$F(t) = \int_0^t f(u)\, du \sim \frac{H}{\Gamma(1+\alpha)}\, t^\alpha \tag{1.2}$$

when $t\to\infty$. The result is still true when $\alpha>0$, $H=0$ if we interpret (1.1) to mean $g(x)=o(x^{-\alpha})$ and (1.2) similarly; and it is still true when $\alpha=0$, $H>0$§.

It should be observed that the theorem is trivial when $\alpha=0$.

The corresponding theorem for moment constants is (in the form in which it first presents itself)

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* Hardy and Littlewood, **1** and **2**. See also Hobson, **3**, 185, or Landau, **5**, 50.

† In the wider sense. We use the word thus throughout.

‡ In the sense of Lebesgue.

§ If α and H were both zero, $f(t)$ would necessarily be null.

THEOREM 2. *If $f(t)$ is positive and integrable over $(\delta, 1)$, for every $\delta > 0$, and $t^n f(t)$ is integrable over $(0, 1)$ for sufficiently large n; and if*

$$g_n = \int_0^1 t^n f(t)\, dt \sim Hn^{-\alpha}, \tag{1.3}$$

where $\alpha > 0$, $H > 0$, when $n \to \infty$ by integral values; then

$$F(t) = \int_{1-t}^1 f(u)\, du \sim \frac{H}{\Gamma(1+\alpha)}\, t^\alpha \tag{1.4}$$

when $t \to 0$. The result is still true when $\alpha > 0$, $H = 0$, if (1.3) and (1.4) be then interpreted as in Theorem 1.

This theorem has no significance when $\alpha = 0$, since $F(t) \to 0$ for any integrable $f(t)$. Its relation to Theorem 1 is better appreciated if we reduce it to a different form. It is plain first, since g_n decreases when n increases, that we may replace n by a continuous x. If then we write $t = e^{-w}$, $e^{-w} f(e^{-w}) = h(w)$, the hypothesis becomes

$$g(x) = \int_0^\infty e^{-xw} h(w)\, dw \sim Hx^{-\alpha}. \tag{1.31}$$

When we make a similar transformation in the conclusion, and observe that

$$\log \frac{1}{1-t} \sim t,$$

it becomes

$$H(t) = \int_0^t h(w)\, dw \sim \frac{H}{\Gamma(1+\alpha)}\, t^\alpha. \tag{1.41}$$

Hence Theorem 2 is equivalent to

THEOREM 3. *If $f(t)$ is positive and integrable over every finite range $(0, T)$, and $e^{-xt} f(t)$ is integrable over $(0, \infty)$ for sufficiently large x; and if (1.1) holds, with $\alpha > 0$, when $x \to \infty$; then (1.2) holds when $t \to 0$.*

This theorem is almost identical verbally with Theorem 1, differing from it only in an exchange of rôles between large and small values of x and t.

2. Theorems 1 and 3 are of just the same "depth" and Theorem 3 may be proved by an argument which runs parallel to that of Theorem 1. In spite of this similarity, it does not seem possible to deduce either, in any obvious manner, from the other. There is, however, another theorem

which is in essentials symmetrical as between large and small values of the variables and from which both Theorems 1 and 3 may be deduced.

Theorem 4. *Suppose that $f(t)$ is positive and integrable over every finite range $(0, T)$, and that*

$$\frac{f(t)}{(t+x)^\rho},$$

where $\rho > 0$, is integrable over $(0, \infty)$ for some (and so for all) $x > 0$. Suppose further that

$$h(x) = \int_0^\infty \frac{f(t)\,dt}{(t+x)^\rho} \sim \frac{H}{x^\sigma}, \tag{2.1}$$

where $0 < \sigma < \rho$, $H > 0$, when $x \to \infty$. Then

$$F(t) = \int_0^t f(u)\,du \sim \frac{H\Gamma(\rho)}{\Gamma(\sigma)\,\Gamma(\rho-\sigma+1)}\, t^{\rho-\sigma} \tag{2.2}$$

when $t \to \infty$.

Further, if (2.1) holds when $x \to 0$, then (2.2) holds when $t \to 0$.

We have stated the theorem with $0 < \sigma < \rho$, $H > 0$. The results are, in fact, true for $0 \leqslant \sigma \leqslant \rho$, $H \geqslant 0$, if we pay regard to the following glosses :—

(i) equations with $H = 0$ are to be interpreted as in Theorem 1;

(ii) when $0 < \sigma = \rho$, we are to suppose $H > 0$ in the first part and $H = 0$ in the second part of the theorem;

(iii) when $0 = \sigma < \rho$, we are to suppose $H = 0$ in the hypothesis of the first part, and $H > 0$ in the hypothesis of the second part, while the conclusion of each part is to be replaced by $F(t) = o(t^\rho)$;

(iv) if $0 = \sigma = \rho$, we are to suppose $H > 0$ in both hypotheses, and interpret both conclusions as meaning $F \to H$.

The results in these special cases are all trivial, and there is only one of them (the case $0 < \sigma = \rho$, $H > 0$, $x \to \infty$, $t \to \infty$) to which we shall refer later. We therefore simplify the discussion by dismissing them at the outset.

3. We observe first that, when $\rho = 1$, $0 < \sigma < 1$, and x and t tend to infinity, Theorem 4 is substantially the same as a theorem of Valiron rediscovered by Titchmarsh*.

* Valiron, **9**, 121–127; Titchmarsh, **7**, 8.

Valiron's theorem* asserts that, *if $f(t)$ is positive and increasing, and vanishes for sufficiently small t, and*

$$\int_0^\infty \frac{f(t)\,dt}{t(t+x)} \sim Ax^{\lambda-1}, \tag{3.1}$$

where $0<\lambda<1$, when $x\to\infty$, then

$$f(t) \sim At^\lambda, \tag{3.2}$$

when $t\to\infty$ †. We shall show that this theorem is deducible from the first part of Theorem 4 (with $\rho=1$), and conversely. Our proof depends on a theorem of Landau‡, which we shall require again later, and which we therefore state as a lemma.

LEMMA *a*. *If $xf'(x)$ increases with x and*

$$f(x) \sim x^a \quad (a>0)$$

when $x\to\infty$, then $f'(x) \sim ax^{a-1}$§.

(1) Suppose that the first part of Theorem 4 is known, and that the conditions of Valiron's theorem are satisfied. Writing $g(t)=t^{-1}f(t)$ in (3.1), and applying Theorem 4, we obtain

$$G(t)=\int_0^t g(u)\,du \sim Au^\lambda.$$

But $tG'=tg=f$ increases with t. Hence Lemma *a* gives $g\sim At^{\lambda-1}$, $f\sim At^\lambda$.

(2) Suppose that Valiron's theorem is known and that the conditions of the first part of Theorem 4 are satisfied, with $\rho=1$‖. It follows from (2.1) that

$$F(t)=\int_0^t f(u)\,du \leqslant 2t\int_0^t \frac{f(u)}{u+t}\,du \leqslant 2t\int_0^\infty \frac{f(u)}{u+t}\,du = O(t^{1-\sigma}),$$

---

* Valiron's theorem, as he proves it, is more general in other respects; we confine ourselves to the case of the theorem directly relevant here.

† Here and in what follows the A's are constants (functions of H, λ, ρ, or σ only) whose values are irrelevant to the explanation. The value of an A can always be determined, if required, by considering the appropriate case $f(x)=x^k$.

‡ See Landau, **4**, 218, or Hardy and Littlewood, **1**, 175.

§ This is, of course, also true when $x\to 0$.

‖ So that $\sigma<1$.

and *a fortiori* $F(t) = o(t)$, for large t. Hence we obtain, by partial integration,

$$\int_0^\infty \frac{f(t)\,dt}{t+x} = \int_0^\infty \frac{F(t)\,dt}{(t+x)^2},$$

and so

$$\int_0^\infty \frac{F(t)\,dt}{(t+x)^2} \sim \frac{H}{x^\sigma}.$$

Integrating with respect to x, and observing that

$$\int_0^x \frac{dy}{(t+y)^2} = \frac{1}{t} - \frac{1}{t+x} = \frac{x}{t(t+x)},$$

we obtain

$$\int_0^\infty \frac{F(t)\,dt}{t(t+x)} \sim Ax^{-\sigma}.$$

From this it follows, by Valiron's theorem, that $F \sim At^{1-\sigma}$.

4. It is convenient to prove three additional lemmas before we go further.

LEMMA β. *If $e^{-xt}f(t)$ is integrable over $(0, \infty)$ for sufficiently large x, and $f(t) \sim Ht^{-\beta}$, where $\beta < 1$, when $t \to 0$, then*

$$g(x) = \int_0^\infty e^{-xt} f(t)\,dt \sim H\Gamma(1-\beta)\,x^{-1+\beta} \tag{4.1}$$

when $x \to \infty$.

LEMMA γ. *If $e^{-xt}f(t)$ is integrable over $(0, \infty)$ for every $x > 0$, and $f(t) \sim Ht^{-\beta}$, where $\beta < 1$, when $t \to \infty$, then (4.1) holds when $x \to 0$.*

These lemmas are of a straightforward (Abelian) character, and it will be sufficient to prove the first of them. We can choose τ so that

$$(1-\epsilon)\,Ht^{-\beta} < f(t) < (1+\epsilon)\,Ht^{-\beta},$$

for $0 < t \leqslant \tau$, and then $g(x)$ lies between

$$(1 \mp \epsilon)\,H\int_0^\tau e^{-xt}t^{-\beta}\,dt \mp \int_\tau^\infty e^{-xt}\,|f(t)|\,dt.$$

The second term is not greater than

$$e^{-\frac{1}{2}x\tau}\int_\tau^\infty e^{-\frac{1}{2}xt}\,|f(t)|\,dt = O(e^{-\frac{1}{2}x\tau}),$$

when τ is fixed, and the integral in the first term is asymptotic to

$\Gamma(1-\beta)x^{-1+\beta}$. It follows that

$$(1-\epsilon)H\Gamma(1-\beta) \leqslant \underline{\lim}\, x^{\beta-1}g(x) \leqslant (1+\epsilon)H\Gamma(1-\beta),$$

which proves the lemma.

LEMMA 8. *Suppose that* $0<s<r$, *that* $f(t)$ *is integrable in every interval* $(0, T)$, *and* $t^{-r}f(t)$ *in every interval* (T, ∞), *where* $T>0$. *Then the relations*

$$F_1(t) = \int_t^\infty u^{-r}f(u)\,du \sim Ht^{-s} \quad (t\to 0) \tag{4.2}$$

and

$$F(t) = \int_0^t f(u)\,du \sim \frac{sH}{r-s}\,t^{r-s} \quad (t\to 0) \tag{4.3}$$

are equivalent.

We take $H=1$. Suppose, first, that (4.2) is satisfied. Then $t^r F_1(t)\to 0$ when $t\to 0$. From this and the fact that F_1 is (except at the origin) an integral, we deduce

$$F(t) = -\int_0^t u^r F_1'(u)\,du = r\int_0^t u^{r-1}F_1(u)\,du - t^r F_1(t)$$

$$\sim r\int_0^t u^{r-s-1}\,du - t^{r-s} = \frac{s}{r-s}\,t^{r-s}.$$

Next suppose (4.3) satisfied. Then

$$t^{-r}F(t) \leqslant t^{-r}\int_0^T f(u)\,du + \int_T^t u^{-r}f(u)\,du,$$

if $0<T<t$. The second term is less than ϵ if $T>T_0(\epsilon)$, and the first is $O(t^{-r})$ when T is fixed and $t\to\infty$. Hence $t^{-r}F(t)\to 0$ when $t\to\infty$. It follows that

$$F_1(t) = \int_t^\infty u^{-r}F'(u)\,du = r\int_t^\infty u^{-r-1}F(u)\,du - t^{-r}F(t)$$

$$\sim \frac{rs}{r-s}\int_t^\infty u^{-s-1}\,du - \frac{s}{r-s}\,t^{-s} = t^{-s},$$

when $t\to 0$.

5. We can now prove (1) that Theorem 1 is a corollary of Theorem 4 (when $x\to\infty$, $t\to\infty$), (2) that Theorem 3 (and therefore Theorem 2) is a corollary of Theorem 4 (when $x\to 0$, $t\to 0$), and (3) that each half of Theorem 4 is a corollary of the other. Thus the complete machinery of a Tauberian proof need not be used more than once.

(1) Suppose that the conditions of Theorem 1 are satisfied, and that $\rho > \alpha$. Then

$$h(y) = \int_0^\infty x^{\rho-1} e^{-xy} g(x)\, dx = \int_0^\infty x^{\rho-1} e^{-xy}\, dx \int_0^\infty e^{-xt} f(t)\, dt$$

$$= \int_0^\infty f(t)\, dt \int_0^\infty x^{\rho-1} e^{-x(t+y)}\, dx = \Gamma(\rho) \int_0^\infty \frac{f(t)\, dt}{(t+y)^\rho},$$

if either repeated integral exists, a condition certainly satisfied, since $g(x)$ is continuous for $x > 0$, tends to zero when $x \to \infty$, and is less than a multiple of $x^{-\alpha}$ when x is small.

It follows from Lemma β and (1.1) that

$$h(y) \sim H\Gamma(\rho-\alpha)\, y^{-\rho+\alpha}$$

when $y \to \infty$. Hence the conditions of Theorem 4 are satisfied, and the theorem gives $F(t) \sim At^\alpha$, where A is a constant which may be identified with the constant of (1.2).

(2) The proof that Theorem 3 is a corollary of the other half of Theorem 4 is the same in principle and may be omitted.

(3) Suppose that Theorem 4 has been proved when $x \to \infty$, $t \to \infty$, and write

$$x = \frac{1}{y}, \quad t = \frac{1}{u}, \quad u^{\rho-2} f\left(\frac{1}{u}\right) = g(u), \quad \rho = r, \quad \rho - \sigma = s.$$

Then $0 < s < r$, and the hypothesis becomes

$$\int_0^\infty \frac{g(u)}{(u+y)^r}\, du \sim \frac{A}{y^s},$$

where now $y \to 0$. The conclusion becomes

$$\int_u^\infty w^{-r} g(w)\, dw \sim Au^{-s}$$

when $u \to \infty$, and Lemma δ shows that this is equivalent to

$$\int_0^u g(w)\, dw \sim Aw^{r-s}$$

(with the appropriate values of the A's). The converse transformation goes similarly. Thus either case of Theorem 4 transforms into the other.

We should add that Titchmarsh* has given deductions of Theorem 1 (with $\alpha < 1$) and Valiron's theorem from one another. The preceding

* Titchmarsh, 8.

analysis shows that his deduction of the first theorem from the second may be simplified, since it is not necessary to use integrals of the Fourier type. No corresponding simplification of the converse deduction seems to be possible.

6. The preceding analysis shows that all our theorems are corollaries of either case of Theorem 4, say that in which the variables tend to infinity. It is therefore very desirable to give as direct and simple a proof of this theorem as possible. Such a proof must naturally follow the lines of our original proof of Theorem 1 or of Valiron's proof of his theorem; but Valiron's argument may be simplified considerably.

It follows from (2 . 1) that

$$F(x) \leqslant (2x)^\rho \int_0^x \frac{f(t)\,dt}{(t+x)^\rho} \leqslant (2x)^\rho \int_0^\infty \frac{f(t)\,dt}{(t+x)^\rho} = O(x^{\rho-\sigma}), \tag{6 . 1}$$

for large x. In particular $F(x) = o(x^\rho)$. Hence, integrating by parts, we obtain

$$h(x) = \rho \int_0^\infty \frac{F(t)\,dt}{(t+x)^{\rho+1}} = \int_0^\infty \frac{f(t)\,dt}{(t+x)^\rho} \sim \frac{H}{x^\sigma}. \tag{6 . 2}$$

We prove first that (6 . 2) *may be differentiated any number of times.*

It follows from (6 . 2) that

$$\frac{d}{dx}(x^{\rho+1}h) = \rho(\rho+1)\int_0^\infty \frac{x^\rho t F(t)\,dt}{(t+x)^{\rho+2}} = \rho(\rho+1)\int_0^\infty \frac{uF(xu)\,du}{(u+1)^{\rho+2}}\,*,$$

an increasing function of x. From this and

$$x^{\rho+1}h \sim Ax^{\rho+1-\sigma},$$

we deduce, by Lemma α†, that

$$\frac{d}{dx}(x^{\rho+1}h) \sim Ax^{\rho-\sigma}.$$

Expanding the left-hand side, inserting the appropriate value of A, and using (6 . 2), we obtain

$$h'(x) = \rho(\rho+1)\int_0^\infty \frac{F(t)\,dt}{(t+x)^{\rho+2}} \sim \frac{\sigma H}{x^{\sigma+1}}.$$

---

* The differentiations under the integral sign here and later are trivial, the derived integrals being uniformly convergent in any interval $x \geqslant \delta > 0$.

† We do not actually require the full force of Lemma α here; it would be enough that the result of the lemma should hold when f' increases.

This justifies one differentiation, and the argument may plainly be repeated.

We thus obtain

$$h^{(q)}(x) = \frac{\Gamma(\rho+q+1)}{\Gamma(\rho)} \int_0^\infty \frac{F(t)\,dt}{(t+x)^{\rho+q+1}} \sim \frac{\Gamma(\sigma+q)}{\Gamma(\sigma)} \frac{H}{x^{\sigma+q}}, \tag{6.3}$$

for $q = 0, 1, 2, \ldots$. Writing down this relation for $q = p, p+1, \ldots, 2p$, and observing that

$$\frac{t^p}{(t+x)^{\rho+2p+1}} = \frac{(t+x-x)^p}{(t+x)^{\rho+2p+1}} = \sum_{s=0}^{p} \binom{p}{s} \frac{(-x)^s}{(t+x)^{\rho+p+s+1}},$$

we find that

$$I_p = \int_0^\infty \frac{t^p F(t)\,dt}{(t+x)^{\rho+2p+1}} \tag{6.4}$$

is asymptotic to a certain multiple of $x^{-\sigma-p}$. The multiple in question may obviously be determined by supposing that $F(t)$ is actually equal to the right-hand side of (2.2). We thus obtain

$$I_p \sim M \int_0^\infty \frac{t^{p+\rho-\sigma}\,dt}{(t+x)^{\rho+2p+1}} = Mx^{-\sigma-p} J_p, \tag{6.5}$$

where

$$M = \frac{H\Gamma(\rho)}{\Gamma(\sigma)\,\Gamma(\rho-\sigma+1)}, \tag{6.61}$$

$$J_p = \int_0^\infty \frac{u^{p+\rho-\sigma}\,du}{(u+1)^{\rho+2p+1}} = \frac{\Gamma(p+\rho-\sigma+1)\,\Gamma(\sigma+p)}{\Gamma(\rho+2p+1)}. \tag{6.62}$$

This is true for all positive integral p.

7. LEMMA 6. *If J_p is the integral in* (6.62), *and $J_{p,1}$, $J_{p,2}$, $J_{p,3}$ the corresponding integrals over the ranges* $(0, 1-\zeta)$, $(1-\zeta, 1+\zeta)$, $(1+\zeta, \infty)$, *where* $0 < \zeta < 1$, *then*

$$J_{p,1} = \eta_p J_p, \qquad J_{p,2} = (1-\eta_p) J_p, \qquad J_{p,3} = \eta_p J_p,$$

where $\eta_p = \eta_p(\zeta) = \eta_p(\rho, \sigma, \zeta) \to 0$ *when* ρ, σ, ζ *are fixed and* $p \to \infty$.

A straightforward application of Stirling's theorem shows that

$$J_p \sim Ap^{-\frac{1}{2}} 2^{-2p} \tag{7.1}$$

when $p \to \infty$. On the other hand, $u(1+u)^{-2}$ has a maximum $\frac{1}{4}$ when $u = 1$, and its values when u is $1-\zeta$ or $1+\zeta$ are each less than a number

$\kappa = \kappa(\zeta) < \frac{1}{4}$. It follows that

$$J_{p,1} = \int_0^{1-\zeta} \frac{u^{p+\rho-\sigma}\,du}{(u+1)^{\rho+2p+1}} < \kappa^p \int_0^{\infty} \frac{u^{\rho-\sigma}\,du}{(u+1)^{\rho+1}} = A\kappa^p. \tag{7.2}$$

The conclusion concerning $J_{p,1}$ follows from (7.1) and (7.2), and it is obvious that a similar argument disposes of $J_{p,3}$.

8. We can now prove Theorem 4. We use the following notation. We denote by η_ζ a function of ζ (and ρ and σ) which tends to zero with ζ; by η_p a function of ζ and p (and ρ and σ) which tends to zero when ζ is fixed and $p \to \infty$; and by η_x a function of ζ, p, and x (and ρ and σ) which tends to zero when ζ and p are fixed and $x \to \infty$.

By (6.4) and (6.5) we have

$$I_p = \int_0^{\infty} \frac{t^p F(t)\,dt}{(t+x)^{\rho+2p+1}} = Mx^{-\sigma-p} J_p (1+\eta_x)^*. \tag{8.1}$$

We write

$$I_p = \int_0^1 + \int_1^{(1-\zeta)x} + \int_{(1-\zeta)x}^{(1+\zeta)x} + \int_{(1+\zeta)x}^{\infty} = I_{p,0} + I_{p,1} + I_{p,2} + I_{p,3}. \tag{8.2}$$

If $t \geqslant 1$, then $F(t) < At^{\rho-\sigma}$, by (6.1). Hence

$$I_{p,1} < A \int_0^{(1-\zeta)x} \frac{t^{p+\rho-\sigma}\,dt}{(t+x)^{\rho+2p+1}} = Ax^{-\sigma-p} J_{p,1},$$

$$I_{p,1} = x^{-\sigma-p} J_p \eta_p, \tag{8.3}$$

by Lemma ϵ; and similarly

$$I_{p,3} = x^{-\sigma-p} J_p \eta_p. \tag{8.4}$$

Also

$$I_{p,0} = O(x^{-\rho-2p-1}) = o(x^{-\sigma-p}) = x^{-\sigma-p}\eta_x. \tag{8.5}$$

From (8.1)–(8.5) we derive

$$I_{p,2} = Mx^{-\sigma-p} J_p (1+\eta_p+\eta_x). \tag{8.6}$$

On the other hand, since $F(t)$ increases with t, we have

$$I_{p,2} \leqslant \frac{F(x+\zeta x)}{(x-\zeta x)^{\rho-\sigma}} \int_{x-\zeta x}^{x+\zeta x} \frac{t^{p+\rho-\sigma}\,dt}{(t+x)^{\rho+2p+1}} = \frac{F(x+\zeta x)}{(x+\zeta x)^{\rho-\sigma}} \left(\frac{1+\zeta}{1-\zeta}\right)^{p-\sigma} x^{-\sigma-p} J_{p,2}.$$

* The η_x here is independent of ζ.

Combining this with (8 . 6), we obtain

$$\frac{F(x+\xi x)}{(x+\zeta x)^{\rho-\sigma}} \geqslant (1+\eta_\zeta)\, x^{\sigma+p}\, \frac{I_{p,2}}{J_{p,2}} \geqslant (1+\eta_\zeta)\, x^{\sigma+p}\, \frac{I_{p,2}}{J_p} \tag{8.7}$$

$$\geqslant M(1+\eta_\zeta+\eta_p+\eta_x).$$

Similarly

$$I_{p,2} \geqslant \frac{F(x-\xi x)}{(x-\xi x)^{\rho-\sigma}} \left(\frac{1-\xi}{1+\xi}\right)^{\rho-\sigma} x^{-\sigma-p}\, J_{p,2},$$

and so

$$\frac{F(x-\xi x)}{(x-\xi x)^{\rho-\sigma}} \leqslant (1+\eta_\zeta)\, x^{\sigma+p}\, \frac{I_{p,2}}{J_{p,2}} \leqslant (1+\eta_\zeta+\eta_p)\, x^{\sigma+p}\, \frac{I_{p,2}}{J_p} \tag{8.8}$$

$$\leqslant M(1+\eta_\zeta+\eta_p+\eta_x).$$

It is plain that (8 . 7) and (8 . 8) are possible only if $F(t) \sim Mt^{\rho-\sigma}$, which proves the theorem. As we have seen, Theorem 4 carries with it Theorems 1–3.

We have ignored the special cases mentioned at the end of § 2. The only one which is relevant is that in which $\sigma = \rho > 0$, $H > 0$, and the variables tend to infinity. The truth of the result in this case follows at once from (6 . 1), which shows that $F(t)$ is bounded.

9. It is known* that the condition $f(t) \geqslant 0$ in Theorem 1 may be replaced by $f(t) \geqslant -Kt^{\alpha-1}$, where K is a positive constant; the extension is immediate when $\alpha > 0$, but less obvious when $\alpha = 0$†. There are corresponding developments of Theorem 4: we confine ourselves to the case in which $x \to \infty$, $t \to \infty$.

THEOREM 5. *If $\rho > \sigma$ in Theorem 4, the condition $f(t) \geqslant 0$ may be replaced by*

$$f(t) \geqslant -Kt^{\rho-\sigma-1}, \tag{9.1}$$

where $K > 0$. The result remains true when $\rho = \sigma$: if $\rho > 0$ and

$$f(t) \geqslant -Kt^{-1}, \tag{9.2}$$

$$h(x) = \int_0^\infty \frac{f(t)\,dt}{(t+x)^\rho} \sim \frac{H}{x^\rho}, \tag{9.3}$$

then

$$\int_0^\infty f(t)\,dt = H. \tag{9.4}$$

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* Hardy and Littlewood, **1**, 185.

† Hardy and Littlewood, **1**, 188.

When $\rho > \sigma$, Theorem 5 is an immediate corollary of Theorem 4; we need only apply Theorem 4 to

$$g(t) = f(t) + Kt^{\rho-\sigma-1}*.$$

The case $\rho = \sigma$ is more delicate, and we require Lemma η below: Lemma ζ is included for the sake of completeness and is not actually used.

LEMMA ζ. *If* (9.4) *is true, then* (9.3) *is true.*

This is the "Abelian" theorem, and follows at once from the relations

$$h(x) = \left(\int_0^T + \int_T^\infty\right) \frac{f(t)\,dt}{(t+x)^\rho} = h_1(x) + h_2(x),$$

$$|h_2(x)| \leqslant x^{-\rho}\left|\int_T^{T'} f(t)\,dt\right| \leqslant \epsilon x^{-\rho} \quad [T' > T \geqslant T_0(\epsilon)],$$

$$h_1(x) \sim x^{-\rho}\int_0^T f(t)\,dt.$$

LEMMA η. *If*

$$G(t) = \int_0^t u f(u)\,du = o(t), \tag{9.5}$$

and, in particular, if

$$f(t) = o(t^{-1}), \tag{9.6}$$

then (9.3) *implies* (9.4).

This is the "o-Tauberian" theorem. To prove it, we write

$$\begin{aligned} x^\rho h(x) &= \int_0^x f(t)\,dt - \int_0^x \left\{1 - \frac{x^\rho}{(t+x)^\rho}\right\} f(t)\,dt + x^\rho \int_x^\infty \frac{f(t)\,dt}{(t+x)^\rho} \\ &= F(x) + l(x) + m(x), \end{aligned} \tag{9.7}$$

say. Here

$$\begin{aligned} l(x) &= -\int_0^x \left\{1 - \frac{x^\rho}{(t+x)^\rho}\right\} \frac{G'(t)}{t}\,dt \\ &= -(1-2^{-\rho})\frac{G(x)}{x} + \rho x^\rho \int_0^x \frac{G(t)\,dt}{t(t+x)^{\rho+1}} - \int_0^x \left\{1 - \frac{x^\rho}{(t+x)^\rho}\right\} \frac{G(t)}{t^2}\,dt. \end{aligned}$$

The first term is $o(1)$. The second is

$$O(x^\rho)\,O(x^{-\rho-1})\int_0^x o(1)\,dt = o(1).$$

* Cf. Hardy and Littlewood, **1**, 185.

In the third we observe that the expression in the curly bracket does not exceed

$$A\left(1-\frac{x}{t+x}\right)=\frac{At}{x},$$

so that the third term is

$$\frac{1}{x}\int_0^x o(1)\,dt=o(1).$$

Hence

$$l(x)=o(1). \tag{9.8}$$

Finally

$$\begin{aligned} m(x) &= x^\rho\int_x^\infty \frac{f(t)\,dt}{(t+x)^\rho} = x^\rho\int_x^\infty \frac{G'(t)\,dt}{t(t+x)^\rho} \\ &= -2^{-\rho}\frac{G(x)}{x}+\rho x^\rho\int_x^\infty \frac{G(t)\,dt}{t(t+x)^{\rho+1}}+x^\rho\int_x^\infty \frac{G(t)\,dt}{t^2(t+x)^\rho} \\ &= o(1)+o\left(x^\rho\int_x^\infty \frac{dt}{t^{\rho+1}}\right)=o(1). \end{aligned} \tag{9.9}$$

From (9.7), (9.8), (9.9), and (9.3) it follows that

$$\int_0^x f(t)\,dt\to H,$$

which proves the lemma.

10. We can now prove Theorem 5. We write

$$\phi(x)=x^{\rho+1}h(x)=\int_0^\infty \frac{x^{\rho+1}}{(t+x)^\rho}f(t)\,dt. \tag{10.1}$$

Then

$$\phi(x)\sim Hx \tag{10.2}$$

and*

$$\begin{aligned} \phi''(x) &= \int_0^\infty \frac{d^2}{dx^2}\left\{\frac{x^{\rho+1}}{(t+x)^\rho}\right\}f(t)\,dt \\ &= \rho(\rho+1)\int_0^\infty \frac{x^{\rho-1}t^2}{(t+x)^{\rho+2}}f(t)\,dt > -A\int_0^\infty \frac{x^{\rho-1}t}{(t+x)^{\rho+2}}\,dt=-\frac{A}{x}. \end{aligned} \tag{10.3}$$

* The differentiations under the integral sign present no difficulty: compare f.n. *, p. 30.

From (10.2) and (10.3) it follows* that

$$\phi'(x) \to H, \tag{10.4}$$

i.e.

$$(\rho+1)x^{\rho}h(x)+x^{\rho+1}h'(x) \to H,$$

and so

$$h'(x) = -\rho\int_0^\infty \frac{f(t)\,dt}{(t+x)^{\rho+1}} \sim -\rho H x^{-\rho-1}.$$

Hence

$$\int_0^\infty \frac{tf(t)\,dt}{(t+x)^{\rho+1}} = \int_0^\infty \frac{f(t)\,dt}{(t+x)^{\rho}} - x\int_0^\infty \frac{f(t)\,dt}{(t+x)^{\rho+1}} = o(x^{-\rho}).$$

But

$$g(t) = tf(t) \geqslant -K.$$

Hence we may apply the case of Theorem 5, already proved, in which ρ is replaced by $\rho+1$ and σ by $\rho < \rho+1$. It follows that

$$G(t) = \int_0^t uf(u)\,du = o(u),$$

and the final conclusion then follows from Lemma η.

11. It is plain that we may prove similarly

THEOREM 6. *Suppose that* $0 \leqslant \sigma \leqslant \rho$, *that*

$$a_n \geqslant -Kn^{\rho-\sigma-1},$$

that

$$h(x) = \sum_1^\infty \frac{a_n}{(n+x)^{\rho}}$$

is convergent for $\rho > 0$, *and that*

$$h(x) \sim Hx^{-\rho},$$

when $x \to \infty$. *Then*

$$\sum_1^\infty a_n = H.$$

[*Added July*, 1929. Important contributions to the theory of Tauberian theorems have been made recently by Schmidt (**6**), Vijayaraghavan (**10**), and Wiener (**11**).

Schmidt proved first that the condition $na_n = O(1)$ of Littlewood's original theorem (afterwards generalized by us to $na_n > -A$) could be replaced by

$$\underline{\lim}\,(s_n - s_m) \geqslant 0,$$

* Cf. Hardy and Littlewood, **1**, 188.

where $s_n = a_0 + a_1 + \ldots + a_n$ and m and n tend to infinity in such a manner that $n/m \to 1$. This condition includes all previously known. A much simpler proof than Schmidt's was given later by Vijayaraghavan. Schmidt also proved what is substantially the corresponding generalization of Theorem 1.

We have proved the corresponding generalization of Theorem 4, in which the condition is

$$\underline{\lim} \frac{F(qx) - F(px)}{x^{\rho-\sigma}} \geqslant 0,$$

when $n \to \infty$, $q > p$, $q/p \to 1$. Our method of proof is unlike those of Schmidt or Vijayaraghavan, but we must reserve its publication until another occasion.

We may, however, take this opportunity of calling attention to Wiener's paper, in which Tauberian theorems generally are investigated from a quite novel point of view.]

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DIVERGENT CONTINUED FRACTIONS AND NON-POLAR SINGULARITIES

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[Received 23 May, 1928.—Read 14 June, 1928.]

1. *Introduction.*

Three previous papers on "Divergent continued fractions and polar singularities"† dealt with the Padé continued fractions of the rows when representing a function meromorphic in a region containing the origin. In the present paper the effects of certain algebraic and logarithmic singularities are considered in regard to functions for which the region of meromorphy is restricted by singularities of this type. The discussion for polar singularities was introduced by a theorem on the region of representation of the Padé continued fractions of the rows (and columns) due to Montessus de Ballore‡. In the same way, the treatment of non-polar singularities must be preceded by the corresponding theorem stated below, which is proved for the cases required by this paper in §§ 6–11.

Montessus' theorem states that, for a function meromorphic in a region containing the origin, the Padé continued fraction of the first row converges within the circle passing through the pole nearest the origin; that of the second row within the circle passing through the second pole (except at the enclosed pole); and so on. The theorem on

† R. Wilson, "Divergent continued fractions and polar singularities", *Proc. London Math. Soc.* (2), 26 (1927), 159–168. Part II, *Proc. London Math. Soc.* (2), 27 (1928), 497–512. Part III, *Proc. London Math. Soc.* (2), 28 (1928), 128–144.

‡ R. Montessus de Ballore, "Sur les fractions continues algébriques", *Rend. di Palermo* 19 (1905), 1–73.

non-polar singularities (of the types considered) states, in addition, that the continued fraction of the row corresponding to the first non-polar singularity converges (except at enclosed poles) within the circle passing through this singularity, and so do the continued fractions of all succeeding rows. In the cases of both theorems the continued fractions diverge outside the corresponding circles.

The order of divergence of the continued fraction belonging to the circle determined by the first non-polar singularity is discussed at enclosed poles and at the non-polar singularity, and the behaviour of the continued fraction is considered at other points on the bounding circle and within it.

A simple algebraic singularity is first taken, then a simple logarithmic singularity, then an algebraic logarithmic singularity. Finally, cases are discussed in which the boundary possesses both polar and non-polar singularities of the different types, and the treatment is extended to include the continued fractions of the succeeding rows. The argument depends upon a combination of the methods employed in the previous papers, and, where possible, calculations will be replaced by references to these papers, one, two, or three asterisks being employed to signify the respective papers.

2. *Method.*

Take a function having poles at ..., $1/\beta_s$, ..., $1/\beta_r$, ... of the respective orders ...m, ... p, ..., and an algebraic singularity at $1/\alpha$ of non-integral order k†. Then, as in § 2**, it is sufficient to consider the function

$$f(z) \equiv \sum_s \frac{B_s}{(1-\beta_s z)^m} + \frac{A}{(1-\alpha z)^k} + \phi(z), \tag{1}$$

where $\phi(z)$ is holomorphic within a circle of radius $1/\rho$ and

$$\rho < |\alpha| < \ldots < |\beta_r| < \ldots < |\beta_s| < \ldots,$$

the number of polar singularities, counted according to multiplicity, being ν.

† It is convenient to take the singularity in the form shown in (1), which includes singularities of the form $A'(1-\alpha z)^{k'}$, the exponent being regarded as negative in this case. If k is a positive integer the singularity becomes polar. The results proved in this paper apply to all three forms, except where otherwise stated.

Let the Taylor expansion of the function in the neighbourhood of the origin be

$$f(z) \equiv c_0 + c_1 z + c_2 z^2 + \ldots + c_n z^n + \ldots .$$

Then, from (1),

$$c_n = \sum_s {}^{-m}c_n B_s \beta_s^n + {}^{-k}c_n A \alpha^n + M_n \rho^n,$$

where $|M_n| < M$, a finite constant, and

$${}^{-m}c_n = m(m+1)\ldots(m+n-1)/n\,!,$$

$${}^{-k}c_n = \frac{\Gamma(k+n)}{\Gamma(k)\,\Gamma(n+1)}.$$

Using Stirling's theorem†

$$\Gamma(n) = \eta(n) \,.\, n^{n-\frac{1}{2}} e^{-n},$$

where $$\eta(n) = (2\pi)^{\frac{1}{2}}\left[1 + \frac{1}{12n} + O\left(\frac{1}{n^2}\right)\right],$$

it is easy to show that

$${}^{-k}c_n = \frac{n^{k-1}}{\Gamma(k)}\left[1 + O\left(\frac{1}{n}\right)\right] \tag{2}$$

for sufficiently large values of n. The polynomials $P_{n,\nu}(z)$, $Q_{n,\nu}(z)$, defined as in § 2*, are evaluated for large values of n. By definition, the continued fraction of the $(\nu+1)$-th row converges if $\lim_{n\to\infty} \frac{P_{n,\nu}(z)}{Q_{n,\nu}(z)}$ or, as we shall write it, $\lim_{n\to\infty} R_{n,\nu}(z)$ exists and is finite. The form of $R_{n,\nu}(z)$ for large values of n will now be discussed at

(*a*) an enclosed pole at $1/\beta_r$,

(*b*) the singularity at $1/\alpha$,

(*c*) all regular points on $|z\alpha| = 1$, and within it.

† Whittaker and Watson, *Modern analysis* (1915), 247.

3. *Evaluation of* $Q_{n,\nu}(1/\beta_r)$.

As in § 3**,

$$Q_{n,\nu}(1/\beta_r) = \frac{\Delta_{1Q}\Delta_{2Q}}{B_r\beta_r^\nu}\,[1+F_n(\rho/\alpha)^n]$$

$$= \frac{1}{B_r\beta_r^\nu}\begin{vmatrix} \dots, 0, & \dots, 0, & \dots, 0 \\ \dots, {}^{-1}c_0\beta_s^{n-\nu+1}, & \dots, {}^{-1}c_{m-1}\beta_s^{n-\nu+1}, & \dots, {}^{-1}c_0\beta_r^{n-\nu+1}, \\ \dots \quad \dots & \dots \quad \dots \quad \dots & \dots \quad \dots \quad \dots \\ \dots \quad \dots & \dots \quad \dots \quad \dots & \dots \quad \dots \quad \dots \\ \dots, {}^{-\nu}c_0\beta_s^n, & \dots, {}^{-1}c_{m-1}\beta_s^n, & \dots, {}^{-\nu}c_0\beta_r^n \end{vmatrix}$$

$$\begin{vmatrix} \dots, 1, & \dots, 0, & \dots, 0 \\ \dots, {}^{-1}c_{p-1}\beta_r^{n-\nu+1}, & \dots, \alpha^{n-\nu+1}, & \dots, 0 \\ \dots \quad \dots & \dots \quad \dots & \dots \quad \dots \\ \dots \quad \dots & \dots \quad \dots & \dots \quad \dots \\ \dots, {}^{-\nu}c_{p-1}\beta_r^n, & \dots, 0, & \dots, \alpha^n \end{vmatrix}$$

$$\times \begin{vmatrix} \dots, {}^{-m}c_{n-\nu}B_s, & \dots, {}^{-1}c_{n-\nu}B_s, & \dots, {}^{-p}c_{n-\nu}B_r \\ \dots, {}^{-m}c_{n-\nu+1}B_s\beta_s, & \dots, {}^{-1}c_{n-\nu+1}B_s\beta_s, & \dots, {}^{-p}c_{n-\nu+1}B_r\beta_r, \\ \dots \quad \dots & \dots \quad \dots \quad \dots & \dots \quad \dots \quad \dots \\ \dots \quad \dots & \dots \quad \dots \quad \dots & \dots \quad \dots \quad \dots \\ \dots, {}^{-m}c_nB_s\beta_s^\nu, & \dots, {}^{-1}c_nB_s\beta_s^\nu, & \dots, {}^{-1}c_nB_r\beta_r^\nu \end{vmatrix}$$

$$\begin{vmatrix} \dots, {}^{-1}c_{n-\nu}B_r, & \dots, {}^{-k}c_{n-\nu+1}A, & \dots, {}^{-k}c_nA \\ \dots, {}^{-1}c_{n-\nu+1}B_r\beta_r, & \dots, {}^{-k}c_{n-\nu+2}A\alpha, & \dots, {}^{-k}c_{n+1}A\alpha \\ \dots \quad \dots & \dots \quad \dots \quad \dots & \dots \quad \dots \quad \dots \\ \dots \quad \dots & \dots \quad \dots \quad \dots & \dots \quad \dots \quad \dots \\ \dots, {}^{-1}c_nB_r\beta_r^\nu, & \dots, {}^{-k}c_{n+1}A\alpha^\nu, & \dots, {}^{-k}c_{n+\nu}A\alpha^\nu \end{vmatrix}$$

$$\times[1+F_n(\rho/\alpha)^n],$$

where $|F_n|<F$, a finite constant, and Δ_{1Q}, Δ_{2Q} are $(\nu+1)$-rowed, 2ν-columned arrays†.

Hence $Q_{n,\nu}(1/\beta_r)$ can be expressed as the sum of 2ν determinant biproducts obtained by multiplying together the corresponding determinants in Δ_{1Q}, Δ_{2Q} in which $\nu-1$ columns are omitted. Of these, the ν biproducts obtained by retaining only one α column are clearly of higher order than the rest.

† The term "matrix" was used in previous papers in this sense.

Using (2), we readily obtain the following result

$$\Delta_{1Q}\Delta_{2Q} = \frac{n^{k-1}}{\Gamma(k)} \begin{vmatrix} \dots, 0, & \dots, 0, & \dots, 0, \\ \dots, {}^{-1}c_0\beta_s^{n-\nu+1}, & \dots, {}^{-1}c_{m-1}\beta_s^{n-\nu+1}, & \dots, {}^{-1}c_0\beta_r^{n-\nu+1}, \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots, {}^{-\nu}c_0\beta_s^n, & \dots, {}^{-\nu}c_{m-1}\beta_s^n, & \dots, {}^{-\nu}c_0\beta_r^n, \end{vmatrix}$$

$$\begin{vmatrix} \dots, 1, & \dots, 0 \\ \dots, {}^{-1}c_{p-1}\beta_r^{n-\nu+1}, & \dots, \alpha^{n-\nu+1} \\ \dots & \dots \\ \dots & \dots \\ \dots, {}^{-\nu}c_{p-1}\beta_r^n, & \dots, \alpha^n \end{vmatrix}$$

$$\times \begin{vmatrix} \dots, {}^{-m}c_{n-\nu}B_s, & \dots, {}^{-1}c_{n-\nu}B_s, & \dots, {}^{-p}c_{n-\nu}B_r, \\ \dots, {}^{-m}c_{n-\nu+1}B_s\beta_s, & \dots, {}^{-1}c_{n-\nu+1}B_s\beta_s, & \dots, {}^{-p}c_{n-\nu+1}B_r\beta_r, \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots, {}^{-m}c_nB_s\beta_s^\nu, & \dots, {}^{-1}c_nB_s\beta_s^\nu, & \dots, {}^{-p}c_nB_r\beta_r^\nu, \end{vmatrix}$$

$$\begin{vmatrix} \dots, {}^{-1}c_{n-\nu}B_r, & \dots, A \\ \dots, {}^{-1}c_{n-\nu+1}B_r\beta_r, & \dots, A\alpha \\ \dots & \dots \\ \dots & \dots \\ \dots, {}^{-1}c_nB_r\beta_r^\nu, & \dots, A\alpha^\nu \end{vmatrix}$$

$$\times[1+(G_n/n)],$$

where $|G_n| < G$, a finite constant.

Writing $\bar{\Delta}_{1Q}$, $\bar{\Delta}_{2Q}$ for the above determinants, it follows that

$$Q_{n,\nu}(1/\beta_r) = \frac{n^{k-1}}{\Gamma(k)} \frac{\bar{\Delta}_{1Q}\bar{\Delta}_{2Q}}{B_r\beta_r^\nu}\ [1+(H_n/n)],$$

where $|H_n| < H$, a finite constant.

4. *Evaluation of* $P_{n,\nu}(1/\beta_r)$.

The evaluation of $P_{n,\nu}(1/\beta_r)$ proceeds as in §§ 4**–5** with an additional contribution to every $1/\beta_r^t\psi_{n-t}(1/\beta_r)$ due to the singularity at $1/\alpha$. It will be noted that

$$c_n = \sum_s {}^{-m}c_nB_s\beta_s^n + P_n\alpha^n,$$

where $|P_n|/n^{k-1} < P$, a finite constant.

The additional contribution thus takes the form

$$\frac{1}{\beta_r^t}\left[1+P_1\left(\frac{\alpha}{\beta_r}\right)+\ldots+P_{n-t}\left(\frac{\alpha}{\beta_r}\right)^{n-t}\right]=\frac{1}{\beta_r^t}\,\chi_{n-t}\left(\frac{1}{\beta_r}\right). \tag{3}$$

We shall now show that its effect on the evaluation of $P_{n,\nu}(1/\beta_r)$ for large values of n is negligible, giving, as it does,

$$\frac{1}{\beta_r^\nu}\,\Delta'_{1P}\Delta'_{2P}=\frac{1}{\beta_r^\nu}\begin{vmatrix} \ldots, 0, & \ldots, 0, & \ldots, 0, & 0, \ldots, 0, 1 \\ \ldots, {}^{-1}c_0\beta_r^{n-\nu+1}, & \ldots, {}^{-1}c_{p-1}\beta_r^{n-\nu+1}, & \ldots, \alpha^{n-\nu+1}, & 0, \ldots, 0, 0 \\ \cdots \quad \cdots & \cdots \quad \cdots & \cdots \quad \cdots & \cdots \quad \cdots \\ \cdots \quad \cdots & \cdots \quad \cdots & \cdots \quad \cdots & \cdots \quad \cdots \\ \ldots, {}^{-\nu}c_0\beta_r^n, & \ldots, {}^{-\nu}c_{p-1}\beta_r^n, & \ldots, 0, & 0, \ldots, \alpha^n, 0 \end{vmatrix}$$

$$\times\begin{vmatrix} \ldots, {}^{-p}c_{n-\nu}B_r, & \ldots, {}^{-1}c_{n-\nu}B_r, & \ldots, {}^{-k}c_{n-\nu+1}A, \\ \ldots, {}^{-p}c_{n-\nu+1}B_r\beta_r, & \ldots, {}^{-1}c_{n-\nu+1}B_r\beta_r, & \ldots, {}^{-k}c_{n-\nu+2}A\alpha, \\ \cdots \quad \cdots & \cdots \quad \cdots & \cdots \quad \cdots \\ \cdots \quad \cdots & \cdots \quad \cdots & \cdots \quad \cdots \\ \ldots, {}^{-p}c_nB_r\beta_r^\nu, & \ldots, {}^{-1}c_nB_r\beta_r^\nu, & \ldots, {}^{-k}c_{n+1}A\alpha^\nu \end{vmatrix}$$

$$\begin{vmatrix} \ldots, {}^{-k}c_nA, & \chi_{n-\nu} \\ \ldots, {}^{-k}c_{n+1}A\alpha, & \chi_{n-\nu+1}\beta_r \\ \cdots \quad \cdots & \cdots \quad \cdots \\ \cdots \quad \cdots & \cdots \quad \cdots \\ \ldots, {}^{-k}c_{n+\nu}A\alpha^\nu, & \chi_n\beta_r^\nu \end{vmatrix},$$

these being $(\nu+1)$-rowed, $(2\nu+1)$-columned arrays.

Since $$\chi_{n-\nu+t}=\chi_{n-\nu}+n^{k-1}\left(\frac{\alpha}{\beta_r}\right)^{n-\nu+t}\eta_t,$$

where $|\eta_t|<\eta$, a finite constant, it follows that, on subtracting from the right-hand columns of Δ'_{2P} corresponding elements of the *column* ${}^{-1}c_{n-\nu}B_r, \ldots, {}^{-1}c_nB_r\beta_r^\nu$, each multiplied by $\chi_{n-\nu}$,

$$\lim_{n\to\infty}\left(\frac{\Delta'_{1P}\Delta'_{1P}}{\Delta_{1Q}\Delta_{2Q}}\right)$$

is finite.

As in § 5**, it can be shown that $\bar{\Delta}_{1P}\bar{\Delta}_{2P}$, the surviving determinant biproduct of $P_{n,\nu}(1/\beta_r)$, is such that

$$\lim_{n\to\infty}\left(\frac{\bar{\Delta}_{1P}\bar{\Delta}_{2P}}{\bar{\Delta}_{1Q}\bar{\Delta}_{2Q}}\right)\Big/\left(\frac{\beta_r}{\alpha}\right)^n$$

is finite, and hence the effect of the contribution (3) is negligible.

Using Lemmas 1–2 of § 5** for the evaluation of the determinants $\bar{\Delta}_{1P}$, $\bar{\Delta}_{2P}$, $\bar{\Delta}_{1Q}$, $\bar{\Delta}_{2Q}$, we obtain the following results, as in § 5**,

$$\frac{P_{n,\nu}(1/\beta_r)}{Q_{n,\nu}(1/\beta_r)} = \frac{\Gamma(k)}{n^{k-1}} \frac{B_r^2}{A} \left(\frac{\beta_r}{\alpha}\right)^{n-\nu+1} \frac{\overset{s}{\Pi}{}'(\beta_r-\beta_s)^m}{\overset{s}{\Pi}{}'(\alpha-\beta_s)^m} \frac{\beta_r^{2\rho-1}[1+O(1/n)]}{(\alpha-\beta_r)^{2\rho-1}},$$

where terms in $(\alpha \sim \beta_r)$ are omitted in Π'.

Hence

$$\lim_{n\to\infty} \frac{R_{n,\nu}(1/\beta_r)}{(\beta_r/\alpha)^n/n^{k-1}} = \frac{B_r^2\,\Gamma(k)}{A(\alpha-\beta_r)^{2\rho-1}} \frac{\alpha^{\nu-1}}{\beta_r^{\nu-2\rho}} \frac{\overset{s}{\Pi}{}'(\beta_r-\beta_s)^{2m}}{\overset{s}{\Pi}{}'(\alpha-\beta_s)^{2m}}. \tag{4}$$

It should be noted that when k is a positive integer this result gives the unevaluated constant of (9)** for $\tau = k-1$.

5. *Value at the algebraic singularity.*

As in § 3,

$$Q_{n,\nu}\left(\frac{1}{\alpha}\right) = \frac{1}{\alpha^\nu}\Delta_{1Q}\Delta_{2Q}\left[1+F_n\left(\frac{\rho}{\alpha}\right)^n\right]$$

$$= \frac{1}{\alpha^\nu}\begin{vmatrix} \dots, 0, & \dots, 0, & \dots, 0, & \dots, 0, & 1 \\ \dots, {}^{-1}c_0\beta_s^{n-\nu+1}, & \dots, {}^{-1}c_{m-1}\beta_s^{n-\nu+1}, & \dots, \alpha^{n-\nu+1}, & \dots, 0, & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots, {}^{-\nu}c_0\beta_s^n, & \dots, {}^{-\nu}c_{m-1}\beta_s^n, & \dots, 0, & \dots, \alpha^n, & 0 \end{vmatrix}$$

$$\times \begin{vmatrix} \dots, {}^{-m}c_{n-\nu}B_s, & \dots, {}^{-1}c_{n-\nu}B_s, \\ \dots, {}^{-m}c_{n-\nu+1}B_s\beta_s, & \dots, {}^{-1}c_{n-\nu+1}B_s\beta_s, \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots, {}^{-m}c_nB_s^\nu, & \dots, {}^{-1}c_nB_s\beta_s^\nu, \end{vmatrix}$$

$$\begin{vmatrix} \dots, {}^{-k}c_{n-\nu+1}A, & \dots, {}^{-k}c_nA, & 1 \\ \dots, {}^{-k}c_{n-\nu+2}A\alpha, & \dots, {}^{-k}c_{n+1}A\alpha, & \alpha \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots, {}^{-k}c_{n+1}A\alpha^\nu, & \dots, {}^{-k}c_{n+\nu}A\alpha^\nu, & \alpha^\nu \end{vmatrix}$$

$$\times[1+F_n(\rho/\alpha)^n],$$

where $|F_n|<F$, a finite constant, and Δ_{1Q}, Δ_{2Q} are $(\nu+1)$-rowed, $(2\nu+1)$-columned arrays.

The surviving determinant biproduct is that containing the right-hand column from each array and no other a columns, and is readily evaluated by means of Lemmas 1–2 of § 5**.

The evaluation of $P_{n,\nu}(1/a)$ is carried out, as in § 6**, with the necessary modification due to the algebraic singularity. The contributions due to the remainder function $\phi(z)$ are ignorable as in § 4* (6)*, while those arising from the enclosed poles are of the same order as $Q_{n,\nu}(1/a)$, as in § 4**.

The contribution to $(1/a^t)\psi_{n-t}(1/a)$ due to the singularity $A/(1-az)^k$ is

$$\frac{A}{a^t}(1+{}^{-k}c_1+{}^{-k}c_2+\ldots+{}^{-k}c_{n-t})={}^{-k-1}c_{n-t}\,\frac{A}{a^t}.$$

This gives to $P_{n,\nu}(1/a)$ an expression similar to $Q_{n,\nu}(1/a)$, except that the right-hand *column* of Δ_{2Q} is changed to

$${}^{-k-1}c_{n-\nu}A,\ {}^{-k-1}c_{n-\nu+1}Aa,\ \ldots,\ {}^{-k}c_nAa^\nu.$$

If $k>0$, the surviving determinant biproduct contains this column (and its correspondent) and no other a columns.

From (2),

$${}^{-k-1}c_{n-\nu+t}=\frac{n^k}{\Gamma(k+1)}\left[1+O\left(\frac{1}{n}\right)\right].$$

Hence

$$\frac{P_{n,\nu}(1/a)}{Q_{n,\nu}(1/a)}=\frac{An^k}{\Gamma(k+1)}\left[1+O\left(\frac{1}{n}\right)\right],$$

and therefore

$$\lim_{n\to\infty}\frac{R_{n,\nu}(1/a)}{n^k}=\frac{A}{\Gamma(k+1)}\quad(k>0).\tag{5}$$

If $k<0$, the evaluation proceeds as in § 6, giving the result

$$\lim_{n\to\infty}R_{n,\nu}\left(\frac{1}{a}\right)=f\left(\frac{1}{a}\right)\quad(k<0).\tag{6}$$

6. *Value on the boundary. Extension of Montessus' theorem.*

The values of $P_{n,\nu}(z)$, $Q_{n,\nu}(z)$ will now be determined at any point of the bounding circle $|za|=1$ for large values of n, except at the point

$z = 1/\alpha$, discussed above.

$$Q_{n,\nu}(z) = \begin{vmatrix} \ldots, 0, & \ldots, 0, & \ldots, 0, & \ldots, 0, & 1 \\ \ldots, {}^{-1}c_0\beta_s^{n-\nu+1}, & \ldots, {}^{-1}c_{n-1}\beta_s^{n-\nu+1}, & \ldots, \alpha^{n-\nu+1}, & \ldots, 0, & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \ldots, {}^{-\nu}c_0\beta_s^n, & \ldots, {}^{-\nu}c_{n-1}\beta_s^n, & \ldots, 0, & \ldots, \alpha^n, & 0 \end{vmatrix}$$

$$\times \begin{vmatrix} \ldots, {}^{-m}c_{n-\nu}B_s, & \ldots, {}^{-1}c_{n-\nu}B_s, \\ \ldots, {}^{-m}c_{n-\nu+1}B_s\beta_s, & \ldots, {}^{-1}c_{n-\nu+1}B_s\beta_s, \\ \cdots & \cdots \\ \cdots & \cdots \\ \ldots, {}^{-m}c_nB_s\beta_s^\nu, & \ldots, {}^{-1}c_nB_s\beta_s^\nu, \end{vmatrix}$$

$$\begin{vmatrix} \ldots, {}^{-k}c_{n-\nu+1}A, & \ldots, {}^{-k}c_nA, & z^\nu \\ \ldots, {}^{-k}c_{n-\nu+2}A\alpha, & \ldots, {}^{-k}c_{n+1}A\alpha, & z^{\nu-1} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \ldots, {}^{-k}c_{n+1}A\alpha^\nu, & \ldots, {}^{-k}c_{n+\nu}A\alpha^\nu, & 1 \end{vmatrix}$$

$$\times [1 + K_n(\rho/\alpha)^n],$$

where $|K_n| < K$, a finite constant.

The surviving determinant biproduct contains the right-hand column and no α columns.

The evaluation of $P_{n,\nu}(z)$ is carried out as in § 9**, but, in addition to the three contributions there determined, a fourth arises from the algebraic singularity at $(1/\alpha)$. The modifications arising in the forms of the arrays due to the singularity affect only the first of these three contributions, giving

$$\frac{P'_{n,\nu}(z)}{Q_{n,\nu}(z)} = \frac{n^{k-1}}{\Gamma(k)}(z\alpha)^n C_n(z),$$

where $\lim_{n\to\infty} C_n(z) = C(z)$, a determinate function of z. Therefore

$$\lim_{n\to\infty} \frac{P'_{n,\nu}(z)}{Q_{n,\nu}(z)} \Big/ n^{k-1}(z\alpha)^n = \frac{C(z)}{\Gamma(k)}.$$

As in § 9**,

$$\lim_{n\to\infty} \frac{P''_{n,\nu}(z)}{Q_{n,\nu}(z)} = \sum_s \left\{ \sum_{t=1}^{m} \frac{B_{s,t}}{(1-\beta_s z)^t} \right\},$$

and

$$\lim_{n\to\infty} \frac{P'''_{n,\nu}(z)}{Q_{n,\nu}(z)} = \phi(z).$$

The contribution to $z^t\psi_{n-t}(z)$ arising from $A/(1-az)^k$ is

$$Az^t[1+{}^{-k}c_1(az)+\ldots+{}^{-k}c_{n-t}(az)^{n-t}] = z^t\chi_{n-t}(z),$$

making a total contribution $P''''_{n,\nu}(z)$ exactly as in $Q_{n,\nu}(z)$ except that, in the right-hand *column* of the second array, $z^\nu, z^{\nu-1}, \ldots, 1$, is replaced by $z^\nu\chi_{n-\nu}(z), z^{\nu-1}\chi_{n-\nu+1}(z), \ldots, \chi_n(z)$.

If $k>1$, it follows that $\lim_{n\to\infty}\frac{P'_{n,\nu}(z)}{Q_{n,\nu}(z)}$ is neither finite nor determinate, while, if $k<1$, the limit is zero.

Further, if $k>1$, $\lim_{n\to\infty}\chi_n(z)$ is not finite, but, if $k<1$, $\lim_{n\to\infty}\chi_n(z)$ converges to $A/(a-z)^k$, and

$$\lim_{n\to\infty}\frac{P''''_{n,\nu}(z)}{Q_{n,\nu}(z)} = \frac{A}{(1-az)^k}.$$

Hence, if $k>1$,

$$\lim_{n\to\infty}R_{n,\nu}(z) = \lim_{n\to\infty}\sum_{r=1}^{4}\frac{P^{r'}_{n,\nu}(z)}{Q_{n,\nu}(z)}$$

does not exist in general, while, if $k<1$,

$$\lim_{n\to\infty}R_{n,\nu}(z) = \sum_s\left\{\sum_{t=1}^{m}\frac{\beta_{s,t}}{(1-\beta_s z)^t}\right\}+\frac{A}{(1-az)^k}+\phi(z)$$

$$= f(z). \tag{7}$$

It will be noted that, at all points within the circle $|za|=1$,

$$\lim_{n\to\infty}\frac{P'_{n,\nu}(z)}{Q_{n,\nu}(z)} = 0,$$

since $|za|<1$, and

$$\lim_{n\to\infty}\frac{P''''_{n,\nu}(z)}{Q_{n,\nu}(z)} = \frac{A}{(1-az)^k},$$

while, as before,

$$\lim_{n\to\infty}\frac{P'''_{n,\nu}(z)}{Q_{n,\nu}(z)} = \phi(z),$$

and, *except at enclosed poles*,

$$\lim_{n\to\infty}\frac{P''_{n,\nu}(z)}{Q_{n,\nu}(z)} = \sum_s\left\{\sum_{t=1}^{m}\frac{B_{s,t}}{(1-\beta_s z)^t}\right\}.$$

Hence at all points *within* the circle, excluding poles,

$$\lim_{n\to\infty}R_{n,\nu}(z) = f(z),$$

while outside the circle this limit does not exist.

This proves Montessus' theorem for the Padé continued fraction of the row corresponding to the first non-polar singularity when this is algebraic.

The cases for which R is integral, the boundary singularity being then a pole, have been fully discussed in the three papers mentioned in § 1.

7. *The boundary singularity logarithmic.*

We shall now consider a function defined in a manner similar to that of § 2, except that the algebraic singularity on the boundary is replaced by a logarithmic singularity of positive integral order q at the point $1/\alpha$. In this case it is sufficient to consider the function†

$$f(z) \equiv \sum_s \frac{B_s}{(1-\beta_s z)^m} + A\left\{\log_e\left(\frac{1}{1-\alpha z}\right)\right\}^q + \phi(z).$$

It is readily shown that c_n, the coefficient of z^n in the Taylor expansion of $f(z)$ in the neighbourhood of the origin, takes the form

$$c_n = \Sigma^{-m} c_n B_s \beta_s^n + c_n' \alpha^n + M_n \rho^n,$$

where

$$c_n' = \frac{Aq(\log_e n)^{q-1}}{n}\left[1+O\left(\frac{1}{\log_e n}\right)\right],$$

and the other symbols have the same significance as in § 2.

With suitable modifications arising from the form of c_n, the discussion follows word for word that of §§ 3–6, and hence only results will be quoted.

At an enclosed pole of order p at $1/\beta_r$

$$\lim_{n\to\infty} \frac{R_{n,\nu}(1/\beta_r)}{n(\beta_r/\alpha)^n(\log_e n)^{q-1}} = \frac{B_r^2}{A} \frac{\alpha^{\nu-1}}{q\beta_r^{\nu-2p}(\alpha-\beta_r)^{2p-1}} \frac{\prod_s{}'(\beta_r-\beta_s)^{2m}}{\prod_s{}'(\alpha-\beta_s)^{2m}}. \tag{8}$$

At the logarithmic singularity at $1/\alpha$

$$\lim_{n\to\infty} \frac{R_{n,\nu}(1/\alpha)}{(\log_e n)^q} = A. \tag{9}$$

At every point of $|z\alpha| = 1$, except at the point $z = 1/\alpha$,

$$\lim_{n\to\infty} R_{n,\nu}(z) = f(z). \tag{10}$$

† The form $\{\log_e(1/1-\alpha z)\}^q$ is chosen because all the coefficients are positive.

At every point within the circle $|z\alpha| = 1$, except at enclosed poles,

$$\lim_{n\to\infty} R_{n,\nu}(z) = f(z),$$

while outside this circle the continued fraction diverges.

8. *The boundary singularity algebraic logarithmic.*

The discussion with an algebraic-logarithmic singularity at $1/\alpha$ for which $f(z)$ has the form

$$f(z) \equiv \sum_s \frac{B_s}{(1-\beta_s z)^m} + \frac{A}{(1-\alpha z)^k}\left\{\log_e\left(\frac{1}{1-\alpha z}\right)\right\}^q + \phi(z),$$

where k may be negative or integral and q is a positive integer, proceeds, as in §§ 3–6, with the following modification.

If the Taylor expansion in the neighbourhood of the origin of

$$\frac{A}{(1-\alpha z)^k}\left\{\log_e\left(\frac{1}{1-\alpha z}\right)\right\}^q \equiv c_0' + c_1' z + c_2' z + \ldots + c_n' z^n + \ldots,$$

then it is readily shown that, for large values of n,

$$c_n' = \frac{A}{\Gamma(k)}\, n^{k-1}(\log_e n)^q\left[1 + O\left(\frac{1}{\log_e n}\right)\right] \quad (k \neq 0, -1, -2, \ldots)$$

and

$$c_n' = (-1)^{-k}(-k)!\, qA\, n^{k-1}(\log_e n)^{q-1}\left[1 + O\left(\frac{1}{\log_e n}\right)\right]$$

$$(k = 0, -1, -2, \ldots).$$

Proceeding as in §§ 3–6, the following results arise.

At an enclosed pole of order p at $1/\beta_r$

$$\lim_{n\to\infty} \frac{R_{n,\nu}(1/\beta_r)}{(\beta_r/\alpha)^n / n^{k-1}(\log_e n)^q} = \frac{B_r^2\,\Gamma(k)\,\alpha^{p-1}}{A\,(\alpha-\beta_r)^{2p-1}\,\beta_r^{\nu-2p}} \frac{\prod_s{}' (\beta_r-\beta_s)^{2m}}{\prod_s{}' (\alpha-\beta_r)^{2m}},$$

$$(k \neq 0, -1, -2, \ldots), \quad (11)$$

with corresponding modifications for the cases in which $k = 0, -1, -2, \ldots$.

At the algebraic logarithmic singularity at $1/\alpha$

$$\left.\begin{aligned} \lim \frac{R_{n,\nu}(1/\alpha)}{n^k(\log_e n)^q} &= \frac{A}{\Gamma(k+1)} \quad (k > 0) \\ \lim R_{n,\nu}\left(\frac{1}{\alpha}\right) &= f\left(\frac{1}{\alpha}\right) \quad (k < 0) \end{aligned}\right\}. \quad (12)$$

At all regular points of the circle $|za|=1$, if $k \geqslant 1$, $\lim_{n\to\infty} R_{n,\nu}(z)$ does not exist in general, while if $k<1$,

$$\lim_{n\to\infty} R_{n,\nu}(z) = f(z).$$

At all points within the circle $|za|=1$, except for enclosed poles,

$$\lim_{n\to\infty} R_{n,\nu}(z) = f(z),$$

while outside the circle the continued fraction diverges.

9. *Extension to algebraic and logarithmic singularities of general form.*

Assuming, as in § 8, that the Taylor expansion in the neighbourhood of the origin of that part of the function which represents the boundary singularity at $1/a$ takes the form

$$\chi(z) = c_0' + c_1' z + c_2' z^2 + \ldots + c_n' z^n + \ldots,$$

we see that the results so far obtained, depending, as they do, on the form $F(n)$ of c_n for large values of n, will hold for any c_n' such that

$$\lim_{n\to\infty} \frac{c_n'}{F(n)} = \text{constant},$$

If to $\chi(z)$ is added any other singularity at $1/a$,

$$\phi(z) = c_0'' + c_1'' z + c_2'' z^2 + \ldots + c_n'' z^n + \ldots,$$

where

$$\lim_{n\to\infty} \frac{c_n''}{c_n'} = 0,$$

then the results will also be true for functions with the boundary singularity $\chi(z)+\phi(z)$. Some cases of immediate interest will now be stated.

Thus the simple algebraic singularity represented by $A/(1-az)^k$ may be replaced by a singularity of the form $\sum_{t=0}^{h} \frac{A_{k-t}}{(1-az)^{k-t}}$. In particular, k may be integral, giving the principal part of the expansion of a meromorphic function with a pole at $1/a$, considered in earlier papers. In general, it is not necessary for the exponents of the separate "elements" to differ by integers, provided that the necessary conditions are satisfied.

Similarly, the simple logarithmic singularity represented by

$$A\,[\log_e(1-az)]^q$$

may be replaced by a singularity of the form

$$\sum_{t=0}^{h} A_{q-t}[\log_e(1-az)]^{q-t} \quad (q \geqslant h),$$

and the algebraic-logarithmic singularity of § 8 may be replaced by

$$\frac{1}{(1-az)^k} \sum_{t=0}^{h} A_{q-t}[\log_e(1-az)]^{q-t} \equiv \frac{1}{(1-az)^k}\{H_q[\log_e(1-az)]\} \quad (q \geqslant h).$$

This, again, may be replaced by the more general form

$$\sum_{t=0}^{h} \frac{1}{(1-az)^{k-t}}\{H_q[1-az)]\}.$$

It should be noted, however, that this extension has been proved to hold for the continued fraction of the $(\nu+1)$-th row only.

10. *The case of a stationary boundary.*

It was shown in § 8** that, with a multiple boundary pole of order k, the continued fractions of k successive rows have their representation bounded by the same circle. The corresponding theorem in the case of a simple algebraic or logarithmic singularity on the boundary is that the continued fractions for the associated row and all succeeding rows have the same circle as boundary. In establishing this theorem difference relations among the coefficients are utilized in a manner similar to that occurring throughout the papers on polar singularities [vide (3)**]. Although this method can be employed in the case of more general singularities, its effectiveness depends on a specific knowledge of the coefficients, and consequently no general result will be stated. The results here appearing for the simple algebraic or logarithmic singularities may be extended to include the more general types

$$\sum_{t=0}^{h} \frac{A_{k-t}}{(1-az)^{k-t}}, \qquad \sum_{t=0}^{h} A_{q-t}[\log_e(1-az)]^{q-t} \quad (q \geqslant h),$$

in which the t's are integral, since the operation of the differences is uniform throughout all the elements which make up the coefficients.

The following results are obtained for a simple algebraic singularity on the boundary.

At an enclosed pole at $1/\beta_r$,

$$\lim_{n\to\infty} \frac{R_{n,\,\nu+\tau}(1/\beta_r)}{n^{2\tau-k+1}(\beta_r/\alpha)^n} \text{ is finite and non-zero} \quad (\tau = 0, 1, 2, \ldots). \quad (14)$$

At the algebraic singularity at $1/a$,

$$\lim_{n\to\infty} \frac{R_{n,\,\nu+\tau}(1/a)}{n^k} = \frac{A}{\Gamma(k+1)} \quad (\tau = 0, 1, 2, \ldots). \tag{15}$$

At any other point of the bounding circle $|za| = 1$, if $k \geqslant 1$,

$$\left.\begin{aligned} &\lim_{n\to\infty} R_{n,\,\nu+\tau}(z) \text{ does not, in general, exist for } \tau = 0, 1, \ldots, \tfrac{1}{2}h-1, \\ &\text{or } \tfrac{1}{2}(h-1) \text{ according as } h, \text{ the greatest integer in } k, \text{ is even or} \\ &\text{odd, while for } \tau \geqslant \tfrac{1}{2}h \text{ or } \tfrac{1}{2}(h+1), \text{ or if } k < 1 \ (\tau = 0, 1, 2, \ldots), \\ &\qquad \lim_{n\to\infty} R_{n,\,\nu+\tau}(z) = f(z) \end{aligned}\right\}. \tag{16}$$

At all regular points within the circle $|za| = 1$,

$$\lim_{n\to\infty} R_{n,\,\nu+\tau}(z) = f(z) \quad (\tau = 0, 1, 2, \ldots),$$

while at external points the continued fractions diverge.

With a simple logarithmic singularity on the boundary the following results arise :

At an enclosed pole at $1/\beta_r$,

$$\lim_{n\to\infty} \frac{R_{n,\,\nu+\tau}(1/\beta_r)}{n^{2\tau+1}(\beta_r/a)^n/(\log_e n)^{q-1}} \text{ is finite and non-zero} \quad (\tau = 0, 1, 2, \ldots). \tag{17}$$

At the logarithmic singularity at $1/a$,

$$\lim_{n\to\infty} \frac{R_{n,\,\nu+\tau}(1/a)}{(\log_e n)^q} = A \quad (\tau = 0, 1, 2, \ldots). \tag{18}$$

At any other point on the boundary $|za| = 1$,

$$\lim_{n\to\infty} R_{n,\,\nu+\tau}(z) = f(z) \quad (\tau = 0, 1, 2, \ldots). \tag{19}$$

At all points within the circle, except at enclosed poles, the continued fractions converge to $f(z)$, while they diverge at external points.

11. *Several singularities on the boundary.*

In the discussion of several boundary poles in the third paper, it was seen that the boundary circle remains the same for l rows of the table, where l is the total multiplicity of the boundary poles. The only cases leading to a complete series of results were those of the continued frac-

tions for the first and last of the l rows. No correlative to the latter exists when there are non-polar singularities on the boundary; in fact, as stated in § 1, the bounding circle remains stationary for all succeeding rows. The results stated below apply only to the continued fraction for the first of these rows. It will be seen that the results (22)***–(28)*** stated for polar singularities on the boundary, hold with one modification† for algebraic singularities. We shall consider, however, the more general case of algebraic-logarithmic singularities on the boundary, as these include also polar logarithmic, algebraic, polar, and logarithmic singularities. The classification turns, as may be expected, on the order of the dominant singularity or singularities on the boundary. Thus the first set of results holds only when there is at least one boundary singularity for which $k \geqslant 1$; in the second set $1 > k > 0$; in the third a logarithmic singularity dominates, while the fourth set concerns only singularities for which $k < 0$. It should be noted that the order of the singularity of exponents (k, q) is higher than that of exponents (k', q'), if $k > k'$, and in the case $k = k'$, when $q > q'$.

If there are several singularities on the bounding circle $|z\alpha| = 1$, of which that at $1/\alpha$ of exponents (k, q) is of the highest order, then, at an enclosed pole at $1/\beta_r$,

$$\lim_{n\to\infty} \frac{R_{n,\nu}(1/\beta_r)}{(\beta_r/\alpha)^n/n^{k-1}(\log_e n)^q} \text{ is finite and non-zero.} \tag{20}$$

If there is more than one singularity of higher order than the others,

$$\overline{\lim_{n\to\infty}} \frac{|R_{n,\nu}(1/\beta_r)|}{|\beta_r/\alpha|^n/n^{k-1}(\log_e n)^q} \text{ is finite and non-zero.} \tag{21}$$

At a boundary singularity different cases arise as follows:

If the singularity at $1/\alpha'$ is of the highest order, or if its exponents (k', q') are so related to those of the dominant singularity (k, q) that either $k' > k-1$ or $k' = k-1$ and $q' > q$, then

$$\lim_{n\to\infty} \frac{R_{n,\nu}(1/\alpha')}{n^{k'}(\log_e n)^{q'}} \text{ is finite and non-zero.} \tag{22}$$

If $k' = k-1$ and $q' = q$, then

$$\overline{\lim_{n\to\infty}} \frac{|R_{n,\nu}(1/\alpha')|}{n^{k-1}(\log_e n)^q} \text{ is finite and non-zero.} \tag{23}$$

† See (22).

If the singularity at $1/a'$ is of lower order, and there is :

(a) only one singularity of the highest order (at $1/a$), then

$$\lim_{n\to\infty} \frac{R_{n,\nu}(1/a')}{(a'/a)^n n^{k-1}(\log_e n)^q} \text{ is finite and non-zero ;} \tag{24}$$

(b) more than one dominant singularity

$$\overline{\lim_{n\to\infty}} \frac{|R_{n,\nu}(1/a')|}{n^{k-1}(\log_e n)^q} \text{ is finite and non-zero.} \tag{25}$$

At all regular points within the circle $|za| = 1$ the continued fraction converges to the function, while on the boundary and at external points it diverges.

It is clear that, for the cases just discussed, the dominating singularity cannot be purely logarithmic. The same is true for the following cases in which $1 > k > 0$ for the dominant singularity.

Results (20) and (21) still hold, as does (22) for $1 > k' \geqslant 0$, but at any boundary singularity for which $k' < 0$,

$$\lim_{n\to\infty} R_{n,\nu}(1/a') = f(1/a').$$

At all regular points within and on the circle the continued fraction converges to $f(z)$.

The apparent inconsistency arising in § 8 has not, so far, exercised any effect. In considering the cases for which a logarithmic singularity is dominant, *i.e.* when $k = 0, -1, -2, \ldots$, it is necessary to restate the results. In this case any algebraic singularities occurring on the boundary must be such that $k < 0$.

If the logarithmic singularity at $1/a$ with exponent q for which $k = 0$ or -1 or $-2, \ldots$ is the boundary singularity of the highest order, then, at an enclosed pole at $1/\beta_\nu$,

$$\lim_{n\to\infty} \frac{R_{n,\nu}(1/\beta_\nu)}{(\beta_\nu/\alpha)^n n^{k-1}(\log_e n)^{q-1}} \text{ is finite and non-zero.} \tag{27}$$

If there are two or more such singularities

$$\overline{\lim_{n\to\infty}} \frac{|R_{n,\nu}(1/\beta_\nu)|}{|\beta_\nu/\alpha|^n n^{k-1}(\log_e n)^{q-1}} \text{ is finite and non-zero.} \tag{28}$$

At any logarithmic singularity on the boundary at $1/\alpha$

$$\lim_{n\to\infty} \frac{R_{n,\nu}(1/\alpha)}{(\log_e n)^q} \text{ is finite and non-zero} \quad (k = 0).$$

At any other singularity on the boundary ($k < 0$) at $1/a$

$$\lim_{n\to\infty} R_{n,\nu}(1/a) = f(1/a). \tag{29}$$

At all regular points within and on the boundary the continued fraction converges to the value of the function, while at external points it diverges. (30)

Finally, if there are only singularities on the boundary for which ($k < 0$; $k \neq -1, -2, \ldots$), results (20) and (21) hold, as do (29) and (30).

The study of the continued fractions of succeeding rows with several singularities on the boundary presents features of peculiar difficulty, even if restricted to simple algebraic (including polar) and logarithmic singularities. The reason for this becomes apparent when it is realized that, in the evaluation of $P_{n,\nu}$, $Q_{n,\nu}$, there will be, in general, several surviving determinant bi-products, and hence, except in special cases, it will not be possible to formulate determinate results.

Three further results, however, can be quoted.

If there is a dominant singularity at $1/a$ and if $k > 0$,

$$\lim_{n\to\infty} \frac{R_{n,\nu+\tau}(1/a)}{n^k(\log_e n)^q} = \frac{A}{\Gamma(k+1)} \quad (\tau = 0, 1, 2, \ldots). \tag{31}$$

When $q = 0$, this gives the result for an algebraic singularity; and when $k = 0$, for a logarithmic singularity.

If the dominant singularity at $1/a$ is such that $k < 0$, then

$$\lim_{n\to\infty} R_{n,\nu+\tau}(1/a) = f(1/a) \quad (\tau = 0, 1, 2, \ldots). \tag{32}$$

Within the bounding circle $|za| = 1$, at regular points

$$\lim_{n\to\infty} R_{n,\nu+\tau}(z) = f(z) \quad (\tau = 0, 1, 2, \ldots). \tag{33}$$

while the continued fractions diverge outside the circle.

12. *Summary of results.*

The results herein stated form a consistent generalization of those obtained for polar singularities, which occur as special cases.

Montessus' theorem on the region of convergence of the continued fractions of the rows associated with successive polar singularities is generalized in the theorem mentioned in § 1. This states that the continued fraction corresponding to the first non-polar singularity of the type considered converges at all regular points within the circle passing

through that singularity and diverges outside. This is also shown to be true when there are several such singularities on the boundary (§ 10), and when there is only one it is proved to hold for the continued fractions of all succeeding rows.

Since the approximants to the continued fraction of the first row of the Padé table are the successive partial sums of the Taylor series of the function, it is of interest to consider the Taylor series of a function having its circle of convergence determined by a single algebraic-logarithmic singularity. If the singularity at $1/a$ has exponents (k, q) and

$$S_n(z) \equiv c_0 + c_1 z + c_2 z^2 + \ldots + c_n z^n,$$

Dienes† has shown that

$$\lim_{n\to\infty} \frac{S_n(1/a)}{n^k (\log_e n)^q} = \frac{A}{\Gamma(k+1)} \quad (k > 0),$$

in the notation of § 8. The behaviour of the series at regular points on the boundary produces the classification natural to § 11. For if $k \geqslant 1$, the series is divergent at regular points on the boundary, while if $0 < k < 1$, it converges there. If $k = 0$ the singularity is logarithmic, and there is also convergence. Finally, if $k < 0$, so that the singularity is not an infinity of the function, then there is convergence at all points of the boundary and

$$\lim_{n\to\infty} S_n(1/a) = f(1/a)‡.$$

These results are reproduced in faithful detail for the continued fractions of the $(\nu+1)$-th and succeeding rows of the Padé table when the function has poles of total multiplicity ν within the circle passing through its first non-polar singularity, such singularities being of the types here considered.

The effectiveness of the generalization from polar singularities is exhibited in particular by the results of § 11 pertaining to the case of several singularities on the boundary. Here, as in the case of poles, the dominant boundary singularity exercises a determinate effect on all the results, and if there is more than one singularity on the boundary of higher order than the others, the results are characterized by indeterminateness.

Again, the results arising from the continued fractions of succeeding rows bear a close relationship to those occurring with a multiple boundary

† P. Dienes, *Leçons sur les singularités des fonctions analytiques* (Paris, 1913), 48.

‡ *Ibid.*, 34-37.

pole (§ 8**), or several boundary poles (§§ 10***–11***). One result stands out as fundamental: at a dominating singularity of exponents (k, q) at $1/a$,

$$\lim_{n\to\infty} \frac{R_{n,\nu+\tau}(1/a)}{n^k(\log_e n)^q} = \frac{A}{\Gamma(k+1)} \quad (k > 0;\ \tau = 0, 1, 2, \ldots),$$

with the corresponding results

$$\lim_{n\to\infty} \frac{R_{n,\nu+\tau}(1/a)}{(\log_e n)^q} = A \qquad (k = 0;\ \tau = 0, 1, 2, \ldots),$$

and

$$\lim_{n\to\infty} R_{n,\nu+\tau}(1/a) = f(1/a) \quad (k < 0;\ \tau = 0, 1, 2, \ldots).$$

Finally, the peculiar nature of the divergence at an enclosed pole completes the connection, the results bearing a striking resemblance to those obtained for polar boundary singularities, if allowance is made for the non-integral order of the non-polar boundary singularities. In particular, with a single non-polar singularity on the boundary, in passing from one row to its successor the order of divergence is increased by integral powers of n^2, as in the case of a multiple boundary pole or several boundary poles. There is, however, this difference, that the process terminates, in the latter cases, after a finite number of rows, while, in the former case, the growth in the order of divergence continues throughout the table.

DETERMINATE FUNCTIONS OF INTERVALS AND THEIR RATE OF INCREASE

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[Received 8 November, 1928.—Read 13 December, 1928.]

Introduction.

1. Many familiar results in the theory of functions of the form $f(x)$ may be exhibited as theorems relating to the additive function of intervals

$$F(a, b) = f(b) - f(a).$$

Burkill† has shown that such results do not always require for their validity the assumption that $F(a, b)$ is additive, and that the theory of functions of intervals which are not necessarily additive often has advantages over the special theory, the restricted nature of which is apt to cause difficulties.

It is an important inquiry, fundamental to the theory of integration, under what conditions the statement

$$Df(x) = 0$$

almost everywhere in an interval implies that the continuous function $f(x)$ is a constant, or that the function $F(a, b)$ is zero throughout the interval. Here $Df(x)$ is a derivative whose nature is specified among the conditions. This question has been treated in detail by many writers‡. A fruitful form of argument is that in which the main result is obtained by combining two sets of conditions, one set sufficient to ensure that $F(a, b) \geqslant 0$ and the other to ensure that $F(a, b) \leqslant 0$.

† 3.

‡ 1, 2. 7.

For a continuous but non-additive function of intervals $F(a, b)$ no success can be expected in a search for conditions under which

$$DF = 0 \tag{1.1}$$

implies that

$$F(a, b) = 0, \tag{1.2}$$

if the conditions assumed for $F(a, b)$ are general. In fact, a continuous function of intervals may have zero as unique derivative at every point of a specified interval without being constant. This is plainly the case with

$$F(a, b) = (b-a)^2.$$

However, the possibility remains that, if the character of the function $F(a, b)$ is less arbitrary, results of the form in question may be attained.

The main object of this paper is to prove pairs of propositions in which the average rate of increase

$$\frac{F(a, b)}{b-a} \tag{1.3}$$

of $F(a, b)$ in an interval is shown to be restricted on one side when $F(a, b)$ and its derivatives satisfy suitable conditions. A combination of the results of this investigation leads to the conclusion that if $F(a, b)$ is not an arbitrary continuous function of intervals, but is, in the terminology adopted, determinate above and below, then (1.1) implies (1.2), where DF is any derivative of the kind usually described as regular.

An extension of the scope of this investigation which naturally suggests itself is an inquiry into conditions determining the average rate of increase of $F(a, b)$ in intervals (a, b) which are not arbitrary but are restricted to have their end-points lying on some closed set, not necessarily continuous. Two sets of such conditions are obtained, corresponding to more and to less restrictive assumptions as to the nature of $F(a, b)$. The first set is used to establish sufficient conditions that a given function of intervals should be determinate above or below.

By the term *derivative* will be understood what is usually described as a regular derivative. Some very elementary illustrations given by Looman† show that irregular derivatives do not possess the properties usually associated with derivatives. Consideration of them thus forms a separate branch of the theory and is not material to the present subject.

† 5, 159–160.

Let x be any point and (a, b) any interval. By the *parameter* of (a, b) relative to x is meant the larger of the two numbers

$$\frac{2|x-a|}{b-a}, \quad \frac{2|x-b|}{b-a}.$$

It is evidently a continuous function of x.

A given number Δ is said to be a derivative of $F(a, b)$ at x if there is some positive number ν such that at least one interval (a, b) of arbitrarily small length and parameter less than ν can be found to make the ratio (1 . 3) lie within any given neighbourhood of Δ.

The intervals whose existence is required by the derivative evidently satisfy all the conditions of Vitali's theorem†. If E is any set of points at each of which there is a derivative, then a countable set of the associated intervals can be found which covers E, with the exception of a set of points of measure zero. The index ν need not be constant from point to point of E and it may be unbounded.

The terms *derivative on the right* or *left*, *derivative restricted to* a given set of points, are used to denote numbers obtained from the ratio (1 . 3) by the ordinary processes.

The main results proved below apply, with one important exception, to a class of non-additive continuous functions of intervals described for convenience as consistent. A necessary preliminary investigation is that which occupies § 4, in which certain properties of continuous and of consistent functions of intervals are deduced from the general theorems I and II of § 3, which deal with point sets associated with an open system of intervals.

A fundamental conception is that of a characteristic point set associated with a function of intervals. Any bounded closed set of points Q, which contains more than two points and is such that

$$\frac{F(a, b)}{b-a} \geqslant \kappa \tag{1 . 4}$$

whenever a and b are consecutive points of Q, while

$$\frac{F(a, b)}{b-a} < \kappa \tag{1 . 5}$$

whenever a and b are non-consecutive points of Q, is a *negative characteristic set* with *index* κ associated with $F(a, b)$. If the inequalities

† 6.

(1.4) and (1.5) are reversed, the definition is that of a *positive characteristic set* with index κ associated with $F(a, b)$.

Any continuous function of intervals has associated characteristic sets. The definition of a consistent function is framed, as will be seen, so as to ensure that the characteristic sets contain perfect characteristic sets.

Functions which have no associated positive or negative characteristic sets of measure zero are described respectively as determinate above or below. The main theorem of this paper shows that, for such functions, the average rate of increase is restricted above or below, as the case may be, by a derivative known almost everywhere.

Preliminary definitions and remarks.

2. In this paper the word "interval" and the notation (a, b), if unqualified by the additions "open" or "closed", refer to a *neutral* interval regarded as consisting simply of two *end-points* a and b; it is always assumed that the numbers a and b are distinct and that the one placed first is the smaller of the two. The difference $b-a$ is the *length* of the interval.

If $a_1 \leqslant a_2 \leqslant b_2 \leqslant b_1$, the interval (a_1, b_1) *contains* (a_2, b_2).

The expression "system of intervals" is used as a correlative to the expression "set of points".

The members of the systems of intervals dealt with here are all supposed to be contained in some finite interval $(\mathbf{a}, \mathbf{b})$ called the *fundamental* interval. The field of operations with intervals is thus the system of all sub-intervals of $(\mathbf{a}, \mathbf{b})$, including $(\mathbf{a}, \mathbf{b})$ itself. Extensions to the case where the fundamental interval is not finite are possible, and the restriction is only made for the sake of convenience.

The exclusion of degenerate intervals whose end-points coincide materially simplifies the exposition. It will be seen that the theory is essentially relative to the system of non-degenerate intervals. An infinite system of intervals, when degenerate intervals are excluded, may not possess a limit; but this is without adverse influence on what follows.

If (a_1, b_1) and (a_2, b_2) are two intervals, the number

$$|a_2-a_1| + |b_2-b_1|$$

is the *distance* between them.

A *neighbourhood* of the interval (a, b) is a system consisting of all those intervals contained in the fundamental interval whose distance from

(a, b) is less than some prescribed positive number. Each positive number determines a corresponding neighbourhood.

A system of intervals Σ is *open* if, whenever (a, b) is a member of Σ, there is a neighbourhood of (a, b) consisting entirely of members of Σ.

A *closed* system of intervals is one whose complement relative to the system of all intervals contained in $(\mathbf{a}, \mathbf{b})$ is open.

Open and closed systems of intervals have many properties in common with bounded open and closed sets of points. Thus, in particular, the common part of a set of closed systems is closed, and the sum of a set of open systems is open.

An interval is said to be *associated with* a given set of points if its end-points are members of the set. A system of intervals is said to be associated with a set of points if all its intervals are associated with the set.

The complementary intervals of a closed set of points determine intervals associated with it. All other associated intervals will be described as *non-complementary* intervals of the set.

It can be verified by standard methods that the system consisting of all the intervals associated with a given closed set is closed.

LEMMA. *If l be the upper bound of the lengths of intervals of a closed system Σ, then an interval of length l belonging to Σ can be determined in a definite way.*

Suppose that the end-points of intervals (α, β) of Σ satisfy

$$a \leqslant \alpha \leqslant b, \qquad c \leqslant \beta \leqslant d.$$

Let p and q bisect the intervals (a, b) and (c, d) respectively.

The system Σ gives rise to four new systems, each consisting of those intervals of Σ whose end-points satisfy one of the four pairs of relations

$$a \leqslant \alpha \leqslant p, \qquad c \leqslant \beta \leqslant q;$$

$$p \leqslant \alpha \leqslant b, \qquad c \leqslant \beta \leqslant q;$$

$$a \leqslant \alpha \leqslant p, \qquad q \leqslant \beta \leqslant d;$$

$$p \leqslant \alpha \leqslant b, \qquad q \leqslant \beta \leqslant d.$$

These systems cannot all be null. Each is closed as the common part of two closed systems. The lengths of the intervals in at least one of the systems have l as their upper bound. Choose the first system in order for which this is the case; let the intervals in which its end-points range

be renamed (a_1, b_1), (c_1, d_1), and let the closed sub-system of Σ defined by these intervals be called Σ_1.

The process may be repeated with Σ_1 in place of Σ, and so on indefinitely, to yield two decreasing sequences of intervals $\{(a_n, b_n)\}$, $\{(c_n, d_n)\}$ such that

$$b_n - a_n = \frac{b-a}{2^n}, \quad d_n - c_n = \frac{d-c}{2^n},$$

and a sequence $\{\Sigma_n\}$ of non-null closed systems such that, for each n, the intervals of Σ_n satisfy

$$a_n \leqslant \alpha \leqslant b_n, \quad c_n \leqslant \beta \leqslant d_n,$$

and have l as the upper bound of their lengths. Clearly

$$c_n - b_n \leqslant l \leqslant d_n - a_n. \tag{2.1}$$

The sequences $\{(a_n, b_n)\}$, $\{(c_n, d_n)\}$ determine uniquely points λ and μ such that, for every n,

$$a_n \leqslant \lambda \leqslant b_n, \quad c_n \leqslant \mu \leqslant d_n.$$

It will be shown that the interval (λ, μ) is of length l and belongs to Σ.

Since

$$a_n, b_n \to \lambda, \quad c_n, d_n \to \mu,$$

it follows from (2.1) that

$$\mu - \lambda = l.$$

The systems Σ_n are sub-systems of Σ. If n is sufficiently large the intervals of Σ_n are at an arbitrarily small distance from (λ, μ). Thus Σ has intervals arbitrarily near (λ, μ) and, since it is closed, it includes (λ, μ).

Take now $a = c = \mathbf{a}$, $b = d = \mathbf{b}$. The interval thus determined clearly satisfies all the requirements of the lemma.

Characteristic point sets associated with systems of intervals.

3. Let Σ be any system of neutral intervals. Consider in relation to it any closed set Q consisting of more than two points. If the non-complementary intervals of Q all belong to Σ while the complementary intervals are excluded from Σ, then Q is said to be a *characteristic set* of the system Σ.

The restriction that a characteristic set must contain at least three

points is made so as to exclude the trivial case of a characteristic set consisting of two points which form an interval not a member of Σ. It ensures that the term characteristic is only applied to sets with at least one non-complementary interval.

Describe as the *complement* of Σ the system $\mathbf{T}$, which consists of all the sub-intervals of (**a**, **b**) which do not belong to Σ. Let a system of intervals be said to be *restricted* to a given set of points when only such intervals as are associated with the set are retained. Then a characteristic set Q of Σ is any closed set containing at least three points such that the operation of restriction to Q reduces Σ to the system of non-complementary intervals and $\mathbf{T}$ to the system of complementary intervals of Q.

THEOREM I. *Each interval of an open system contains a characteristic set of the system.*

Let Ω be the open system and (a, b) any one of its intervals. Unless there is at least one sub-interval of (a, b) which does not belong to Ω, the closed interval (a, b) is a characteristic set and there is nothing further to be proved.

The sub-intervals of (a, b) plainly form a closed system. If the open system Ω is subtracted from this, the result is a closed system, which may be denoted by Γ. The case which remains to be considered is that in which Γ contains at least one interval.

Let l_0 be the upper bound of the lengths of the intervals of Γ. Since Γ is closed, an interval (a_1, b_1) belonging to it, and of length l_0, can be determined. No interval of Γ other than (a_1, b_1) can contain (a_1, b_1), for its length would be greater than l_0, contradicting a hypothesis. Thus the intervals of Γ, both of whose end-points are excluded from closed (a_1, b_1), must lie in one or other of the closed intervals which remain when open (a_1, b_1) is subtracted from closed (a, b). Let Q_1 denote the set so obtained. It consists of two closed intervals; at least one of these is non-degenerate. Thus Q_1 contains more than two points.

If Γ is restricted to Q_1, the result is a closed system containing (a_1, b_1). Now (a_1, b_1) is an isolated member of this system, for it is the only member which is not associated with one or other of the components of Q_1. If (a_1, b_1) is removed from this system, the remainder is still closed. It will be called Γ_1.

It is seen at once that, if Γ_1 is null, Q_1 is a characteristic set; there is no more to be proved.

If Γ_1 is not null, the process can be repeated with Γ_1 instead of Γ.

Suppose now that the process has been continued to the stage which yields l_{n-1}, (a_n, b_n), Q_n, Γ_n, without encountering a system Γ_r which is null. Then l_{n-1} is the upper bound of the lengths of the intervals of Γ_{n-1}, (a_n, b_n) an interval of Γ_{n-1} which has length l_{n-1}, Q_n the closed set which remains when the open interval (a_n, b_n) is subtracted from Q_{n-1}, and Γ_n the system consisting of all members of Γ_{n-1} which are associated with Q_n, except (a_n, b_n).

The following facts can be established by induction.

As n increases the number l_{n-1}, the set Q_n and the system Γ_n decrease.

The intervals

$$(a_1, b_1),\ (a_2, b_2),\ \ldots,\ (a_n, b_n) \tag{3.1}$$

form a non-overlapping system.

The interval (a_n, b_n) is an isolated member of the system which results when Γ_{n-1} is restricted to Q_n.

The set Q_n is obtained by subtraction of the open intervals (3.1) from closed (a, b), and consists of a finite number of closed intervals.

The system Γ_n is closed and consists of all the intervals of Γ, other than the intervals (3.1), which are associated with Q_n.

For $n \geqslant 2$ the closed intervals of which Q_n is composed may all be degenerate; but in this case Q_n contains the points a_1, b_1, a_2, b_2, not more than two of which can coincide, so that Q_n still contains at least three points.

It is easy to see that, if Γ_n is null, Q_n is a characteristic set.

The only case which remains to be considered is that in which Γ_n contains intervals for every value of n. The set

$$Q = Q_1 Q_2 \ldots Q_n \ldots$$

will be found to be a characteristic set.

Since the sequence $\{Q_n\}$ decreases, Q is a closed set containing at least one point. Its complementary intervals are

$$(a_1, b_1),\ (a_2, b_2),\ \ldots,\ (a_n, b_n),\ \ldots, \tag{3.2}$$

whose end-points all belong to Q, which thus contains more than two points.

The intervals (3.2) do not overlap, so that

$$\sum_{n=1}^{\infty} (b_n - a_n) \leqslant b - a,$$

and $l_n \to 0$ as $n \to \infty$.

Suppose now that α and β are non-consecutive points of Q. They must belong to Q_n for all sufficiently large values of n, and they cannot be consecutive points of Q_n, since the complementary intervals of Q_n are also complementary intervals of Q. If n is sufficiently large,

$$l_n < \beta - \alpha,$$

and (α, β) cannot belong to Γ_n. It must belong to Ω, for otherwise it would belong to Γ and therefore to Γ_n, since every interval of Γ whose end-points are non-consecutive points of Q_n belongs to Γ_n.

On the other hand, the complementary intervals of Q are all taken from Γ and so do not belong to Ω.

The set Q has all the properties required.

It will be noticed that the characteristic set obtained by the use of the above process always contains the end-points a and b of the interval of Ω in which it is constructed.

A characteristic set is said to be *restricted* to a given set E, if it is a sub-set of E.

THEOREM II. *If Ω is an open system of intervals, and E a closed set such that there is an interval of Ω having both end-points on E, while no complementary interval of E is a member of Ω, then Ω has a characteristic set restricted to E.*

Let (a, b) be an interval of Ω whose end-points belong to E. The system of intervals having both end-points on E is closed. Its complement with respect to the system of all intervals contained in (a, b) is open. Let this system be added to Ω and denote the resulting open system by Ω^*. The interval (a, b) belongs to this system. Theorem I shows that Ω^* has a characteristic set Q whose extreme points are a and b.

It will be shown that Q is a sub-set of E and that it is a characteristic set of Ω.

The complementary intervals of Q are not members of Ω^*; consequently they have both their end-points on E. The boundary of Q consists of the end-points of complementary intervals and their limits, with the addition of the points a and b, if these have not already been included. It is clear that the boundary of Q is a sub-set of E. Thus the interior of a complementary interval of E either contains no points of Q or consists entirely of points of Q. The latter case, however, is not possible, since the end-points of the interval would also belong to Q, forming an interval necessarily belonging to Ω^*, contrary to hypothesis. No point of Q is interior to a complementary interval of E, and Q is a sub-set of E.

Again, an interval whose end-points are non-consecutive points of Q is a member of Ω^* and has both end-points on E. It is therefore a member of Ω. A complementary interval of Q is not a member of Ω^*, nor of Ω. The system Ω has Q as a characteristic set.

Characteristic sets associated with continuous and with consistent functions of intervals.

4. Characteristic sets associated with functions of intervals have already been defined.

The functions of intervals which are considered in this and the following sections are supposed to be finite and to be defined in the field consisting of all non-degenerate intervals contained in some finite fundamental interval.

The function of intervals $F(a, b)$ is said to be continuous if the difference between the upper and lower bounds of $F(a, b)$ is arbitrarily small in a sufficiently small neighbourhood of each interval of the field of definition.

It should be observed that a function, which is continuous according to the definition just given, may possibly be defined for degenerate intervals, and take values for these which destroy the property of continuity on any definition framed to include degenerate intervals. This would, in fact, occur in many cases of interest. For this reason, among others, degenerate intervals have been excluded.

Let (a, γ), (γ, β) and (a, β) be any two abutting intervals and their sum. A function which is such that, whenever

$$\frac{F(a, \gamma)}{\gamma - a} \leqslant \kappa, \quad \frac{F(\gamma, \beta)}{\beta - \gamma} \leqslant \kappa,$$

then also

$$\frac{F(a, \beta)}{\beta - a} \leqslant \kappa,$$

is said to be *consistent above* relative to the *index* κ. The corresponding property with the direction of the inequalities reversed characterizes a function which is *consistent below* relative to the *index* κ.

A function of intervals which is consistent above relative to all indices is described as *consistent above*; if it is consistent below relative to all indices it is described as *consistent below*.

An example of a class of functions consistent above is provided by the

functions which increase by subdivision. For, if

$$F(\alpha, \gamma) \leqslant \kappa(\gamma-\alpha), \quad F(\gamma, \beta) \leqslant \kappa(\beta-\gamma),$$

and if the property of increase by subdivision holds, so that

$$F(\alpha, \gamma)+F(\gamma, \beta) \geqslant F(\alpha, \beta),$$

then clearly

$$F(\alpha, \beta) \leqslant \kappa(\beta-\alpha),$$

and the function is consistent above relative to every index κ.

Similarly, functions of intervals which decrease by subdivision are consistent below.

The characteristic sets associated with consistent functions have an important special property. Let Q be a positive characteristic set with index κ associated with a function $F(a, b)$ which is consistent above relative to the index κ. Then Q cannot have more than two isolated points, and these must be extreme points, so that Q, which contains more than two points, contains a perfect positive characteristic set with index κ associated with the function.

To see this, suppose that γ is an isolated point of Q, not an extreme point. There are two abutting complementary intervals (α, γ), (γ, β) of Q. By the property of a characteristic set,

$$\frac{F(\alpha, \gamma)}{\gamma-\alpha} \leqslant \kappa, \quad \frac{F(\gamma, \beta)}{\beta-\gamma} \leqslant \kappa,$$

$$\frac{F(\alpha, \beta)}{\beta-\alpha} > \kappa.$$

The property of consistency above relative to the index κ is contradicted, and it follows that Q has no isolated points, except possibly its extreme points.

Characteristic sets associated with a continuous consistent function $F(a, b)$ have the further property that, if (α, β) is a complementary interval of such a set,

$$\frac{F(\alpha, \beta)}{\beta-\alpha} = \kappa,$$

where κ is the index of the set.

Suppose that Q is a positive characteristic set with index κ, that (α, β) is a complementary interval of Q, and that

$$\frac{F(\alpha, \beta)}{\beta-\alpha} < \kappa.$$

It has just been shown that the extreme points of Q are its only possible isolated points; it follows that α and β are not both isolated points of Q. Hence an interval (α', β') can be found at an arbitrarily small distance from (α, β), such that α' and β' are non-consecutive points of Q. Since Q is a characteristic set of $F(a, b)$,

$$\frac{F(\alpha', \beta')}{\beta' - \alpha'} > \kappa$$

and the continuity of $F(a, b)/(b-a)$, and hence of $F(a, b)$, is contradicted.

THEOREM IIIa. *If $F(a, b)$ is a continuous function of intervals, and if (c, d) is an interval such that*

$$\frac{F(c, d)}{d-c} > \kappa,$$

then (c, d) contains a positive characteristic set with index κ associated with the function $F(a, b)$.

If, in addition, the function $F(a, b)$ is consistent above relative to the index κ, then (c, d) contains a perfect positive characteristic set with index κ.

Consider the system Ω of intervals (α, β) such that

$$\frac{F(\alpha, \beta)}{\beta - \alpha} > \kappa.$$

Since $F(a, b)$ and therefore $F(a, b)/(b-a)$ is continuous, this system is open relative to the fundamental interval in which $F(a, b)$ is defined.

The interval (c, d) belongs to Ω and therefore contains a characteristic set Q of Ω, consisting of the points c, d and at least one other point. If α and β are non-consecutive points of Q,

$$\frac{F(\alpha, \beta)}{\beta - \alpha} > \kappa,$$

while, if α and β are consecutive points of Q,

$$\frac{F(\alpha, \beta)}{\beta - \alpha} \leqslant \kappa.$$

Thus Q is a positive characteristic set with index κ associated with $F(a, b)$.

If the function $F(a, b)$ is consistent above relative to the index κ, a perfect characteristic set of the kind required may be obtained from Q by the omission from it of its isolated extremes, at most two in number.

Theorem IIIB, which deals with negative characteristic sets, can now be formulated and proved without difficulty. It can be deduced from Theorem IIIA by writing $-F(a, b)$ for $F(a, b)$ and $-\kappa$ for κ.

The theorem which now follows is an extension giving sufficient conditions for the existence of restricted characteristic sets.

THEOREM IVA. *If $F(a, b)$ is a continuous function of intervals and if (c, d) is an interval such that*

$$\frac{F(c, d)}{d-c} > \kappa,$$

while E is a closed point set whose extreme points are c and d, and whose complementary intervals,

$$(c_1, d_1),\ (c_2, d_2),\ \ldots,\ (c_n, d_n),\ \ldots.$$

are such that

$$\frac{F(c_n, d_n)}{d_n-c_n} \leqslant \kappa \quad (n = 1, 2, \ldots),$$

then $F(a, b)$ has a positive characteristic set with index κ restricted to E.

If, in addition, $F(a, b)$ is consistent above relative to the index κ, then there is a perfect positive characteristic set with index κ restricted to E.

Consider the open system Ω of intervals (α, β) such that

$$\frac{F(\alpha, \beta)}{\beta-\alpha} > \kappa.$$

This system includes the interval (c, d), while all the complementary intervals of E are excluded from it. Theorem II shows that there is a characteristic set Q of Ω restricted to E, and, just as in the proof of IIIA, it is seen that Q is a positive characteristic set with index κ associated with $F(a, b)$.

If the function $F(a, b)$ is consistent above relative to the index κ, a perfect characteristic set of the kind required may be obtained from Q by the omission from it of at most two points.

Theorem IVB, which deals with negative characteristic sets, can be formulated and proved without difficulty.

Theorem III yields information about the derivatives of a consistent function.

If the continuous function $F(a, b)$ is consistent above relative to the index κ, and if

$$\frac{F(c, d)}{d-c} > \kappa, \tag{4.1}$$

then (c, d) contains a perfect set Q at each point of which the derivatives of $F(a, b)$ restricted to Q are not less than κ.

A corresponding deduction may be made in the case of a function consistent below relative to the index κ.

From Theorem IV it appears that, if (4.1) holds, with $F(a, b)$ as described, and if c and d are the extreme points of a closed set E for whose complementary intervals (c_n, d_n)

$$\frac{F(c_n, d_n)}{d_n-c_n} \leqslant \kappa,$$

then E contains a perfect set Q. Also, at each point of Q, the derivatives of $F(a, b)$ restricted to Q are not less than κ.

Determinate functions.

5. THEOREM V. *If a continuous function of intervals $F(a, b)$, for which*

$$\frac{F(c, d)}{d-c} = \rho,$$

has a derivative less than κ, where $\kappa < \rho$, at points of a measurable subset E of open (c, d), then associated with $F(a, b)$ there is in closed (c, d) a positive characteristic set of index κ, whose measure is not greater than $d-c-mE$.

If the function has at points of E a derivative greater than κ, where $\kappa > \rho$, then $F(a, b)$ has in closed (c, d) a negative characteristic set of index κ, whose measure is not greater than $d-c-mE$.

Only the first part of the theorem will be considered in detail, the proof of the second being similar to that of the first.

Associated with each point x of E there is a family of sub-intervals (a, b) of (c, d), for each of which

$$F(a, b) \leqslant \kappa(b-a),$$

and this family satisfies the conditions of Vitali's theorem. This shows that E can be covered, except possibly for a set of measure zero, by a

countable number

$$(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n), \ldots \tag{5 . 1}$$

of non-overlapping intervals taken from the families.

Let S be the set of points interior to the intervals (5 . 1) and let T be the complement of S with respect to closed (c, d). Then T is a closed set containing c and d. Its measure is not greater than $d-c-mE$. The intervals (5 . 1) are its complementary intervals.

Since

$$\frac{F(c, d)}{d-c} > \kappa,$$

while for all values of n

$$\frac{F(a_n, b_n)}{b_n-a_n} \leqslant \kappa,$$

the conditions of Theorem IV are satisfied, and there is a positive characteristic set of $F(a, b)$ restricted to T. This is a characteristic set of the kind required.

The result just proved applies in particular to the case where E is the set of all points of open (c, d) at which $F(a, b)$ has a derivative less than κ. It is only necessary to show that the set in question is measurable.

If $F(a, b)$ is a function of intervals defined in $(\mathbf{a}, \mathbf{b})$ *and if κ is any number, then the set of points of open* $(\mathbf{a}, \mathbf{b})$ *at which $F(a, b)$ has a derivative less than κ is measurable.*

Let $\Sigma_{p, q}$ be the system of all intervals (α, β) such that

$$\beta-\alpha < \frac{1}{q},$$

$$\frac{F(\alpha, \beta)}{\beta-\alpha} < \kappa-\frac{1}{p},$$

where p and q are positive integers. Denote by $E_{n, p, q}$ the set of points x of open $(\mathbf{a}, \mathbf{b})$ such that at least one interval of $\Sigma_{p, q}$ has, with respect to x, a parameter less than n, where n is a positive integer. It will be shown that the operations $\lim_{n\to\infty} \lim_{p\to\infty} \lim_{q\to\infty}$ applied to $E_{n, p, q}$ give rise to a measurable set E, and that this is the set of points of open $(\mathbf{a}, \mathbf{b})$ at which $F(a, b)$ has a derivative less than κ.

The set $E_{n, p, q}$ is open. Any point x_0 of the set has parameter less than n with respect to some interval of $\Sigma_{p, q}$, and, the parameter with respect to this particular interval being a continuous function of x, there is a neighbourhood of x_0 consisting entirely of points of $E_{n, p, q}$.

The sequence $\{E_{n,p,q}\}$ decreases, for fixed n and p, as $q \to \infty$. Thus, if

$$E_{n,p} = \lim_{q \to \infty} E_{n,p,q} = E_{n,p,1} E_{n,p,2} \dots E_{n,p,r} \dots,$$

$E_{n,p}$ is measurable as a product of open sets.

The new set $E_{n,p}$ consists entirely of points at which $F(a, b)$ has a derivative not greater than $\kappa-(1/p)$, and it contains all the points at which there is a derivative less than $\kappa-(1/p)$ with parameter less than n.

The sequence $\{E_{n,p}\}$ increases with p if n is fixed. Thus, if

$$E_n = \lim_{p \to \infty} \lim_{q \to \infty} E_{n,p,q} = E_{n,1} + E_{n,2} + \dots + E_{n,r} + \dots,$$

E_n is measurable, as a sum of measurable sets. It consists entirely of points at which there is a derivative less than κ, and contains all the points at which there is a derivative less than κ with parameter less than n.

The sequence $\{E_n\}$ increases. Thus, if

$$E = \lim_{n \to \infty} \lim_{p \to \infty} \lim_{q \to \infty} E_{n,p,q} = E_1 + E_2 + \dots + E_n + \dots,$$

E is measurable. It consists entirely of points at which there is a derivative less than κ, and contains all such points.

It will be observed that E is a sum of products of open sets. This fixes its place in Borel's system.

A continuous function of intervals is said to be *determinate above* in any interval in which no positive characteristic set of measure zero is associated with it. It is *determinate below* in any interval in which no negative characteristic set of measure zero is associated with it.

A continuous functions of intervals which is consistent above is determinate above if it has no perfect positive characteristic set of measure zero; for such a function has no positive characteristic set which does not contain a perfect characteristic set of the same kind. A corresponding statement holds for functions which are consistent below.

A function which is determinate above or below in a particular interval is evidently determinate above or below in all sub-intervals of that interval.

THEOREM VIa. *If $F(a, b)$ is determinate above in the interval (c, d) and has almost everywhere in (c, d) a derivative*

$$DF \leqslant h,$$

then

$$\frac{F(c, d)}{d-c} \leqslant h.$$

Suppose, if possible, that

$$\frac{F(c, d)}{d-c} = \rho > h,$$

while

$$DF \leqslant h$$

almost everywhere in (c, d).

Take any number κ such that $h < \kappa < \rho$. Then Theorem V shows at once that $F(a, b)$ has a positive characteristic set of index κ and measure zero. This is impossible, since $F(a, b)$ is determinate above, and the theorem is proved.

The corresponding Theorem VIb on functions which are determinate below can be stated and proved without difficulty.

An immediate consequence of VI is that

If the function $F(a, b)$ is determinate above and below in the interval (c, d), and if it has a regular derivative

$$DF = h$$

almost everywhere in (c, d), then

$$\frac{F(a, b)}{b-a} = h$$

for all intervals (a, b) contained in (c, d).

In the particular case $h = 0$ the conclusion is

$$F(a, b) = 0.$$

The average rate of increase between the extreme points of a closed set.

6. THEOREM VIIa. *If Q is a closed set whose extreme points are c and d and whose complementary intervals are*

$$(c_1, d_1), (c_2, d_2), \ldots, (c_n, d_n), \ldots, \tag{6.1}$$

and if $F(a, b)$ is a continuous function of intervals consistent above relative to the index κ, such that for all the intervals (6.1)

$$\frac{F(c_n, d_n)}{d_n - c_n} \leqslant \kappa,$$

while at every point of Q which is not isolated on the right there is a right-hand derivative of $F(a, b)$ restricted to Q less than κ, then

$$\frac{F(c, d)}{d-c} \leqslant \kappa. \tag{6 . 2}$$

Suppose that (6 . 2) is false.

Consider the set of points x of Q such that

$$\frac{F(c, x)}{x-c} \leqslant \kappa. \tag{6 . 3}$$

The continuity of $F(a, b)$ shows that the points x form a closed set, and, since d is not a member of this set, it has as its right-hand extreme a point ξ of Q on the left of d. Now either ξ is one of the points c_n or it is a limit of points of Q on its right. The hypotheses on complementary intervals and on right-hand derivatives show that in either case a point η of Q, on the right of ξ, can be found, such that

$$\frac{F(\xi, \eta)}{\eta-\xi} \leqslant \kappa. \tag{6 . 4}$$

Since ξ is an x satisfying (6 . 3), and $F(a, b)$ is consistent above relative to the index κ, it follows from (6 . 4) that

$$\frac{F(c, \eta)}{\eta-c} \leqslant \kappa.$$

This shows that η, which lies on the right of ξ, is an x, contrary to the definition of ξ. Thus (6 . 2) cannot be false, and the theorem is proved.

The companion result VIIB on functions consistent below can easily be obtained.

A slight modification of the proof of VIIA is sufficient to establish VIIC; this is the same as VIIA, except that in the hypothesis on derivatives the word "left" is substituted for "right".

The companion result on functions consistent below may be referred to as VIID.

THEOREM VIIIA. *If Q is a closed set whose extreme points are c and d and whose complementary intervals are*

$$(c_1, d_1), (c_2, d_2), \ldots, (c_n, d_n) \ldots,$$

and if $F(a, b)$ is a continuous function of intervals which increases by

subdivision and is such that

$$\sum_{n=1}^{\infty} |F(c_n, d_n)| < \infty, \tag{6.5}$$

while at every point of Q which is not isolated on the right there is at least one right-hand derivative not greater than h, then

$$F(c, d) \leqslant h.mQ + \sum_{n=1}^{\infty} F(c_n, d_n).$$

It is sufficient to prove that

$$F(c, d) \leqslant \kappa.mQ + \sum_{n=1}^{\infty} F(c_n, d_n)$$

for every number κ greater than h.

By considering

$$F(a, b) - \kappa(b-a),$$

the general case can be reduced to that in which $\kappa = 0$. Thus, under the assumption (6.5) and that of at least one negative right-hand derivative at every point of Q that is not isolated on the right, it is required to prove that

$$F(c, d) \leqslant \sum_{n=1}^{\infty} F(c_n, d_n). \tag{6.6}$$

Let $\phi(a, b)$ be the function defined by

$$\phi(a, b) = \sum_{a \leqslant c_n,\ d_n \leqslant b} |F(c_n, d_n)|.$$

This function is additive for intervals associated with Q. It is also continuous in Q. For let ϵ be any positive number. In view of (6.5) a positive integer ν can be found such that

$$\sum_{n=\nu+1}^{\infty} |F(c_n, d_n)| < \epsilon. \tag{6.7}$$

If now δ is the smallest of the numbers

$$d_1 - c_1,\ d_2 - c_2,\ \ldots,\ d_\nu - c_\nu,$$

then clearly

$$\phi(a, b) < \epsilon,$$

whenever a and b are points of Q such that

$$0 < b - a < \delta,$$

for the terms of which $\phi(a, b)$ is composed are taken from the series in (6.7).

As a consequence, the set consisting of the points x of Q such that

$$F(c, x) \leqslant \phi(c, x)$$

is closed relative to Q and therefore absolutely. It has a right-hand extreme ξ belonging to Q.

If possible, suppose that

$$F(c, d) > \phi(c, d). \tag{6.8}$$

Then ξ cannot be d. Either it is one of the points c_n or it has points of Q on its immediate right.

If $\xi = c_r$, then, since $F(a, b)$ increases by subdivision,

$$\begin{aligned} F(c, d_r) &\leqslant F(c, c_r) + F(c_r, d_r) \\ &\leqslant \phi(c, c_r) + \phi(c_r, d_r) \\ &= \phi(c, d_r), \end{aligned}$$

which is impossible, since d_r lies to the right of ξ.

On the other hand, if ξ is not isolated on the right, there are points η of Q on the right of ξ such that

$$\frac{F(\xi, \eta)}{\eta - \xi} < 0,$$

and so

$$F(\xi, \eta) < 0.$$

Now

$$\begin{aligned} F(c, \eta) &\leqslant F(c, \xi) + F(\xi, \eta) \\ &\leqslant \phi(c, \xi) \\ &\leqslant \phi(c, \eta), \end{aligned}$$

which again is impossible, since η lies on the right of ξ.

The inequality (6.8) cannot hold, and

$$F(c, d) \leqslant \phi(c, d).$$

From this, which manifestly holds when c and d are replaced by any two points of Q, the final result can be deduced.

For subtract from (c, d) the first ν complementary intervals, where ν is determined as already described, and let

$$(\gamma_1, \delta_1), (\gamma_2, \delta_2), \ldots, (\gamma_s, \delta_s)$$

be the intervals which remain, degenerate intervals being omitted. Then,

because $F(a, b)$ increases by subdivision,

$$\begin{aligned} F(c, d) &\leqslant \sum_{n=1}^{\nu} F(c_n, d_n) + \sum_{r=1}^{s} F(\gamma_r, \delta_r) \\ &\leqslant \sum_{n=1}^{\nu} F(c_n, d_n) + \sum_{r=1}^{s} \phi(\gamma_r, \delta_r) \\ &\leqslant \sum_{n=1}^{\infty} F(c_n, d_n) + \sum_{n=\nu+1}^{\infty} |F(c_n, d_n)| + \sum_{r=1}^{s} \phi(\gamma_r, \delta_r) \\ &< \sum_{n=1}^{\infty} F(c_n, d_n) + \epsilon + \epsilon. \end{aligned}$$

Since ϵ is arbitrary, (6.6) follows.

Theorem VIIIB, which deals with functions that decrease by subdivision, can now be formulated and proved without difficulty.

If the word "left" replaces "right" in the hypotheses on derivatives in VIIIA and B, two further results are obtained. These may be called Theorems VIIIC and D. Their proofs present no difficulty.

Sufficient conditions that a function should be determinate.

7. THEOREM IXA. *A continuous function of intervals which increases by subdivision cannot have a perfect positive characteristic set of measure zero unless it has positive infinity as a derivative on both sides at the points of a residual of the characteristic set.*

As has been observed in § 4, the function is consistent above relative to every index. Thus every positive characteristic set contains a perfect positive characteristic set.

Let $F(a, b)$ be the function and Q a perfect positive characteristic set of measure zero.

Take any two non-consecutive points c, d of Q and let q be the portion of Q obtained by taking the common part of Q and closed (c, d). If κ is the index of Q, and

$$(c_1, d_1),\ (c_2, d_2),\ \ldots,\ (c_n, d_n),\ \ldots$$

are the complementary intervals of q, the properties of a characteristic set yield

$$\frac{F(c, d)}{d-c} > \kappa,$$

$$\frac{F(c_n, d_n)}{d_n - c_n} = \kappa \quad (n = 1, 2, \ldots).$$

Now q is of measure zero, so that

$$d-c = \sum_{n-1}^{\infty} (d_n-c_n),$$

which gives at once

$$F(c, d) > \sum_{n=1}^{\infty} F(c_n, d_n)$$

$$= h.mq + \sum_{n=1}^{\infty} F(c_n, d_n),$$

where h is any finite number, since q is of measure zero.

Theorem VIIIA shows that q must contain at least one point at which $F(a, b)$ has a right-hand derivative greater than h.

Now let h be an arbitrary positive integer, and consider the points α of q which are left-hand end-points of intervals (α, β) such that

$$(7.1) \qquad \begin{cases} \dfrac{F(\alpha, \beta)}{\beta-\alpha} > h, \\ \beta-\alpha < \dfrac{1}{h}. \end{cases}$$

It has just been shown that there is at least one such point. Since $F(a, b)$ is continuous, all points α sufficiently near this point also have the required property. Thus there is a portion of q consisting of points α satisfying (7.1).

This, however, is the condition that the set consisting of the points of Q which satisfy (7.1) should be completely dense in Q. Denote it by Q_h.

The common part

$$R = Q_1 Q_2 \dots Q_h \dots$$

is by definition a residual of Q. At each point of it there is clearly an infinite right-hand derivative.

In a similar way it can be proved that there is a residual S, at each of whose points there is an infinite left-hand derivative. The common part RS is a residual which has the required properties.

Theorem IXB, which gives the corresponding result involving negative characteristic sets, can be formulated and proved without difficulty.

THEOREM XA. *A continuous function of intervals which increases by subdivision is determinate above if the set of points at which it has positive infinity as a derivative on both sides does not contain a perfect set.*

If the function is not determinate above there must be a perfect positive characteristic set of measure zero, and Theorem IX shows that $+\infty$ is a derivative on both sides at the points of a residual of this set. This residual, if obtained by the methods of IXA, is a Borel set, and, as such, being non-countable, contains a perfect set.

Theorem XB, which gives the corresponding result involving functions which decrease by subdivision, is similar.

The two results just proved combine to show that a continuous additive function of intervals is determinate above and below if there is no perfect set at each of whose points the function has an infinite derivative of fixed sign on both sides.

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ON SERIES ALLIED TO THE HYPERGEOMETRIC SERIES WITH ARGUMENT −1

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[Received 2 February, 1928.—Read 16 February, 1928.]

1. *Introduction.*

In the course of a discussion of transformations of generalized hypergeometric series I obtained recently two formulae involving the ordinary hypergeometric series with the argument −1. One of these formulae* was general, the other† was subject to the condition that both the series involved must terminate. The question arose whether the latter formula was a special case of some more general formula, and thus I was led to consider the classification of the series which could be expressed in terms of one or more hypergeometric series of type ${}_2F_1(-1)$. The transformations used depend mostly on the first of the formulae which have been mentioned, and the series involved are of the types ${}_3F_2(1)$ and ${}_6F_5(-1)$. There is, however, another group in which the principal formula is due to Fox, and the series are of the type ${}_4F_3(1)$. The relation of this group to the other is discussed in the last paragraphs of the paper.

2. *The hypergeometric series* ${}_2F_1(-1)$.

In the discussion of hypergeometric series from the point of view of differential equations 24 related series are obtained as solutions of the same equation. Of the 24 four have the same argument, say x, and another four the argument $1/x$. When $x = -1$ the series assume the

* Whipple, **1**, Equation (5.32).

† Whipple, **1**, Equation (6.53) and the remark after (6.6).

following forms. The reason for the notation $P(0)$ $N(0)$, $P(4)$ $N(4)$ will appear later.

$$(2.11)\quad P(0) \equiv {}_2F_1\left[\begin{matrix} a, b; \\ c \end{matrix} -1\right] = 2^{c-a-b}\,{}_2F_1\left[\begin{matrix} c-a, c-b; \\ c \end{matrix} -1\right],$$

$$(2.12)\quad N(0) \equiv {}_2F_1\left[\begin{matrix} a-c+1, b-c+1; \\ 2-c \end{matrix} -1\right] = 2^{c-a-b}\,{}_2F_1\left[\begin{matrix} 1-a, 1-b; \\ 2-c \end{matrix} -1\right],$$

$$(2.13)\quad P(4) \equiv {}_2F_1\left[\begin{matrix} a, a-c+1; \\ a-b+1 \end{matrix} -1\right] = 2^{c-a-b}\,{}_2F_1\left[\begin{matrix} c-b, 1-b; \\ a-b+1 \end{matrix} -1\right],$$

$$(2.14)\quad N(4) \equiv {}_2F_1\left[\begin{matrix} b, b-c+1; \\ b-a+1 \end{matrix} -1\right] = 2^{c-a-b}\,{}_2F_1\left[\begin{matrix} c-a, 1-a; \\ b-a+1 \end{matrix} -1\right].$$

By taking a special case of the series involving $1/(1-x)$ we find that

$$(2.2)\qquad P(0) = \frac{1}{2^a}\,{}_2F_1\left[\begin{matrix} a, c-b; \\ c \end{matrix} \tfrac{1}{2}\right].$$

The relations involving three series lead to equations such as

$$(2.32)\qquad \frac{P(0)}{\Gamma(c)} = \frac{\Gamma(b-a)}{\Gamma(c-a)\,\Gamma(b)}\,P(4) + \frac{\Gamma(a-b)}{\Gamma(c-b)\,\Gamma(a)}\,N(4).$$

3. *The characteristics of a reducible* ${}_3F_2(+1)$.

The formula which was the starting point of our investigation* is

$$(3.1)\quad {}_2F_1\left[\begin{matrix} a, b; \\ c \end{matrix} -1\right] = \frac{\Gamma(c)\,\Gamma(\frac{1}{2}b+\frac{1}{2}c)}{\Gamma(\frac{1}{2}c-\frac{1}{2}b)\,\Gamma(b+c)}\,{}_3F_2\left[\begin{matrix} b, \frac{1}{2}(b+c-a), \frac{1}{2}(1+b+c-a) \\ b+c-a, \frac{1}{2}(1+b+c) \end{matrix} \right].$$

The ${}_3F_2$ has parameters related in a special way. To study the other reducible ${}_3F_2$'s it is convenient to utilize the notation which has been introduced previously† for the discussion of relations between ${}_3F_2$'s. It will be remembered that in this notation six parameters $r_0, r_1, \ldots, r_5$ are required such that $\sum_0^5 r = 0$.

With the r's are associated α's and β's such that

$$\alpha_{lmn} = \tfrac{1}{2}+r_l+r_m+r_n\,; \quad \beta_{mn} = 1+r_m-r_n.$$

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* Whipple, **1**, Equation (5.32).

† Whipple, **2**. The reader will find it necessary to construct a table like Table I, p. 106, of this paper.

Then the function $Fp(u\,;\ v,\ w)$ is defined by the equation

$$(3.2)\qquad Fp(u;\ v,\ w) \equiv \frac{1}{\Gamma(\alpha_{xyz})\,\Gamma(\beta_{vu})\,\Gamma(\beta_{wu})}\ F\begin{bmatrix}\alpha_{vwx},\ \alpha_{vwy},\ \alpha_{vwz}\\ \beta_{vu},\ \beta_{wu}\end{bmatrix}.$$

The 10 functions $Fp(u\,;\ v,\ w)$ obtained by permutations of $w,\ w,\ x,\ y,\ z$ are all equal*. By virtue of this equality we write $Fp(u)$ as the common value of the 10 functions. In our example we take

$$(3.3)\qquad Fp(0\,;\ 4,\ 5) \equiv \frac{1}{\Gamma(\frac{1}{2}c-\frac{1}{2}b)\,\Gamma(b+c-a)\,\Gamma(\frac{1}{2}+\frac{1}{2}b+\frac{1}{2}c)} \times {}_3F_2\begin{bmatrix}b,\ \frac{1}{2}(b+c-a),\ \frac{1}{2}(1+b+c-a)\\ b+c-a,\ \frac{1}{2}(1+b+c)\end{bmatrix},$$

and thereby fix all the r's.

Quoting (2.11), we have

$$(3.31)\qquad Fp(0) \equiv Fp(0\,;\ 4,\ 5) = \frac{2^{b+c-1}}{\Gamma(b+c-a)\,\Gamma(c)\,\Gamma(\frac{1}{2})}\ P(0).$$

The special relations satisfied by the parameters r are easily found to be

$$(3.4)\qquad r_3-r_2 = r_5-r_1 = \tfrac{1}{2}.$$

The symmetry between the characteristics 3 and 5, 2 and 1 will be noted.

Our original identity (3.1) can be written in the form

$$(3.5)\qquad {}_3F_2\begin{bmatrix}b,\ \frac{1}{2}(b+c-a),\ \frac{1}{2}(1+b+c-a)\\ b+c-a,\ \frac{1}{2}(1+b+c)\end{bmatrix} = \frac{\Gamma(\frac{1}{2}c-\frac{1}{2}b)\,\Gamma(c+b)}{\Gamma(c)\,\Gamma(\frac{1}{2}c+\frac{1}{2}b)}\ P(0).$$

By utilizing the relation $Fp(0\,;\ 1,\ 4) \equiv Fp(0)$ we find that

$$(3.51)\qquad {}_3F_2\begin{bmatrix}b,\ \frac{1}{2}(b+c-a),\ \frac{1}{2}(b+c-a-1)\\ b+c-a,\ \frac{1}{2}(b+c)\end{bmatrix} = \frac{\Gamma(\frac{1}{2}+\frac{1}{2}c-\frac{1}{2}b)\,\Gamma(c+b)}{\Gamma(c)\,\Gamma(\frac{1}{2}+\frac{1}{2}c+\frac{1}{2}b)}\ P(0).$$

Similarly, by considering $Fp(0\,;\ 1,\ 2)$, $Fp(0\,;\ 1,\ 3)$ and $Fp(0\,;\ 3,\ 5)$, we obtain the identities

$$(3.6)\qquad {}_3F_2\begin{bmatrix}\frac{1}{2}(c-b),\ \frac{1}{2}(b+c-a-1),\ \frac{1}{2}a\\ \frac{1}{2}(b+c),\ c-\frac{1}{2}a\end{bmatrix} = 2^a\,\frac{\Gamma(\frac{1}{2}c+\frac{1}{2}b)\,\Gamma(c-\frac{1}{2}a)}{\Gamma(\frac{1}{2}b+\frac{1}{2}c-\frac{1}{2}a)\,\Gamma(c)}\ P(0),$$

$$(3.61)\qquad {}_3F_2\begin{bmatrix}\frac{1}{2}(c-b),\ \frac{1}{2}(b+c-a),\ \frac{1}{2}(a+1)\\ \frac{1}{2}(b+c),\ c-\frac{1}{2}a+\frac{1}{2}\end{bmatrix} = 2^a\,\frac{\Gamma(\frac{1}{2}c+\frac{1}{2}b)\,\Gamma(\frac{1}{2}+c-\frac{1}{2}a)}{\Gamma(\frac{1}{2}+\frac{1}{2}b+\frac{1}{2}c-\frac{1}{2}a)\,\Gamma(c)}\ P(0),$$

* This is in virtue of an identity discovered by Thomae.

$$(3.62)\quad {}_3F_2\left[\begin{matrix}\frac{1}{2}(1+c-b),\ \frac{1}{2}(b+c-a+1),\ \frac{1}{2}(a+1)\\ \frac{1}{2}(b+c+1),\ c-\frac{1}{2}a+\frac{1}{2}\end{matrix}\right]$$

$$= 2^a \frac{\Gamma(\frac{1}{2}c+\frac{1}{2}b+\frac{1}{2})\,\Gamma(\frac{1}{2}+c-\frac{1}{2}a)}{(b+c-a-1)\,\Gamma(\frac{1}{2}b+\frac{1}{2}c-\frac{1}{2}a)\,\Gamma(c)}\,P(0).$$

Finally, by considering $Fp(0\,;\,1,\,5)$ we find that

$$(3.7)\qquad {}_3F_2\left[\begin{matrix}b,\ \frac{1}{2}a,\ \frac{1}{2}(a+1)\\ \frac{1}{2}(b+c),\ \frac{1}{2}(b+c+1)\end{matrix}\right] = \frac{\Gamma(c-a)\,\Gamma(b+c)}{\Gamma(b+c-a)\,\Gamma(c)}\,P(0).$$

The identities (3.6) and (3.62) are unaltered by the substitution of $c-b$ for a and $c-a$ for b; the other four identities are altered by this substitution. Thus the ten ${}_3F_2$'s related to $P(0)$ through $Fp(0)$ are accounted for.

The neatest of the foregoing identities is (3.7). This identity can be established readily by integrating with respect to x both sides of the equation

$$(1-x^2)^{-b}(1-x)^{c-a+b-1}x^{a-1} = (1+x)^{-b}(1-x)^{c-a-1}x^{a-1},$$

after expanding the factors $(1-x^2)^{-b}$ and $(1+x)^{-b}$ in powers of x. If (3.7) is proved in this way then the preceding identities, including (3.1), can be regarded as corollaries.

4. *The three classes of associated ${}_3F_2$'s.*

Having found how the series defined by $Fp(0)$ can be expressed in terms of $P(0)$, we proceed to inquire whether any of the associated Fp's and Fn's can be expressed in terms of $N(0)$, $P(4)$ and $N(4)$, an Fn being equivalent to the corresponding Fp with the signs of the r's all changed.

We notice that neither r_0 nor r_4 occurs explicitly in the conditional equations (3.4), so that $Fp(4)$ may be expected to have properties similar to $Fp(0)$. Moreover, the conditional equations may be written

$$(-r_2)-(-r_3) = (-r_1)-(-r_5) = \tfrac{1}{2},$$

so that we may expect $Fn(0)$ and $Fn(4)$ to have similar properties also.

The same argument shows that there will be a second class of associated functions $Fp(3)$, $Fp(5)$, $Fn(2)$, $Fn(1)$, and a third class $Fp(2)$, $Fp(1)$, $Fn(3)$, $Fn(5)$.

5. *Associated ${}_3F_2$'s of the first class.*

The reduction of the ${}_3F_2$'s of the first class is effected without difficulty by change of letters in the primary formula (3.1) or (3.3). Thus we

have not only

$$(5.1) \qquad Fp(0) = \frac{2^{b+c-1}}{\Gamma(b+c-a)\,\Gamma(c)\,\Gamma(\frac{1}{2})}\,P(0),$$

but

$$(5.2) \qquad Fp(4) = \frac{2^{2a-b-c+1}}{\Gamma(2-b-c+a)\,\Gamma(a-b+1)\,\Gamma(\frac{1}{2})}\,P(4),$$

$$(5.3) \qquad Fn(0) = \frac{2^{2+a-2c}}{\Gamma(2-b-c+a)\,\Gamma(2-c)\,\Gamma(\frac{1}{2})}\,N(0),$$

and

$$(5.4) \qquad Fn(4) = \frac{2^{2b-a}}{\Gamma(b+c-a)\,\Gamma(b-a+1)\,\Gamma(\frac{1}{2})}\,N(4).$$

The convenience of the notation adopted in § 2 will be noted.

6. *Associated ${}_3F_2$'s of the second class.*

As representative of the second class we tabulate the various forms of $Fp(3)$.

$$(6.1) \quad Fp(3;\ 1,\ 2) = \frac{1}{\Gamma(\frac{1}{2})\,\Gamma(1-\frac{1}{2}c+\frac{1}{2}b)\,\Gamma\{\frac{1}{2}(1+a+b-c)\}} \times {}_3F_2\left[\begin{matrix}\frac{1}{2}a,\ \frac{1}{2}(1-b-c+a),\ \frac{1}{2}(b+c-a-1)\\ \frac{1}{2},\ \frac{1}{2}(1+a+b-c)\end{matrix}\right],$$

$$(6.11) \quad Fp(3;\ 2,\ 5) = \frac{1}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}-\frac{1}{2}c+\frac{1}{2}b)\,\Gamma\{\frac{1}{2}(2+a+b-c)\}} \times {}_3F_2\left[\begin{matrix}\frac{1}{2}a,\ 1-\frac{1}{2}(b+c-a),\ \frac{1}{2}(b+c-a)\\ \frac{1}{2},\ \frac{1}{2}(2+a+b-c)\end{matrix}\right],$$

$$(6.2) \quad Fp(3;\ 0,\ 2) = \frac{1}{\Gamma(\frac{1}{2})\,\Gamma(b)\,\Gamma(\frac{3}{2}-c+\frac{1}{2}a)} \times {}_3F_2\left[\begin{matrix}\frac{1}{2}(1-a),\ \frac{1}{2}(1-b-c+a),\ \frac{1}{2}(2-b-c+a)\\ \frac{1}{2},\ \frac{3}{2}-c+\frac{1}{2}a\end{matrix}\right],$$

$$(6.21) \quad Fp(3;\ 4,\ 2) = \frac{1}{\Gamma(\frac{1}{2})\,\Gamma(1-c+a)\,\Gamma(\frac{1}{2}+b-\frac{1}{2}a)} \times {}_3F_2\left[\begin{matrix}\frac{1}{2}(1-a),\ \frac{1}{2}(b+c-a-1),\ \frac{1}{2}(b+c-a)\\ \frac{1}{2},\ \frac{1}{2}+b-\frac{1}{2}a\end{matrix}\right],$$

$$(6.3) \quad Fp(3;\ 1,\ 5) = \frac{2^{a+b-c}}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}-\frac{1}{2}a)\,\Gamma(1+a+b-c)} \times {}_3F_2\left[\begin{matrix}b,\ \frac{1}{2}a,\ 1-c+a\\ \frac{1}{2}(1+a+b-c),\ \frac{1}{2}(2+a+b-c)\end{matrix}\right],$$

$$(6.4)\quad Fp(3;\ 0,\ 1) = \frac{1}{\Gamma\{\frac{1}{2}(b+c-a)\}\,\Gamma(\frac{3}{2}-c+\frac{1}{2}a)\,\Gamma\{\frac{1}{2}(1+a+b-c)\}} \times {}_3F_2\left[\begin{matrix}\frac{1}{2}(1-b-c+a),\ \frac{1}{2}(1+b-c),\ 1-c+a \\ \frac{3}{2}-c+\frac{1}{2}a,\ \frac{1}{2}(1+a+b-c)\end{matrix}\right],$$

$$(6.41)\quad Fp(3;\ 0,\ 5) = \frac{1}{\Gamma\{\frac{1}{2}(b+c-a-1)\}\,\Gamma(\frac{3}{2}-c+\frac{1}{2}a)\,\Gamma\{\frac{1}{2}(2+a+b-c)\}} \times {}_3F_2\left[\begin{matrix}\frac{1}{2}(2-b-c+a),\ \frac{1}{2}(2+b-c),\ 1-c+a \\ \frac{3}{2}-c+\frac{1}{2}a,\ \frac{1}{2}(2+a+b-c)\end{matrix}\right],$$

$$(6.5)\quad Fp(3;\ 0,\ 4) = \frac{1}{\Gamma(\frac{1}{2}a)\,\Gamma(\frac{3}{2}-c+\frac{1}{2}a)\,\Gamma(\frac{1}{2}+b-\frac{1}{2}a)} \times {}_3F_2\left[\begin{matrix}\frac{1}{2}(1-a),\ \frac{1}{2}(1+b-c),\ 1+\frac{1}{2}(b-c) \\ \frac{3}{2}-c+\frac{1}{2}a,\ \frac{1}{2}+b-\frac{1}{2}a\end{matrix}\right].$$

To express an ${}_3F_2$ of the second class in terms of hypergeometric series we make use of the linear relations* between the Fp and Fn functions and then substitute from the equations of § 5. In this way we can show that

$$(6.6)\quad Fp(3) = 2^{a+b-c}\,\frac{\Gamma(\frac{1}{2}+\frac{1}{2}a)\,\Gamma(c-b)}{\pi^{\frac{3}{2}}}\left[\frac{\cos\pi(\frac{1}{2}a-b)}{\Gamma(c)\,\Gamma(1-c+a)}\,P(0) - \frac{\cos\pi(\frac{1}{2}a-c)}{\Gamma(b)\,\Gamma(1-b+a)}\,P(4)\right],$$

$$(6.7)\quad Fp(3) = \frac{\Gamma(c-a)}{\Gamma(\frac{1}{2})}\left[\frac{2^{a+b-c}}{\Gamma(c)\,\Gamma(1+b-c)\,\Gamma(\frac{1}{2}-\frac{1}{2}a)}\,P(0) - \frac{2^{1+b-c}\cos\pi(c-\frac{1}{2}a)}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}a)\,\Gamma(1+b-a)}\,N(4)\right],$$

$$(6.8)\quad Fp(3) = \frac{2^{a+b-c}\,\Gamma(1-c)}{\Gamma(\frac{1}{2})\,\Gamma(1+b-c)\,\Gamma(\frac{1}{2}-\frac{1}{2}a)\,\Gamma(1+a-c)}\,P(0) - \frac{2^{1+b-c}\cos\pi(c-\frac{1}{2}a)\,\Gamma(c-1)}{\pi\,\Gamma(\frac{1}{2}a)\,\Gamma(b)}\,N(0),$$

and

$$(6.9)\quad Fp(3) = \frac{2^{b-c}\,\Gamma(1-\frac{1}{2}a)}{\pi}\left[\frac{N(0)}{\Gamma(b)\,\Gamma(2-c)} + \frac{N(4)}{\Gamma(1-c+a)\,\Gamma(1+b-a)}\right].$$

We may note here that, when b is a negative integer, the second term on the right of (6.8) vanishes, and the second formula referred to in § 1 is obtained by combining (6.3) and (6.8).

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* Whipple, 2, Equations 10–17. Note that the sign in Equation 16 should be +.

7. *Associated ${}_3F_2$'s of the third class.*

As representative of the third class we tabulate the various forms of $Fp(2)$.

$$(7.1)\quad Fp(2;\ 1,\ 3) = \frac{1}{\Gamma(\frac{3}{2})\,\Gamma(1-\frac{1}{2}c+\frac{1}{2}b)\,\Gamma\{\frac{1}{2}(2+a+b-c)\}} \times {}_3F_2\left[\begin{matrix}\frac{1}{2}(1+a),\ \frac{1}{2}(2-b-c+a),\ \frac{1}{2}(b+c-a)\\ \frac{3}{2},\ 1+\frac{1}{2}(a+b-c)\end{matrix}\right],$$

$$(7.11)\quad Fp(2;\ 3,\ 5) = \frac{1}{\Gamma(\frac{3}{2})\,\Gamma(\frac{1}{2}-\frac{1}{2}c+\frac{1}{2}b)\,\Gamma\{\frac{1}{2}(3+a+b-c)\}} \times {}_3F_2\left[\begin{matrix}\frac{1}{2}(1+a),\ \frac{1}{2}(3-b-c+a),\ \frac{1}{2}(1+b+c-a)\\ \frac{3}{2},\ \frac{1}{2}(3+a+b-c)\end{matrix}\right],$$

$$(7.2)\quad Fp(2;\ 0,\ 3) = \frac{1}{\Gamma(\frac{3}{2})\,\Gamma(b)\,\Gamma(2-c+\frac{1}{2}a)} \times {}_3F_2\left[\begin{matrix}1-\frac{1}{2}a,\ \frac{1}{2}(2-b-c+a),\ \frac{1}{2}(3-b-c+a)\\ \frac{3}{2},\ 2-c+\frac{1}{2}a\end{matrix}\right],$$

$$(7.21)\quad Fp(2;\ 4,\ 3) = \frac{1}{\Gamma(\frac{3}{2})\,\Gamma(1-c+a)\,\Gamma(1+b-\frac{1}{2}a)} \times {}_3F_2\left[\begin{matrix}1-\frac{1}{2}a,\ \frac{1}{2}(b+c-a),\ \frac{1}{2}(1+b+c-a)\\ \frac{3}{2},\ 1+b-\frac{1}{2}a\end{matrix}\right],$$

$$(7.3)\quad Fp(2;\ 1,\ 5) = \frac{2^{a+b-c+1}}{\Gamma(\frac{1}{2})\,\Gamma(1-\frac{1}{2}a)\,\Gamma(2+a+b-c)} \times {}_3F_2\left[\begin{matrix}b,\ \frac{1}{2}(1+a),\ 1-c+a\\ \frac{1}{2}(2+a+b-c),\ \frac{1}{2}(3+a+b-1)\end{matrix}\right],$$

$$(7.4)\quad Fp(2;\ 0,\ 1) = \frac{1}{\Gamma\{\frac{1}{2}(1+b+c-a)\}\,\Gamma(2-c+\frac{1}{2}a)\,\Gamma\{\frac{1}{2}(2+a+b-c)\}} \times {}_3F_2\left[\begin{matrix}1-\frac{1}{2}(b+c-a),\ \frac{1}{2}(1+b-c),\ 1-c+a\\ 2-c+\frac{1}{2}a,\ 1+\frac{1}{2}(a+b-c)\end{matrix}\right],$$

$$(7.41)\quad Fp(2;\ 0,\ 5) = \frac{1}{\Gamma\{\frac{1}{2}(b+c-a)\}\,\Gamma(2-c+\frac{1}{2}a)\,\Gamma\{\frac{1}{2}(3+a+b-c)\}} \times {}_3F_2\left[\begin{matrix}\frac{1}{2}(3-b-c+a),\ \frac{1}{2}(2+b-c),\ 1-c+a\\ 2-c+\frac{1}{2}a,\ \frac{1}{2}(3+a+b-c)\end{matrix}\right],$$

$$(7.5)\quad Fp(2;\ 0,\ 4) = \frac{1}{\Gamma(\frac{1}{2}+\frac{1}{2}a)\,\Gamma(2-c+\frac{1}{2}a)\,\Gamma(1+b-\frac{1}{2}a)} \times {}_3F_2\left[\begin{matrix}1-\frac{1}{2}a,\ 1+\frac{1}{2}(b-c),\ \frac{1}{2}(1+b-c)\\ 2-c+\frac{1}{2}a,\ 1+b-\frac{1}{2}a\end{matrix}\right].$$

To express an ${}_3F_2$ of the third class in terms of hypergeometric series we proceed in the same way and obtain the following equations :—

$$(7.6)\quad Fp(2) = \frac{2^{a+b-c+1}}{b+c-a-1}\,\frac{\Gamma(\frac{1}{2}a)\,\Gamma(c-b)}{\pi^{\frac{3}{2}}}\left[\frac{-\sin\pi(\frac{1}{2}a-b)}{\Gamma(c)\,\Gamma(1-c+a)}\,P(0) + \frac{\sin\pi(\frac{1}{2}a-c)}{\Gamma(b)\,\Gamma(1-b+a)}\,P(4)\right],$$

$$(7.7)\quad Fp(2) = \frac{2}{b+c-a-1}\,\frac{\Gamma(c-a)}{\Gamma(\frac{1}{2})}\left[\frac{2^{a+b-c}}{\Gamma(c)\,\Gamma(1+b-c)\,\Gamma(1-\frac{1}{2}a)}\,P(0) + \frac{2^{1+b-c}\sin\pi(\frac{1}{2}a-c)}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}+\frac{1}{2}a)\,\Gamma(1+b-a)}\,N(4)\right],$$

$$(7.8)\quad Fp(2) = \frac{2}{1+a-b-c}\left[\frac{2^{a+b-c}\,\Gamma(1-c)}{\Gamma(\frac{1}{2})\,\Gamma(1+b-c)\,\Gamma(1-\frac{1}{2}a)\,\Gamma(1+a-c)}\,P(0) - \frac{2^{1+b-c}\sin\pi(\frac{1}{2}a-c)\,\Gamma(c-1)}{\pi\,\Gamma(\frac{1}{2}+\frac{1}{2}a)\,\Gamma(b)}\,N(0)\right],$$

$$(7.9)\quad Fp(2) = \frac{2}{b+c-a-1}\,\frac{2^{b-c}\,\Gamma(\frac{1}{2}-\frac{1}{2}a)}{\pi} \times\left[\frac{N(0)}{\Gamma(b)\,\Gamma(2-c)} - \frac{N(4)}{\Gamma(1-c+a)\,\Gamma(1+b-a)}\right].$$

8. *Reduction of a pair of ${}_3F_2$'s to the form ${}_2F_1(-1)$.*

By eliminating $P(4)$ from the equations (6 . 6) and (7 . 6) we find that

$$(8.1)\quad \frac{\sin\pi(c-\frac{1}{2}a)}{\Gamma(\frac{1}{2}+\frac{1}{2}a)}\,Fp(3) + \frac{a-b-c+1}{2}\,\frac{\cos\pi(c-\frac{1}{2}a)}{\Gamma(\frac{1}{2}a)}\,Fp(2) = \frac{2^{a+b-c}}{\Gamma(\frac{1}{2})\,\Gamma(1+b-c)\,\Gamma(1+a-c)\,\Gamma(c)}\,P(0).$$

Similarly from (6 . 9) and (7 . 9) we find that

$$(8.2)\quad \frac{Fp(3)}{\Gamma(1-\frac{1}{2}a)} - \frac{a-b-c+1}{2}\,\frac{Fp(2)}{\Gamma(\frac{1}{2}-\frac{1}{2}a)} = \frac{2^{1+b-c}}{\pi\,\Gamma(b)\,\Gamma(2-c)}\,N(0).$$

These are perhaps the simplest of the relations between pairs of Fp functions. In substituting for the Fp's the appropriate ${}_3F_2$'s we choose pairs which will combine into single series. Thus in the case of (8 . 1) we use (6 . 2) and (7 . 2) and obtain the identity

$$(8.3)\quad \sum_{r=0}^{\infty}\frac{(1+a-b-c)_r}{r!}\,\frac{\Gamma(c-\frac{1}{2}a-\frac{1}{2}r-\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{1}{2}a-\frac{1}{2}r)} = \frac{2^{a+b-c+1}\,\pi\,\Gamma(b)}{\Gamma(1+b-c)\,\Gamma(1+a-c)\,\Gamma(c)\sin\pi(a-2c)}\,P(0).$$

For substitution in (8 . 2) we may use the same pair of identities and obtain the equation*

$$(8.4)\quad \sum_{r=0}^{\infty}(-1)^r\frac{(1+a-b-c)_r}{r!}\frac{\Gamma(\frac{1}{2}-\frac{1}{2}a+\frac{1}{2}r)}{\Gamma(\frac{3}{2}-c+\frac{1}{2}a+\frac{1}{2}r)}=2^{1+b-c+a}\frac{\Gamma(1-a)}{\Gamma(2-c)}N(0).$$

Writing $1-a$ for a, $1-b$ for b, and $1-c$ for c, we find in place of this last equation

$$(8.41)\quad \sum_{r=0}^{\infty}(-1)^r\frac{(c+b-a-1)_r}{r!}\frac{\Gamma(\frac{1}{2}a+\frac{1}{2}r)}{\Gamma(c-\frac{1}{2}a+\frac{1}{2}r)}=2\,.\,\frac{\Gamma(a)}{\Gamma(c)}P(0).$$

Similar results can be obtained by selecting other expressions for $Fp(3)$ and $Fp(4)$.

Equations of a somewhat different type, in which a single series is expressed in terms of two hypergeometric series, can also be deduced. Thus (6 . 1) and (7 . 1), combined with the aid of (6 . 9) and (7 . 9), give

$$(8.5)\quad \sum_{r=0}^{\infty}\frac{\Gamma\{\frac{1}{2}(a+r)\}\,\Gamma\{\frac{1}{2}(1-b-c+a+r)\}\,\Gamma\{\frac{1}{2}(b+c-a-1+r)\}}{r!\,\Gamma\{\frac{1}{2}(1+a+b-c+r)\}}(2x)^r$$

$$=\frac{2^{b-c+3}\pi\,\Gamma(\frac{1}{2})\,\Gamma(1-\frac{1}{2}c+\frac{1}{2}b)}{(b+c-a-1)\sin\pi(b+c-a)\sin\pi a}$$

$$\times\left[\frac{N(0)}{\Gamma(b)\,\Gamma(2-c)}\sin\tfrac{1}{2}\pi(b+c)+\frac{N(4)}{\Gamma(1-c+a)\,\Gamma(1+b-a)}\sin\tfrac{1}{2}\pi(b+c-2a)\right],$$

when $x=+1$, and the same expression with the sines in the square brackets interchanged when $x=-1$.

Similarly from (6 . 11) and (7 . 11) we derive the results :

$$(8.6)\quad \sum_{r=0}^{\infty}\frac{\Gamma\{\frac{1}{2}(a+r)\}\,\Gamma\{\frac{1}{2}(2-b-c+a+r)\}\,\Gamma\{\frac{1}{2}(b+c-a+r)\}}{r!\,\Gamma\{\frac{1}{2}(2+a+b-c+r)\}}(2x)^r$$

$$=\frac{2^{b-c+2}\pi\,\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}-\frac{1}{2}c+\frac{1}{2}b)}{\sin\pi(b+c-a)\sin\pi a}$$

$$\times\left[\frac{N(0)}{\Gamma(b)\,\Gamma(2-c)}\cos\tfrac{1}{2}\pi(b+c)+\frac{N(4)}{\Gamma(1-c+a)\,\Gamma(1+b-c)}\cos\tfrac{1}{2}\pi(b+c-2a)\right],$$

when $x=+1$ and the same expression with the cosines interchanged when $x=-1$.

9. *Reducible well poised series.*

By the name well poised series† I designate a generalized hypergeometric series such that the parameters after the first occur in pairs

* (8.4) was given by Fox for the case $1+a-b-c$ a negative integer [Fox, **1**, Equation (2.7)].

† Whipple, **3**.

each with the same sum, this being equal to the sum of the first parameter and unity. The well poised series ${}_6F_5(-1)$ is associated with allied series ${}_3F_2$, the typical relation is brought out by the notation* :—

$$(9.1)\quad Fp(0;\ 1) \equiv \frac{\Gamma(\delta_{01}+1)}{\Gamma(\beta_{20})\,\Gamma(\beta_{30})\,\Gamma(\beta_{40})\,\Gamma(\beta_{50})} \times {}_6F_5\left[\begin{matrix}\delta_{01},\ 1+\frac{1}{2}\delta_{01},\ \alpha_{345},\ \alpha_{245},\ \alpha_{235},\ \alpha_{234}\\ \frac{1}{2}\delta_{01},\ \beta_{20},\ \beta_{30},\ \beta_{40},\ \beta_{50}\end{matrix};\ -1\right],$$

where

$$1+\delta_{01} = \alpha_{345}+\beta_{20} = \alpha_{245}+\beta_{30} = \alpha_{235}+\beta_{40} = \alpha_{234}+\beta_{50}.$$

The relation is

$$(9.11)\qquad Fp(0;\ 1) = Fp(0;\ 4,\ 5),$$

where the latter expression is defined in accordance with (3 . 2) above.

When the ${}_3F_2$'s represented by $Fp(0)$ are reducible the ${}_6F_5$'s must also be reducible. Accordingly, by evaluating $Fp(0;\ 1)$, $Fp(0;\ 3)$, and $Fp(0;\ 4)$, we obtain the following identities :—

If $\kappa = \frac{1}{2}(b+3c-2a-1)$, then

$$(9.2)\quad {}_6F_5\left[\begin{matrix}\kappa,\ 1+\frac{1}{2}\kappa,\ \frac{1}{2}(1+b+c-a),\ \frac{1}{2}(b+c-a),\ \frac{1}{2}(1-b+c),\ c-a\\ \frac{1}{2}\kappa,\ c-\frac{1}{2}a,\ \frac{1}{2}+c-\frac{1}{2}a,\ b+c-a,\ \frac{1}{2}(1+b+c)\end{matrix};\ -1\right] = 2^{a+b-c}\,\frac{\Gamma(2c-a)\,\Gamma\{\frac{1}{2}(b+c-a)\}}{\Gamma(c)\,\Gamma\{\frac{1}{2}(1+b+3c-2a)\}}\,P(0).$$

If $l = b+c-\frac{1}{2}a-1$, then

$$(9.3)\quad {}_6F_5\left[\begin{matrix}l,\ 1+\frac{1}{2}l,\ \frac{1}{2}(b+c-a),\ \frac{1}{2}(b+c-a-1),\ b,\ \frac{1}{2}a\\ \frac{1}{2}l,\ \frac{1}{2}(b+c),\ \frac{1}{2}(b+c+1),\ c-\frac{1}{2}a,\ b+c-a\end{matrix};\ -1\right] = \frac{\Gamma(b+c)\,\Gamma(c-\frac{1}{2}a)}{\Gamma(b+c-\frac{1}{2}a)\,\Gamma(c)}\,P(0),$$

$$(9.4)\quad {}_6F_5\left[\begin{matrix}c-\frac{1}{2},\ \frac{1}{2}c+\frac{3}{4},\ \frac{1}{2}(c-b),\ \frac{1}{2}(1+c-b),\ \frac{1}{2}a,\ \frac{1}{2}(1+a)\\ \frac{1}{2}c-\frac{1}{4},\ \frac{1}{2}(1+b+c),\ \frac{1}{2}(b+c),\ \frac{1}{2}+c-\frac{1}{2}a,\ c-\frac{1}{2}a\end{matrix};\ -1\right] = 2^a\,\frac{\Gamma(b+c)\,\Gamma(2c-a)}{\Gamma(b+c-a)\,\Gamma(2c)}\,P(0).$$

This last result may be written with advantage in the form

$$(9.41)\quad \sum_{r=0}^{\infty}(-1)^r(2c-1+4r)\,\frac{(c-\frac{1}{2})_r(c-b)_{2r}(a)_{2r}}{r!\,(c+b)_{2r}(2c-a)_{2r}} = 2^a\,\frac{\Gamma(b+c)\,\Gamma(2c-a)}{\Gamma(b+c-a)\,\Gamma(2c-1)}\,P(0).$$

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* *Loc. cit.*, § 15.

Corresponding with the associated ${}_3F_2$'s of the second and third classes we have associated well-poised ${}_6F_5$'s. These are of no great interest individually; but, as in § 8, we can combine the associated series in pairs. Substitution for $Fp(3;\ 4)$ and $Fp(2;\ 4)$ in (8.1) gives the identity

$$(9.5)\quad \sum_{r=0}^{\infty} (\tfrac{1}{2}+a-c+r)\,\frac{(1+a-b-c)_r}{r!} \times \frac{\Gamma(1+2a-2c+r)\,\Gamma(c-\tfrac{1}{2}a-\tfrac{1}{2}-\tfrac{1}{2}r)\,\Gamma(\tfrac{1}{2}a+\tfrac{1}{2}r)}{\Gamma(1+a+b-c+r)} = \frac{2^{a-2c+2}\,\pi^{\frac{3}{2}}\,\Gamma(a)}{\Gamma(1+b-c)\,\Gamma(c)\sin\pi(a-2c)}\,P(0).$$

From the analogue of (8.2), viz.

$$\frac{Fn(2)}{\Gamma(\frac{1}{2}+\frac{1}{2}a)}+\frac{1+a-b-c}{2}\,\frac{Fn(3)}{\Gamma(\frac{1}{2}a)} = \frac{2^a P(0)}{\pi\,\Gamma(c)\,\Gamma(1-b)},$$

we can derive two identities, one by using $Fn(2;\ 0)$, $Fn(3;\ 0)$, the other by using $Fn(2;\ 4)$ and $Fn(3;\ 4)$. These are

$$(9.6)\quad \sum_{r=0}^{\infty} (\tfrac{1}{2}-b+r)\,\frac{(1+a-b-c)_r}{r!} \times \frac{\Gamma(1-2b+r)}{\Gamma(\frac{1}{2}+\frac{1}{2}a-\frac{1}{2}r)\,\Gamma(1-b+\frac{1}{2}a+\frac{1}{2}r)\,\Gamma(1+c-a-b+r)} = \frac{2^{2a-2-b}}{\Gamma(\frac{1}{2})\,\Gamma(c)}\,P(0)$$

and

$$(9.7)\quad \sum_{r=0}^{\infty} (-1)^r(c-a-\tfrac{1}{2}+r)\,\frac{(b+c-a-1)_r}{r!} \times \frac{\Gamma(2c-2a-1+r)}{\Gamma(\frac{1}{2}+\frac{1}{2}a-\frac{1}{2}r)\,\Gamma(c-\frac{1}{2}a+\frac{1}{2}r)\,\Gamma(1+c-a-b+r)} = \frac{2^{b+c-2}\,\Gamma(c-a)}{\Gamma(\frac{1}{2})\,\Gamma(1-b)\,\Gamma(c)}\,P(0).$$

A transformation discovered by C. Fox.

10. All the transformations which have been under discussion depend on the primary formula enunciated in § 3. Attention must now be called to an independent formula, derived from an equation which is due to Fox.

This equation* is

$$(10.1)\quad {}_2F_1\left[\begin{matrix}2a,\ 2\beta\ ;\ \frac{1}{2}\\ a+\beta+\frac{1}{2}-m\end{matrix}\right]=\frac{\Gamma(a+\beta+\frac{1}{2})\,\Gamma(a+\beta+\frac{1}{2}-m)}{\Gamma(\frac{1}{2})\,\Gamma(2a)\,\Gamma(2\beta)}\,2^{2a+2\beta-2}$$
$$\times\sum_{r=0}(-2)^r\,\frac{(-m)_r}{r!}\,\frac{\Gamma(a+\frac{1}{2}r)\,\Gamma(\beta+\frac{1}{2}r)}{\Gamma(r+a+\beta+\frac{1}{2}-m)}.$$

In Fox's statement m is a positive integer, but that proviso is not essential in his proof.

Making the substitutions $2a=a$, $2\beta=c-b$, $a+\beta+\frac{1}{2}-m=c$, and using (2 . 2), we see that

$$(10.2)\quad P(0)\equiv {}_2F_1\left[\begin{matrix}a,\ b\ ;\ -1\\ c\end{matrix}\right]$$
$$=2^{c-b-2}\,\frac{\Gamma\{\frac{1}{2}(a+c-b-1)\}\,\Gamma(c)}{\Gamma(\frac{1}{2})\,\Gamma(a)\,\Gamma(c-b)}$$
$$\times\sum_{r=0}^{\infty}(-2)^r\,\frac{(-m)_r}{r!}\,\frac{\Gamma(\frac{1}{2}a+\frac{1}{2}r)\,\Gamma\{\frac{1}{2}(c-b+r)\}}{\Gamma(c+r)},$$

where $m\equiv\frac{1}{2}(a-b-c+1)$.

This equation is equivalent to

$$(10.3)\quad \frac{2^a P(0)}{\Gamma\{\frac{1}{2}(a+c-b+1)\}\,\Gamma(\frac{1}{2})}$$
$$=\frac{1}{\Gamma(\frac{1}{2}a+\frac{1}{2})\,\Gamma(\frac{1}{2}c-\frac{1}{2}b+\frac{1}{2})}\,{}_4F_3\left[\begin{matrix}-\frac{1}{2}m,\ \frac{1}{2}(1-m),\ \frac{1}{2}a,\ \frac{1}{2}(c-b)\\ \frac{1}{2},\ \frac{1}{2}c,\ \frac{1}{2}(c+1)\end{matrix}\right]$$
$$+\frac{2m}{c}\,\frac{1}{\Gamma(\frac{1}{2}a)\,\Gamma(\frac{1}{2}c-\frac{1}{2}b)}\,{}_4F_3\left[\begin{matrix}\frac{1}{2}(1-m),\ \frac{1}{2}(2-m),\ \frac{1}{2}(a+1),\ \frac{1}{2}(c-b+1)\\ \frac{3}{2},\ \frac{1}{2}(c+1),\ \frac{1}{2}(c+2)\end{matrix}\right].$$

In general, no further transformation is possible. In Fox's case, m being a positive integer, the ${}_4F_3$'s terminate and, as the condition that the sum of the parameters in the lower line exceeds by unity the sum of those in the upper line is satisfied, these are Saalschützian series. It follows that numerous new forms could be derived from them. In view of the limitation to terminating series, these are of minor interest and they need not be written out.

Added July, 1929.

Some terminating series.

11. In consequence of the relation

$$r_1+r_3=r_2+r_5,$$

* Fox, **1**, Equation (2.6).

which is implied by (3.4), several of the ${}_3F_2$'s which occur in this paper are "nearly poised". When a "nearly poised" ${}_3F_2(1)$ is a terminating series it can be transformed into a Saalschützian ${}_4F_3$. For this purpose the following formulae* are available, n and m being integers.

$$(11.1)\quad {}_3F_2\left[\begin{matrix}a,\ b,\ -n\\ \kappa-b,\ \kappa+n\end{matrix}\right]=\frac{(\kappa)_n(\frac{1}{2}\kappa-b)_n}{(\frac{1}{2}\kappa)_n(\kappa-b)_n}\,{}_4F_3\left[\begin{matrix}b,\ -n,\ \frac{1}{2}\kappa-\frac{1}{2}a,\ \frac{1}{2}+\frac{1}{2}\kappa-\frac{1}{2}a\\ \kappa-a,\ \frac{1}{2}+\frac{1}{2}\kappa,\ b-\frac{1}{2}\kappa-n+1\end{matrix}\right],$$

$$(11.2)\quad {}_3F_2\left[\begin{matrix}x,\ y,\ -n\\ 1+x-y,\ w\end{matrix}\right]$$

$$=\frac{(w-x)_n}{(w)_n}\,{}_4F_3\left[\begin{matrix}1+x-w,\ \frac{1}{2}x,\ \frac{1}{2}+\frac{1}{2}x-y,\ -n\\ 1+x-y,\ 1+\frac{1}{2}(x-w-n),\ \frac{1}{2}(1+x-w-n)\end{matrix}\right],$$

$$(11.21)\quad =\frac{(w-n)_n}{(w-x-n)_n}\,{}_4F_3\left[\begin{matrix}w-y,\ \frac{1}{2}x,\ \frac{1}{2}(1+x),\ -n\\ 1+x-y,\ \frac{1}{2}(w-n),\ \frac{1}{2}(1+w-n)\end{matrix}\right],$$

$$(11.3)\quad {}_3F_2\left[\begin{matrix}-m,\ y,\ u\\ 1-m-y,\ w\end{matrix}\right]$$

$$=\frac{(w+u)_m}{(w)_m}\,{}_4F_3\left[\begin{matrix}w-y,\ u,\ \frac{1}{2}(1-m),\ -\frac{1}{2}m\\ 1-m-y,\ \frac{1}{2}(w+u),\ \frac{1}{2}(1+w+u)\end{matrix}\right].$$

In each case further transformations into other Saalschützian series or into "well-poised" ${}_7F_6$'s could be made.

For the formulae to be applicable the ${}_3F_2$'s must terminate. With that proviso (11.1) can be used to transform the series on the left of (3.6) and (3.62), and either (11.2) or (11.3) to transform those on the left of (6.2), (6.21), (6.41), (7.2), (7.21), and (7.4).

As an example we write $a=-2n$ in (3.6) and deduce with the aid of (11.1) that

$$(11.4)\quad {}_4F_3\left[\begin{matrix}\frac{1}{2}(c-b),\ \frac{1}{4}(c-b+1-2n),\ \frac{1}{4}(c-b+3-2n),\ -n\\ \frac{1}{2}(c-b+1-2n),\ \frac{1}{2}(1+c),\ 1-n-\frac{1}{2}b\end{matrix}\right]$$

$$=\frac{1}{2^{2n}}\frac{(\frac{1}{2}c)_n}{(\frac{1}{2}b)_n}\,{}_2F_1\left[\begin{matrix}b,\ -2n\\ c\end{matrix};\ -1\right].$$

Writing down one of the well-poised series derived† from this ${}_4F_3$, we

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* 11.1 and 11.2 are from Whipple, **1**, Equations 5.2 and 6.52. 11.21 is derived from 11.2 by the Saalschützian transformation (Whipple, **4**, Equation 10.11). 11.3 follows from 11.21, since an equation of the m-th degree in u satisfied by all negative integral values of u must be an identity.

† Whipple, **4**, Equation (2.1).

have the equation

$$(11.41)\quad {}_7F_6\left[\begin{matrix}\frac{1}{2}-n,\ -n,\ 1\frac{1}{4}-\frac{1}{2}n,\ \frac{1}{2}(1+b),\ 1-n-\frac{1}{2}c, \\ \qquad\qquad\qquad\qquad\qquad \frac{1}{4}(c-b+1-2n),\ \frac{1}{4}(c-b+3-2n) \\ \frac{3}{2},\ \frac{1}{4}-\frac{1}{2}n,\ 1-n-\frac{1}{2}b,\ \frac{1}{2}(1+c), \\ \qquad\qquad\qquad\qquad\qquad \frac{1}{4}(b-c+5-2n),\ \frac{1}{4}(b-c+3-2n)\end{matrix}\right]$$

$$= \frac{1-2n+b-c}{1+2n+b-c}\ \frac{(-\frac{1}{2})_n(\frac{1}{2}c)_n}{\{\frac{1}{2}(1-b+c)\}_n(\frac{1}{2}b)_n}\ {}_2F_1\left[\begin{matrix}b,\ -2n\ ;\ -1 \\ c\end{matrix}\right].$$

Fox's formula (10.3) can now be brought into relation with the general theory. We have

$$(11.5)\quad Fn(1\ ;\ 0,\ 5) = \frac{1}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}c+\frac{1}{2}b)\,\Gamma(a-c+1)} \times {}_3F_2\left[\begin{matrix}-m,\ -m+\frac{1}{2},\ \frac{1}{2}(c-b) \\ \frac{1}{2},\ \frac{1}{2}(c+b)\end{matrix}\right]$$

and

$$(11.51)\quad Fn(5\ ;\ 0,\ 1) = \frac{1}{\Gamma(\frac{3}{2})\,\Gamma\{\frac{1}{2}(1+c+b)\}\,\Gamma(a-c+1)} \times {}_3F_2\left[\begin{matrix}1-m,\ \frac{1}{2}-m,\ \frac{1}{2}(1+c-b) \\ \frac{3}{2},\ \frac{1}{2}(1+c+b)\end{matrix}\right],$$

where $m \equiv \frac{1}{2}(a-b-c+1)$.

These series transform, by (11.3), into those on the right of (10.3), and that equation is found to represent the identity* which connects $Fp(0)$, $Fn(1)$, and $Fn(5)$. The demonstration requires that m shall be a positive integer. The theorem of Carlson's which is often of service in generalizing such formulae is not applicable in this case. It is only Fox's proof which tells us that (10.1) holds and therefore (10.3) also when the series do not terminate.

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C. Fox, **1.** "The expression of hypergeometric series in terms of similar series", *Proc. London Math. Soc.* (2), 26 (1927), 201–210.

* Cf. Whipple, **2**, Equation (10).

ADDITIONAL NOTE ON CERTAIN INTEGRAL EQUATIONS

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[Received 5 August, 1928.—Read 14 June, 1928.]

1. In a paper published in the *Proceedings* in 1924* we discussed the solution of the equations

$$\text{(B)} \qquad f(x) = \frac{1}{\pi}\int_0^\infty \frac{\sin t}{t}\,\{f(x+t)+f(x-t)\}\,dt,$$

$$\text{(K)} \qquad f'(x) = \tfrac{1}{2}\int_0^\infty \frac{J_1(t)}{t}\,\{f(x+t)-f(x-t)\}\,dt,$$

$$\text{(M)} \qquad f'(x) = \tfrac{1}{2}\int_0^\infty \frac{e^{-t}}{t}\,\{f(x+t)-f(x-t)\}\,dt,$$

of Bateman, Kapteyn, and Milne, using two different methods, a "real" method applicable only when $f(x)$ belongs to the class L^2†, and a "complex" method of wider scope. There was a mistake in our treatment of (M) by the first method, which we corrected in a later note‡.

We return to the subject here for two reasons. In the first place, we can simplify our "real" method by adopting a suggestion of Dr. Norbert Wiener, and we take the opportunity of simplifying the proof and incorporating our own correction simultaneously. This part of the paper (§ 2) is then complete in itself. Secondly, there is a passage later in our first paper, in the proof of Theorem 7, to which we have not referred before, where the same difficulty which necessitated our correction recurs again in a different form. We deal with this point in two different ways.

* G. H. Hardy and E. C. Titchmarsh, "Solutions of some integral equations considered by Bateman, Kapteyn, Littlewood, and Milne", *Proc. London Math. Soc.* (2), 23 (1924), 1–26. We refer to this paper as BKLM.

† f is measurable and f^2 integrable over $(-\infty, \infty)$.

‡ G. H. Hardy and E. C. Titchmarsh, "Correction of an error in a paper on certain integral equations", *Proc. London Math. Soc.* (2), 24 (1925), xxxi–xxxiii (*Records* for February, 1925).

In § 3 we show how to complete our original proof. Secondly, in § 4, we give a new, and in some respects simpler, method of dealing with this part of the proof of Theorem 7. This section is alternative to §§ 3.21 and 3.26 of BKLM and § 3 of the present paper. The "complex" argument of BKLM remains unchanged.

The case where $f(x)$ belongs to L^2.

2.1. THEOREM 1. *The necessary and sufficient condition that a function $f(x)$ of L^2 should satisfy* (B) *or* (K) *is that it should be of the form*

$$f(x) = \int_0^1 \{\lambda(u) \cos xu + \mu(u) \sin xu\}\, du,$$

where $\lambda(u)$ and $\mu(u)$ belong to L^2.

THEOREM 2. *The only solution of* (M) *belonging to L^2 is $f(x) = 0$.*

If $f(x)$ is a solution, so is $\frac{1}{2}\{f(x) \pm f(-x)\}$, so that any solution is the sum of an even and an odd solution. The argument is substantially the same in these two cases, and we may suppose $f(x)$ to be *even*.

2.2. *Solution of* (B). It follows from the theory of Fourier transforms* that there is a function $\lambda(u)$ of L^2 such that

$$\int_0^a \lambda(u) \cos xu\, du \to f(x) \tag{2.21}$$

in mean square when $a \to \infty$ †. Since $t^{-1} \sin t$ belongs to L^2, we have

$$\begin{aligned} f(x) &= \frac{1}{\pi} \int_0^\infty \frac{\sin t}{t} \{f(x+t) + f(x-t)\}\, dt \\ &= \lim_{a \to \infty} \frac{2}{\pi} \int_0^\infty \frac{\sin t}{t}\, dt \int_0^a \lambda(u) \cos xu \cos tu\, du \\ &= \lim_{a \to \infty} \frac{2}{\pi} \int_0^a \lambda(u) \cos xu\, du \int_0^\infty \frac{\sin t \cos ut}{t}\, dt, \end{aligned}$$

the inversion of the order of integration being justified by "bounded

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* See, for example, E. W. Hobson, *Theory of functions of a real variable*, 2 (ed. 2, 1926), 742–752.

† In BKLM we used the representation

$$f(x) = \frac{d}{dx} \int_0^\infty \lambda(u) \frac{\sin xu}{u}\, du.$$

Wiener's suggestion was that we should use (2.21) instead.

convergence''*. The inner integral is $\frac{1}{2}\pi$ if $u<1$ and 0 if $u>1$, so that

$$f(x) = \int_0^1 \lambda(u) \cos xu \, du.$$

2 . 3. *Solution of* (K). Using (2 . 21) again, we obtain

$$\begin{aligned} f'(x) &= \tfrac{1}{2} \int_0^\infty \frac{J_1(t)}{t} \{f(x+t)-f(x-t)\} \, dt \\ &= -\lim_{a\to\infty} \int_0^\infty \frac{J_1(t)}{t} \, dt \int_0^a \lambda(u) \sin xu \sin tu \, du \\ &= -\lim_{a\to\infty} \int_0^a \lambda(u) \sin xu \, du \int_0^\infty \frac{J_1(t)}{t} \sin ut \, dt, \end{aligned}$$

the inversion here being justified by absolute convergence. This gives

$$f'(x) = -\int_0^1 u\lambda(u) \sin xu \, du - \int_1^\infty \frac{\lambda(u) \sin xu}{u+\sqrt{(u^2-1)}} \, du\dagger.$$

Now $f(t)$ and $t^{-1}J_1(t)$ belong to L^2, and so the integrals

$$\int_0^\infty \frac{J_1(t)}{t} f(x \pm t) \, dt$$

are continuous‡. Hence f' is continuous, and its integral is f. Integrating, we obtain

$$f(x) = \int_0^1 \lambda(u) \cos xu \, du + \int_1^\infty \frac{\lambda(u) \cos xu}{u\{u+\sqrt{(u^2-1)}\}} \, du$$

(apart from a constant which must obviously be 0). Since the Fourier transform of $f(x)$ is $\lambda(u)$, $\lambda(u)$ must be null for $u>1$, and we obtain the result.

2 . 4. *Solution of* (M). Since $f(x)$ belongs to L^2,

$$g(x) = \frac{1}{\pi} \int_0^\infty \{f(x+t)-f(x-t)\} \, \frac{dt}{t}$$

exists for almost all x and belong to L^2§.

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* See BKLM, 10.

† See G. N. Watson, *Theory of Bessel functions*, 405.

‡ See E. W. Hobson, *loc. cit.*, 1 (ed. 3, 1927), 639.

§ f and $-g$ are ''conjugate''. For the theory of conjugate functions, see M. Riesz: (1) ''Les fonctions conjuguées et les séries de Fourier'', *Comptes rendus*, 178, 1464 (28 April 1924); (2) ''Sur les fonctions conjuguées, *Math. Zeitschrift*, 27 (1927), 218–244; (3) ''Sur les maxima des formes bilinéaires et sur les fonctionelles linéaires'', *Acta Math.*, 49 (1927), 465–497: E. C. Titchmarsh, ''Reciprocal formulae involving series and integrals'', *Math. Zeitschrift*, 25 (1926), 321–347: G. H. Hardy, ''On Hilbert transforms'', *Messenger of Math.* 54 (1925), 20–27 and 81–18: and other memoirs there referred to.

Hence

$$f'(x)-\tfrac{1}{2}\pi g(x) = \tfrac{1}{2}\int_0^\infty \frac{e^{-t}-1}{t}\{f(x+t)-f(x-t)\}\,dt$$

$$= \lim_{a\to\infty}\int_0^\infty \frac{1-e^{-t}}{t}\,dt\int_0^a \lambda(u)\sin xu\sin tu\,du$$

$$= \lim_{a\to\infty}\int_0^a \lambda(u)\sin xu\,du\int_0^\infty \frac{1-e^{-t}}{t}\sin ut\,dt$$

$$= \int_0^\infty \lambda(u)(\tfrac{1}{2}\pi-\arctan u)\sin xu\,du,$$

the inversion being justified by bounded convergence. But

$$\int_0^a \lambda(u)\sin xu\,du \to -g(x)$$

in mean, and so

$$-\int_0^a \lambda(u)\arctan u\sin xu\,du \to f'(x)$$

in mean. Now $f'-\frac{1}{2}\pi g$ is continuous (as in § 2 . 3), and g belongs to L^2, so that f' belongs to L^2. Also f' is finite everywhere, by hypothesis. It follows* that f is the integral of f'. Integrating, we obtain

$$\int_0^\infty \lambda(u)\arctan u\,\frac{1-\cos xu}{u}\,du = f(0)-f(x).$$

From the last equation, and the fact that $f(x)$ is the transform of $\lambda(u)$, it follows that

$$\int_0^\infty \lambda(u)\left(1-\frac{\arctan u}{u}\right)\cos xu\,du = f(0)-\int_0^\infty \lambda(u)\,\frac{\arctan u}{u}\,du,$$

where the integral on the left-hand side is a mean square integral and that on the right an ordinary integral. Since the left-hand side belongs to L^2, and the right-hand side is independent of x, both must vanish and

$$\lambda(u)\left(1-\frac{\arctan u}{u}\right)$$

must be a null function. It follows that $\lambda(u)$ is null and $f(x)$ identically zero.

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* See E. W. Hobson, *loc. cit.*, 1 (ed. 3, 1927), 600.

2 . 5. We may add that the solutions which precede may be extended to the case in which $f(x)$ belongs to any Lebesgue class L^p, where $1 < p \leqslant 2$. The arguments are the same, but the theorems concerning transforms and conjugate functions on which they rest are naturally more difficult*. In this case the $\lambda(u)$ and $\mu(u)$ of the conclusion belong to $L^{p'}$, where $p' = p/(p-1)$.

The general case of Milne's equation.

3 . 1. We have to prove

THEOREM 7†. *If $f(x)$ satisfies* (M), *and*

$$f(x) = O(e^{A|x|}), \tag{3 . 11}$$

where $0 < A < 1$, then $f(x)$ is a quadratic.

We proved this first on the assumption that $f(x)$ is indefinitely differentiable, and so far our proof requires no change‡. We then removed this restriction by an artifice, which demands a justification of the inversion of the order of integration in the repeated integral

$$\int_0^\infty \frac{e^{-t}}{t}\,dt \int_{-\infty}^{\infty} \{f(x+u+t)-f(x+u-t)\} \operatorname{sech} u\,du.$$

This inversion is not at all trivial; the point is similar to that which arose before, but there are additional complications.

3 . 2. There is no real loss of generality in taking x here to be 0, in which case what we have to prove is that

$$\int_0^\infty \frac{e^{-t}}{t}\,dt \int_{-\infty}^{\infty} \{f(u+t)-f(u-t)\} \operatorname{sech} u\,du = \int_{-\infty}^{\infty} \operatorname{sech} u\,du \int_0^\infty \frac{e^{-t}}{t} \{f(u+t)-f(u-t)\}\,dt \tag{3 . 21}$$

or

$$\int_{-\infty}^{\infty} \frac{e^{-|t|}}{t}\,dt \int_{-\infty}^{\infty} \frac{f(u+t)}{\cosh u}\,du = \int_{-\infty}^{\infty} \frac{du}{\cosh u} \int_{-\infty}^{\infty} \frac{e^{-|t|}}{t} f(u+t)\,dt, \tag{3 . 22}$$

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* See E. C. Titchmarsh, "A contribution to the theory of Fourier transforms", *Proc. London Math. Soc.* (2), 23 (1924), 279–289, and the works by Riesz, Titchmarsh, and Hobson already referred to.

† We follow the numeration of BKLM.

‡ We correct two misprints: in the enunciation of Lemma δ, α should be β, and the last line on p. 19 should be

$$|x-x_n| \leqslant 2^{-2}+2^{-3}+\ldots+2^{-n-1} < \tfrac{1}{2}.$$

where the integrals in t are principal values both at the origin and at infinity, *i.e.* are defined as

$$\lim_{\epsilon\to 0,\ T\to\infty}\left(\int_{-T}^{-\epsilon}+\int_{\epsilon}^{T}\right).$$

We write $$f(x)=e^{a|x|}g(x),$$

where $0<a<1$ and $g(x)$ belongs to L^2; this is possible after (3.11). If we do this we obtain

$$\text{(3.23)}\quad \int_{-\infty}^{\infty}\frac{dt}{t}\int_{-\infty}^{\infty}e^{-|t|+a|u+t|-a|u|}\,\frac{e^{a|u|}}{\cosh u}\,g(u+t)\,du = \int_{-\infty}^{\infty}\frac{e^{a|u|}}{\cosh u}\,du\int_{-\infty}^{\infty}e^{-|t|+a|u+t|-a|u|}\,g(u+t)\,\frac{dt}{t}.$$

Now it is easily verified that

$$0\leqslant w(t,u)=1-e^{-|t|+a|u+t|-a|u|}\leqslant 1-e^{-(1+a)|t|}.$$

It follows that the integral

$$\iint\frac{w(t,u)}{t}\,\frac{e^{a|u|}}{\cosh u}\,g(u+t)\,dt\,du$$

is absolutely convergent; for we may compare it with

$$\int_{-\infty}^{\infty}\frac{e^{a|u|}}{\cosh u}\,du\int_{-\infty}^{\infty}\frac{1-e^{-(1+a)|t|}}{t}\,|g(u+t)|\,dt,$$

and the integral in t converges uniformly in u. Hence

$$\text{(3.24)}\quad \int_{-\infty}^{\infty}\frac{dt}{t}\int_{-\infty}^{\infty}w(t,u)\,\frac{e^{a|u|}}{\cosh u}\,g(u+t)\,du = \int_{-\infty}^{\infty}\frac{e^{a|u|}}{\cosh u}\,du\int_{-\infty}^{\infty}w(t,u)\,g(u+t)\,\frac{dt}{t}.$$

Comparing (3.23) and (3.24), we see that (3.23) will follow from

$$\text{(3.25)}\quad \int_{-\infty}^{\infty}\frac{dt}{t}\int_{-\infty}^{\infty}\frac{e^{a|u|}}{\cosh u}\,g(u+t)\,du = \int_{-\infty}^{\infty}\frac{e^{a|u|}}{\cosh u}\,du\int_{-\infty}^{\infty}g(u+t)\,\frac{dt}{t}.$$

3.3. The proof of (3.25) depends upon the theory of conjugate functions. We state the result required in a more general form than we really need; but it has a certain interest of its own.

LEMMA *a*. *Suppose that $g(u)$ belongs to L^2 and that $-h(u)$ is its conjugate, and that $k(u)$ also belongs to L^2. Then the functions*

$$\text{(3.31)}\quad G(t)=\int_{-\infty}^{\infty}k(u)\,g(u+t)\,du,\qquad H(t)=\int_{-\infty}^{\infty}k(u)\,h(u+t)\,du,$$

are continuous, and

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{G(t)}{t-x}\,dt = H(x) \tag{3.32}$$

for every x (so that G and $-H$ are also conjugate).

The continuity of G and H is a known result, which has been used already in § 2. Since all values of x are on the same footing, we may take $x=0$. We have then to prove that

$$\left(\int_{-T}^{-\epsilon}+\int_{\epsilon}^{T}\right)\frac{G(t)}{t}\,dt \to \pi\int_{-\infty}^{\infty}k(u)\,h(u)\,du, \tag{3.33}$$

when $\epsilon\to 0$ and $T\to\infty$.

Substituting from (3.31) for $G(t)$, and inverting the order of integration (by absolute convergence), we see that the left-hand side of (3.33) is equal to

$$\int_{-\infty}^{\infty}k(u)\,du\left(\int_{-T}^{-\epsilon}+\int_{\epsilon}^{T}\right)\frac{g(u+t)}{t}\,dt = \pi\int_{-\infty}^{\infty}k(u)\{h_\epsilon(u)-h_T(u)\}\,du,$$

where

$$\pi h_\epsilon(u) = \left(\int_{-\infty}^{-\epsilon}+\int_{\epsilon}^{\infty}\right)\frac{g(u+t)}{t}\,dt. \tag{3.34}$$

The result of the lemma will follow if we can show that $h_\epsilon(u)$ tends to $h(u)$ in mean square, and $h_T(u)$ to zero, *i.e.* that

$$\int_{-\infty}^{\infty}\{h_\epsilon(x)-h(x)\}^2\,dx\to 0 \tag{3.35}$$

when $\epsilon\to 0$, and

$$\int_{-\infty}^{\infty}\{h_T(x)\}^2\,dx\to 0 \tag{3.36}$$

when $T\to\infty$.

3.4. To prove this, we use the formula*

$$h_\epsilon(x) = \frac{1}{\pi^2}\int_{-\infty}^{\infty}h(y+x)\log\left|\frac{y+\epsilon}{y-\epsilon}\right|\frac{dy}{y}. \tag{3.41}$$

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* This formula, which is very important in the theory of conjugate functions, is a corollary of "Parseval's theorem for conjugate functions", the theorem that, if g and $-h$, G and $-H$, are conjugate (and all belong to L^2), then

$$\int_{-\infty}^{\infty}g(y)\,G(y)\,dy = \int_{-\infty}^{\infty}h(y)\,H(y)\,dy. \tag{P}$$

If we take $G(y)$ to be $(y-x)^{-1}$ when $|y-x|>\epsilon$ and zero otherwise, we find by an elementary calculation that

$$H(y) = \frac{1}{\pi(y-x)}\log\left|\frac{y-x+\epsilon}{y-x-\epsilon}\right|.$$

These values of G and H, substituted in (P), give (3.37).

For (P) see, for example, Titchmarsh, *loc. cit.*, 322, formula (5*b*); Hardy, *loc. cit.*, 84, Theorem B. Riesz does not state the formula explicitly, though it is naturally implied in his work.

The value of the right-hand side of (3.41) when $h(x)=1$ is 1. Hence

$$h_\epsilon(x)-h(x)=\frac{1}{\pi^2}\int_{-\infty}^{\infty}\{h(x+y)-h(x)\}\log\left|\frac{y+\epsilon}{y-\epsilon}\right|\frac{dy}{y},$$

$$\int_{-\infty}^{\infty}\{h_\epsilon(x)-h(x)\}^2dx$$

$$\leqslant\int_{-\infty}^{\infty}dx\left[\frac{1}{\pi^2}\int_{-\infty}^{\infty}\{h(x+y)-h(x)\}^2\log\left|\frac{y+\epsilon}{y-\epsilon}\right|\frac{dy}{y}\cdot\frac{1}{\pi^2}\int_{-\infty}^{\infty}\log\left|\frac{y+\epsilon}{y-\epsilon}\right|\frac{dy}{y}\right]$$

$$=\frac{1}{\pi^2}\int_{-\infty}^{\infty}\log\left|\frac{y+\epsilon}{y-\epsilon}\right|\frac{dy}{y}\int_{-\infty}^{\infty}\{h(x+y)-h(x)\}^2dx.$$

The x integral is bounded in y and tends to zero with y, and (3.35) follows.

To prove (3.36), let $h(x)=\phi(x)+\psi(x)$, where $\phi(x)$ is equal to $h(x)$ in $(-X, X)$ and to zero outside, and $\psi(x)$ conversely; and let ϕ_T, ψ_T be derived from ϕ, ψ as h_T is from h. Then

$$\int_{-\infty}^{\infty}h_T^2dx\leqslant 2\int_{-\infty}^{\infty}\phi_T^2dx+2\int_{-\infty}^{\infty}\psi_T^2dx.$$

The argument used above shows that

$$\int_{-\infty}^{\infty}\psi_T^2dx\leqslant\frac{1}{\pi^2}\int_{-\infty}^{\infty}\log\left|\frac{y+T}{y-T}\right|\frac{dy}{y}\int_{-\infty}^{\infty}\{\psi(x+y)\}^2dx$$

$$=\int_{-\infty}^{\infty}\{\psi(x)\}^2dx=\left(\int_{-\infty}^{-X}+\int_{X}^{\infty}\right)\{h(x)\}^2dx<\eta,$$

for any $\eta>0$, if X is sufficiently large. Also

$$\int_{-\infty}^{\infty}\phi_I^2dx=\int_{-\infty}^{\infty}dx\left(\frac{1}{\pi^2}\int_{-\infty}^{\infty}\phi(u)\log\left|\frac{u-x+T}{u-x-T}\right|\frac{du}{u-x}\right)^2$$

$$\leqslant A\int_{-\infty}^{\infty}dx\int_{-X}^{X}\left(\frac{1}{u-x}\log\left|\frac{u-x+T}{u-x-T}\right|\right)^2du$$

$$=A\int_{-X}^{X}du\int_{-\infty}^{\infty}\left(\frac{1}{u-x}\log\left|\frac{u-x+T}{u-x-T}\right|\right)^2du$$

$$=\frac{A}{T}\int_{-X}^{X}du\int_{-\infty}^{\infty}\left(\frac{1}{w}\log\left|\frac{w+1}{w-1}\right|\right)^2dw<\frac{AX}{T},$$

where the A's are independent of T and X. We can make this also less than η, when X has been fixed, by choice of T. This completes the proof of (3.36) and so of the lemma.

The lemma may be extended to show that if g (and so h) belong to L^{1+p}, and k to L^{1+q}, where $p>0$, $q>0$, $pq\leqslant 1$, so that (in accordance

with a theorem of Young*) G and H belong to L^{ϖ}, where

$$\varpi = \frac{(1+p)(1+q)}{1-pq}$$

if $pq < 1$, and are continuous if $pq = 1$, then G and H are conjugate, the formula expressing either in terms of the other holding almost always when $pq < 1$ and always when $pq = 1$.

3 . 4. To deduce (3 . 25) from the lemma, we take the $g(u)$ of the lemma to be the $g(u)$ of (3 . 25), and $k(u) = e^{a|u|} \operatorname{sech} u$. Then the left-hand side of (3 . 25) is

$$\int_{-\infty}^{\infty} \frac{dt}{t} \int_{-\infty}^{\infty} k(u)\, g(u+t)\, du = \int_{-\infty}^{\infty} \frac{G(t)}{t}\, dt,$$

and the right-hand side is

$$\pi \int_{-\infty}^{\infty} k(u)\, h(u)\, du = \pi H(0),$$

so that (3 . 25) becomes (3 . 32) with $x = 0$. This completes our original proof of Theorem 7.

4. *New proof of Theorem* 7. This also depends on the theory of conjugate functions. We use the following result, which does not seem to have been stated anywhere exactly in the form we require.

4 . 1. LEMMA β. *If $f(x)$ belongs to L^2, and*

$$g(x) = \int_{x-1}^{x+1} \frac{f(t)}{t-x}\, dt,$$

then there is an absolute constant K such that

$$\int_{\xi-1}^{\xi+1} \{g(x)\}^2\, dx \leqslant K^2 \int_{\xi-2}^{\xi+2} \{f(t)\}^2\, dt. \tag{4 . 11}$$

Let

$$g_1(x) = \int_{\xi-2}^{\xi+2} \frac{f(t)}{t-x}\, dt.$$

Then

$$\int_{\xi-1}^{\xi+1} \{g_1(x)\}^2\, dx \leqslant \int_{-\infty}^{\infty} \{g_1(x)\}^2\, dx = \pi^2 \int_{\xi-2}^{\xi+2} \{f(t)\}^2\, dt. \tag{4 . 12}$$

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* W. H. Young, "On a class of parametric integrals", *Proc. Royal Soc.* (A), 85 (1911), 401–414, and "Sur la généralisation du théorème de Parseval", *Comptes rendus*, 155, 30 (1 July 1912).

Also for $\xi-1<x<\xi+1$

$$|g_1(x)-g(x)| \leqslant \left|\int_{x+1}^{\xi+2} \frac{f(t)}{t-x}\,dt\right| + \left|\int_{\xi-2}^{x-1} \frac{f(t)}{t-x}\,dt\right|$$

$$\leqslant \left\{\int_{x+1}^{\xi+2} \frac{dt}{(t-x)^2} \int_{x+1}^{\xi+2} [f(t)]^2\,dt\right\}^{\frac{1}{2}} + \left\{\int_{\xi-2}^{x-1} \frac{dt}{(t-x)^2} \int_{\xi-2}^{x-1} [f(t)]^2\,dt\right\}^{\frac{1}{2}}$$

$$\leqslant 2\left\{\int_{\xi-2}^{\xi+2} [f(t)]^2\,dt\right\}^{\frac{1}{2}}.$$

Hence

$$\int_{\xi-1}^{\xi+1} [g_1(x)-g(x)]^2\,dx \leqslant 4\int_{\xi-2}^{\xi+2} [f(t)]^2\,dt, \tag{4.13}$$

and the result follows from (4.12) and (4.13).

4.2. We now begin the proof of Theorem 7 again at the beginning. The equation (M) gives

$$f'(x) = \tfrac{1}{2}\int_0^1 \frac{f(x+t)-f(x-t)}{t}\,dt$$

$$+\tfrac{1}{2}\left[\int_0^1 \frac{e^{-t}-1}{t}\{f(x+t)-f(x-t)\}\,dt + \int_1^\infty \frac{e^{-t}}{t}\{f(x+t)-f(x-t)\}\,dt\right]$$

$$= \phi(x)+\psi(x),$$

say. Now $\psi(x)$ is bounded in a finite interval; more precisely

$$|\psi(x)| \leqslant \tfrac{1}{2}\int_0^1 \{|f(x+t)|+|f(x-t)|\}\,dt \tag{4.21}$$

$$+\tfrac{1}{2}\int_1^\infty e^{-t}\{|f(x+t)|+|f(x-t)|\}\,dt$$

$$< Ce^{A(|x|+1)} + \frac{Ce^{A|x|}}{1-A} < \frac{2Ce^{A(|x|+1)}}{1-A}.$$

Hence also

$$\int_{x-1}^{x+1} \{\psi(u)\}^2\,du < \frac{8C^2e^{2A(|x|+2)}}{(1-A)^2}. \tag{4.22}$$

Also $f(x)$ belongs to L^2 over any finite interval, and so, by Lemma β,

$$\int_{x-1}^{x+1} \{\phi(u)\}^2\,du \leqslant \tfrac{1}{4}K^2\int_{x-2}^{x+2} \{f(t)\}^2\,dt < K^2C^2e^{2A(|x|+2)}. \tag{4.23}$$

We conclude that $f'(x)$ belongs to L^2 over any finite interval, and therefore that $f(x)$ satisfies a Lipschitz condition of order $\frac{1}{2}$; more precisely,

for $|t|<1$,

$$|f(x+t)-f(x-t)| = \left|\int_{x-t}^{x+t} f'(u)\,du\right| \leqslant \left[\int_{x-t}^{x+t} du \int_{x-1}^{x+1} \{f'(u)\}^2\,du\right]^{\frac{1}{2}}$$

$$\leqslant \left[4t\int_{x-1}^{x+1} \{\phi(u)\}^2\,du\right]^{\frac{1}{2}} + \left[4t\int_{x-1}^{x+1} \{\psi(u)\}^2\,du\right]^{\frac{1}{2}}$$

$$< A_1 Ce^{A|x|}\sqrt{t}, \tag{4.24}$$

by (4.22) and (4.23), the constant A_1 depending on A only.

It now follows that $f'(x)$ is bounded in a finite interval; more precisely, by (4.24),

$$|\phi(x)| < \tfrac{1}{2}\int_0^1 A_1 Ce^{A|x|}\,t^{-\frac{1}{2}}\,dt = A_1 Ce^{A|x|},$$

and this and (4.21) give

$$|f'(x)| \leqslant |\phi(x)| + |\psi(x)| < A_2 Ce^{A|x|}, \tag{4.25}$$

say. Thus $f'(x)$ satisfies an inequality similar to that satisfied by $f(x)$.

4.3. We prove secondly that $f'(x)$ also is a solution of (M), *i.e.*

$$f''(x) = \tfrac{1}{2}\int_0^\infty \frac{e^{-t}}{t}\,\{f'(x+t)-f'(x-t)\}\,dt. \tag{4.31}$$

This can be obtained by differentiating (M), provided that the integral on the right of (4.31) is uniformly convergent. The uniform convergence at infinity follows from (4.25). Also

$$f'(x+t)-f'(x-t)$$

$$= \tfrac{1}{2}\int_0^\infty \frac{e^{-u}}{u}\,\{f(x+t+u)-f(x+t-u)-f(x-t+u)+f(x-t-u)\}\,du$$

$$= \tfrac{1}{2}\int_0^\infty \frac{e^{-u}}{u}\,F(x,\,t,\,u)\,du, \tag{4.32}$$

say. Now

$$|f(x+t+u)-f(x+t-u)| = \left|\int_{x+t-u}^{x+t+u} f'(v)\,dv\right| \leqslant 2u\,A_2 Ce^{A(|x|+|t|+|u|)},$$

and a similar result holds for the other difference. Hence

$$|F(x,\,t,\,u)| \leqslant 4u\,A_2 Ce^{A(|x|+|t|+|u|)}.$$

Since $F(x,\,t,\,u)$ is symmetrical in t and u, we may replace the factor u on the right by t. Multiplying these two inequalities and taking the square root

$$|F(x,\,t,\,u)| \leqslant 4\sqrt{(ut)}\,A_2 Ce^{A(|x|+|t|+|u|)}.$$

Inserting this in (4 . 32), we see that $f'(x)$ satisfies a Lipschitz condition of order $\frac{1}{2}$, and so that (4 . 31) is uniformly convergent at $t = 0$.

4 . 4. Our argument can now be repeated. We see that

$$|f''(x)| < A_2^2 C e^{A|x|},$$

and that $f''(x)$ satisfies M, and so generally. Then, as in Lemma δ of BKLM, $f(x)$ is an integral function, and from § 3 . 22 of BKLM onwards our proof proceeds as before.

4 . 5. We add, in conclusion, that the more general equation

$$f'(x) = \lambda \int_0^\infty \frac{e^{-t}}{t} \{f(x+t)-f(x-t)\}\, dt$$

has been investigated by E. Hopf*. The theory is not much changed by the generalization of λ, but the form of the solutions is different. If we assume that $f(x) = O(e^{A|x|})$, where $A < 1$, and that λ is real, then there are no solutions when $\lambda < 0$, and otherwise only the solutions

$$f(x) = a+be^{\alpha x}+ce^{-\alpha x},$$

where α and $-\alpha$ are the two roots of

$$\alpha = \lambda \log \frac{1+\alpha}{1-\alpha}$$

(other than $\alpha = 0$) whose real part lies between -1 and 1. These roots are real when $\lambda < \frac{1}{2}$ and pure imaginary when $\lambda > \frac{1}{2}$. When $\lambda = \frac{1}{2}$ there is a treble root at the origin and the solution degenerates.

* E. Hopf, "Über eine Klasse singulärer Integralgleichungen: Bemerkungen zur Methode von Hardy und Titchmarsh", *Journal London Math. Soc.*, 4 (1929), 23–27.

SOME ASPECTS OF THE INVARIANT ALGEBRA ASSOCIATED WITH LINES

By W. SADDLER.

[Received 25 October, 1928.—Read 8 November, 1928.]

1. In dealing with points and lines in three-dimensional space it is usual to typify the point coordinate as

$$x = [x_1, x_2, x_3, x_4]$$

with cogredient variables y, a, b, ξ, ..., and the line coordinate as

$$p = [p_{14}, p_{24}, p_{34}, p_{23}, p_{31}, p_{12}]$$

$$= \left\| \begin{matrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{matrix} \right\|,$$

so that

$$p_{ik} = x_i y_k - x_k y_i ;$$

in dual plane coordinates these are expressible as

$$\pi_{ik} = u_i v_k - u_k v_i,$$

such that (except for a numerical factor)

$$\pi_{lm} = p_{ik} \quad (i, k, l, m = 1, 2, 3, 4).$$

Without ambiguity we may thus write

$$p = (xy) = (uv) = \pi.$$

A linear complex has the equation

$$C = \Sigma C_{ik} p_{ik} = \Sigma C_{ik} \pi_{lm},$$

or, by symbolizing (following Clebsch) the C_{ik}, namely

$$C_{ik} = C_i C'_k - C_k C'_i = (CC')_{ik},$$

the linear complex takes the form

$$(Ap) = (Bp) = (Cp) = \ldots,$$

and its invariant is given by

$$\Sigma C_{ik} C_{lm} = (AB) = (aa'bb').$$

When the complex is special its directrix will have the coordinates A_{ik}, B_{ik}, ... The invariant algebra associated with lines and linear complexes thus deals with bracket factors of the type (AB).

It is proposed to consider here the geometry of certain bracket factors of this type, with corresponding extensionals for higher space, and for this purpose it is convenient to introduce a certain quadrilinear binary correspondence which will be built up in two distinct ways.

2. A quadrilinear binary correspondence, connecting four binary fields with variable parameters x, y, z, t, can be represented by the form*

$$\Sigma a_{iklm} x_i y_k z_l t_m = 0 \quad (i, k, l, m = 1, 2),$$

or, symbolically, as

$$a_x b_y c_z d_t = a'_x b'_y c'_z d'_t = (a_1x_1+a_2x_2)(b_1y_1+b_2y_2)(c_1z_1+c_2z_2)(d_1t_1+d_2t_2), \quad \text{(i)}$$

where $$a_{iklm} = a_i b_k c_l d_m.$$

Such a correspondence arises, in the first instance, by considering two quadrics, one in point, and the other in plane coordinates when the points and planes are represented by generator coordinates. Taking the first quadric as

$$\xi_1\xi_4 - \xi_2\xi_3 = 0,$$

and the second in its general form involving the variables u_i, we may express their equations in generator coordinates as

$$\begin{aligned} \xi_1 &= xy\,; & u_1 &= a_1zt+a_2z+a_3t+a_4, \\ \xi_2 &= x\,; & u_2 &= b_1zt+b_2z+b_3t+b_4, \\ \xi_3 &= y\,; & u_3 &= c_1zt+c_2z+c_3t+c_4, \\ \xi_4 &= 1\,; & u_4 &= d_1zt+d_2z+d_3t+d_4. \end{aligned}$$

When a point on the first is incident with a plane of the second a correspondence of the type (i) arises.

The (2, 2)-correspondence between the fields (x, y) on the quartic curve of intersection of the two quadrics is given by the relation

$$(cc')(dd')\, a_x a'_x b_y b'_y = 0, \quad \text{(ii)}$$

* Segre, "On a quadrilinear correspondence between forms of the first species", *Annali di Mat.* (3), 29 (1921), 105–140.

with a similar meaning for the analogous (2, 2) form connecting (z, t). We shall return to this again.

Now consider four skew lines, A_{ik}, B_{ik}, C_{ik}, D_{ik}, and let the points on them at their points of intersection with the two transversals be

$$\alpha, \alpha' ; \ \beta, \beta' ; \ \gamma, \gamma' ; \ \delta, \delta' ;$$

so that $$(\alpha\beta\gamma\delta) = 0, \quad (\alpha'\beta'\gamma'\delta') = 0,$$

and, incidentally, all factors of the type $(\alpha\beta\gamma\delta')$ vanish. (iii)

A point on the first line $\alpha\alpha'$ is taken as

$$(x\alpha_1+x'\alpha_1', \ x\alpha_2+x'\alpha_2', \ x\alpha_3+x'\alpha_3', \ x\alpha_4+x'\alpha_4'),$$

or, briefly, $x\alpha+x'\alpha'$, where α, α' are fixed points and x/x' is a variable parameter. A similar parametric representation is taken for the other lines.

Four points, one on each of the four lines, namely

$$x\alpha+x'\alpha', \quad y\beta+y'\beta', \quad z\gamma+z'\gamma', \quad t\delta+t'\delta',$$

will be coplanar if the determinant

$$(x\alpha+x'\alpha', \ y\beta+y'\beta', \ z\gamma+z'\gamma', \ t\delta+t'\delta') = 0,$$

or, non-homogeneously,

$$(\alpha'\beta'\gamma\delta)\,xy+(\alpha'\beta\gamma'\delta)\,xz+(\alpha'\beta\gamma\delta')\,xt+(\alpha\beta'\gamma'\delta)\,yz$$
$$+(\alpha\beta'\gamma\delta')\,yt+(\alpha\beta\gamma'\delta')\,zt = 0, \quad \text{(iv)}$$

the other terms vanishing by virtue of (iii).

This is a quadrilinear binary correspondence between the four fields x, y, z, t, and may be written as

$$fyz+gzx+hxy+uxt+vyt+wzt = 0. \quad \text{(v)}$$

Since we may replace

$$\alpha \text{ by } \lambda\beta+\lambda'\delta, \qquad \alpha' \text{ by } \rho\beta'+\rho'\delta',$$
$$\gamma \text{ by } \mu\beta+\mu'\delta, \qquad \gamma' \text{ by } \sigma\beta'+\sigma'\delta'$$

where $$\lambda, \lambda' ; \ \mu, \mu' ; \ \rho, \rho' ; \ \sigma, \sigma'$$

are parameters, the invariants of the correspondence will thus be associated with the cross ratios of the two transversals.

The correspondence (v) is a special one, for it has the points α, β, γ, δ as three by three singular ternaries and similarly for the points α', β', γ', δ'. This follows from the fact that any three of the four points will satisfy the correspondence irrespective of the element of the remain-

ing field. For the general correspondence, however, a point in each of the three fields always determines a unique point in the remaining domain. Equation (v) is exactly similar in this respect to (i) for the two quadrics, when these quadrics have double contact, or cut in two conics with distinct intersections, for here also two sets of singular ternaries occur.

The invariants of such a correspondence will now be considered.

Consider it in its symbolic form

$$a_x b_y c_z d_t = 0.$$

The (1, 1)-correspondence between (x, y) denoted by $P_{12} = 0$, for a fixed z, t, will lie in a net if the four (1, 1) forms given by

$$a_x b_y c_1 d_1 = 0, \quad a_x b_y c_1 d_2 = 0, \quad a_x b_y c_2 d_1 = 0, \quad a_x b_y c_2 d_2 = 0,$$

are linearly connected, and conversely : with the same condition $P_{34} = 0$ will lie in a net. Similar considerations will hold for $P_{13} = 0$ and $P_{42} = 0$, $P_{14} = 0$ and $P_{23} = 0$.

Consider these as applied to (v).

$P_{12} = 0$ will lie in a net if

$$hw(vg - uf) = 0, \quad \text{and} \quad h = (\alpha'\beta'\gamma\delta) \neq 0, \quad w = (\alpha\beta\gamma' t') \neq 0,$$

hence $$(vg - uf) = (\alpha\beta'\gamma\delta')(\alpha'\beta\gamma'\delta) - (\alpha'\beta\gamma\delta')(\alpha\beta'\gamma'\delta) = 0,$$

or, on reduction,

$$(AC)(BD) - (BC)(AD) = 0 = L. \qquad \text{(vi)}$$

Hence if the quadrilinear correspondence is such that the (1, 1)-*correspondences between* (x, y) *or* (z, t) *lie in a net, the four lines satisfy the invariant relationship*

$$(AC)(BD) - (BC)(AD) = 0.$$

Interpreted in terms of the ratios of the two secants, this equation $L = 0$ is

$$\frac{\mu\lambda'}{\mu'\lambda} = \frac{1}{1 - \frac{\sigma'\rho}{\sigma\rho'}}.$$

Similarly $P_{13} = 0$ and $P_{24} = 0$ will lie in a net if

$$(AB)(CD) - (BC)(AD) = 0 = M,$$

and $P_{14} = 0$ and $P_{23} = 0$ will lie in a net if

$$(AC)(BD) - (AB)(CD) = 0 = N.$$

Since $$-L+M+N=0,$$

if $P_{12}=0$ and $P_{13}=0$ stand in a net, so does $P_{14}=0$.

There are *six* (2, 2)-correspondences of the type

$$(cc')(dd')\,a_x a'_x b_y b'_y = 0.$$

x, y will satisfy such a form when the (1, 1)-correspondence $P_{34}=0$ between the other two fields is degenerate.

For (v) this (2, 2) form is given by

$$gux^2+fvy^2+(fu+gv-hw)\,xy = 0. \qquad \text{(vii)}$$

It is thus reducible, and similarly for the other (2, 2) forms. Compare this with (i) in connection with the two quadrics. The quartic curve of intersection breaks up into two conics, and the parametric representations of points on the two conics are given by the factors of (vii).

All these (2, 2) forms will give two coincident (1, 1) forms if

$$(uf+gv-hw)^2-4hfuw = 0,$$

or, on substitution for the values of the coefficients and reduction,

$$\sqrt{\{(AB)(CD)\}}+\sqrt{\{(BC)(AD)\}}+\sqrt{\{(CA)(BD)\}} = 0^* \qquad \text{(viii)}$$

or $$\frac{\lambda\mu'}{\lambda'\mu} = \frac{\rho\sigma'}{\rho'\sigma}.$$

Again, the (2, 2) form (vii) will break up into two apolar (1, 1) forms; viz., two forms of the type

$$a_x b_y = 0, \quad a'_x b'_y = 0,$$

such that $$(aa')(bb') = 0$$

if $$(fu+gv-hw) = 0,$$

or, if the lines satisfy the relation

$$(AC)(BD)+(BC)(AD)-(AB)(CD) = 0$$

or $$(ACBD) = 0 = \underset{a,\,a'}{\overset{b,\,b'}{\Omega}}\,(aCb)(b'Da') = 0\dagger,$$

where

$$\underset{a,\,a'}{\overset{b,\,b'}{\Omega}}\,(aCb)(b'Da') = (aCb)(b'Da')-(a'Cb)(b'Da)+(a'Cb')(bDa)-(aCb')(bDa'),$$

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* In another connection see H. F. Baker, *Proc. Edinburgh Math. Soc.* (44), 2 (1925–6), 134.

† H. W. Turnbull, "On the invariant theory of mixed quaternary forms", *Proc. London Math. Soc.* (2), 25 (1925), 309.

$A = aa'$, $B = bb'$, and Ω denotes the simultaneous alternations in aa' and in bb'.

This invariant bracket factor can be regarded as the extensional of the ternary type

$$(aa'.bb'.cc') \quad \text{or} \quad (ABC).$$

Similarly, the (2, 2)-correspondences in (y, z) or (x, t), viz.

$$hfy^2+uwz^2+(uf+hw-gv)yz = 0,$$

will give two apolar (1, 1) forms if

$$(uf+hw-gv)^* = (ABCD) = 0.$$

This last, expressed in terms of the ratio of the two secants, is merely

$$\frac{\lambda\mu'}{\lambda'\mu}+\frac{\rho\sigma'}{\rho'\sigma} = 0.$$

It is noteworthy that the vanishing of the expression

$$uf+hw-gv,$$

as applied to the case of the form arising from the two quadrics, means that the two conics, forming the degenerate space quartic, are conjugate with respect to the first quadric.

Again, the symmetrical invariant

$$(AB)(CD)+(BC)(AD)+(CA)(BD) = 0 \qquad \text{(ix)}$$

is obtained thus.

The form $a_x b_y c_z d_t = 0$

is self-apolar, if $(aa')(bb')(cc')(dd') = 0,$

or $uf+vg+wh = 0,$

or $(a'\beta\gamma\delta')(a\beta'\gamma'\delta)+(a\beta'\gamma\delta')(a'\beta\gamma'\delta)+(a'\beta'\gamma\delta)(a\beta\gamma'\delta') = 0,$

which, on reduction, is result (ix) and is equivalent to

$$\frac{\sigma\rho'-\sigma'\rho}{\sigma\rho'+\sigma'\rho} = \tfrac{1}{3}\frac{\lambda\mu'+\lambda'\mu}{\lambda\mu'-\lambda'\mu}.$$

In the case of the two quadrics this invariant (ix) gives the condition that a tetrahedron may be inscribed in the one to be self-conjugate to the other.

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* It is to be observed that this is the Δ invariant of the $(2-2)$ form. Turnbull, "Geometrical interpretation of the $(2-2)$ form", *Proc. Royal Soc. Edinburgh*, 44 (1924), 23–50.

3. How far do such invariant expressions apply when, instead of four lines, we consider four linear complexes

$$(Ap) = 0, \quad (Bp) = 0, \quad (Cp) = 0, \quad (Dp) = 0\ ?$$

Consider a system of two terms

$$(Ap)+\lambda(Bp) = 0.$$

The two special complexes of the system are given by the values of λ which satisfy the equation

$$(AA')+2\lambda(AB)+\lambda^2(BB') = 0.$$

Call these two values λ_1 and λ_2; thus the directrices of the special complexes of the system have coordinates

$$A_{ik}+\lambda_1 B_{ik}, \quad A_{ik}+\lambda_2 B_{ik}.$$

Take now the four lines as being the directrices of the special complexes of

$$(Ap)+\lambda(Bp) = 0, \quad \text{viz.} \quad A, B, \tag{x}$$

and

$$(Cp)+\mu(Dp) = 0, \quad \text{viz.} \quad C, D. \tag{xi}$$

Let X, Y be written for the non-vanishing expressions

$$(AA')(BB')-(AB)^2, \quad (CC')(DD')-(CD)^2.$$

The invariant $L = 0$ [Equation (vi)] for these four lines takes the form

$$\frac{4\sqrt{(X.Y)}}{(BB')(DD')}[(AC)(BD)-(BC)(AD)] = 0.$$

Hence the vanishing of the invariant L applies equally well to the directrices of the special complexes of the two systems (x) and (xi); but the other invariants M and N will be completely changed and would only apply to different combinations of the four initial complexes.

Consider again the irrational invariant (viii), namely

$$\sqrt{\{(AB)(CD)\}}+\sqrt{\{(BC)(AD)\}}+\sqrt{\{(CA)(BD)\}} = 0^*.$$

This is known to be the condition that the line D should touch the quadric whose generators are A, B, C, and whose equation is

$$(xABCx) = \underset{aa'}{\Omega}\,(xaB)(a'Cx) = 0. \tag{xii}$$

Use now for the four lines the directrices of the special complexes (x)

* Baker, *loc. cit.*

and (xi), and the invariant now rationalized has the form

$$\frac{16XY}{(BB')^2(DD')^2}\cdot[(AA')(BB')(CC')(DD')-(AB)^2(CC')(DD')-(CD)^2(AA')(BB')$$
$$-(AC)^2(BB')(CC')-(BC)^2(AA')(DD')-(AD)^2(CC')(DD')$$
$$-(BD)^2(AA')(CC')+2(AC)(BC)(AB)(DD')$$
$$+2(AC)(AD)(CD)(BB')+2(BC)(BD)(CD)(AA')$$
$$+2(AD)(BD)(AB)(CC')+(AB)^2(CD)^2+(AC)^2(BD)^2$$
$$+(AD)^2(BC)^2-2(AB)(AC)(CD)(BD)-2(AC)(BD)(AD)(BC)$$
$$-2(AB)(CD)(BC)(AD)]=0^*. \qquad \text{(xiii)}$$

The vanishing of the factor in the square brackets is thus the condition that the quadric, whose generators of the same system are the directrices of the special complexes of the system of two terms

$$(Ap)+\lambda(Bp)=0$$

and one of the directrices of the other system of two terms

$$(Cp)+\lambda(Dp)=0,$$

should be touched by the other directrix of this second system of two terms.

Now the condition (xiii) is symmetrical in the letters A, B, C, D, and hence will not be altered if the four complexes are combined in pairs in any order and their respective directrices taken for the lines in question.

Thus if one of the directrices of the special complexes of a system of two terms formed out of any two of the four initial complexes is a tangent to the quadric formed, as in (xii), *out of the other directrix and the directrices of the special complexes of the remaining two terms, the same condition will hold good no matter how the complexes are paired off.*

4. Consider again the invariant (viii) and the quadric (xii), namely

$$(xABCx)=0,$$

whose line equation is

$$\sqrt{\{(AB)(Cp)\}}+\sqrt{\{(BC)(Ap)\}}+\sqrt{\{(CA)(Bp)\}}=0. \qquad \text{(xiv)}$$

* This is no other than the known invariant of 4 linear complexes $|(AA)(AB)(AC)(AD)|$.

The irrational invariant (viii), when expressed in the form

$$(ABDC)^2 - 4(AB)(BD)(CA)(CD) = 0,$$

has several analogous results for space of higher dimensions, but such results are not expressible in the irrational form given above.

Thus, for the special quadric in four dimensions,

$$(xA_2B_3C_3x) = 0 = \Omega_{aa'}(xabb'b'')(a'cc'c''x) = 0,$$

where the line A_2 and the planes B_3 and C_3 lie on the quadric, the line D_2 will touch it if

$$(A_2B_3D_2C_3)^2 - 4(A_2B_3)(A_2C_3)(D_2B_3)(D_2C_3) = 0,$$

a result symmetrical in A_2, D_2, where

$$(A_2B_3D_2C_3) = \overset{dd'}{\Omega}_{aa'}(aB_3d)(a'C_3d').$$

In five dimensions the quadric

$$(xA_2B_2C_2D_2E_2x) = 0 = \Omega_{aa'}(xabb'cc')(a'dd'ee'x)$$

will touch the line F_2 if

$$(A_2B_2C_2F_2D_2E_2)^2 - 4(F_2B_2C_2)(A_2B_2C_2)(F_2D_2E_2)(A_2D_2E_2) = 0,$$

where $$(A_2B_2C_2F_2D_2E_2) = \overset{ff'}{\Omega}_{aa'}(aB_2C_2f)(f'D_2E_2a').$$

These forms can be extended in a similar manner to higher dimensions.

5. Two quadrics

$$a_x^2 = a_x'^2 = 0 \quad \text{and} \quad b_x^2 = b_x'^2 = 0,$$

whose line equations are given by

$$(Ap)^2 = 0 \quad \text{and} \quad (Bp)^2 = 0,$$

are such that a tetrahedron whose edges touch the one will be conjugate to the other, if

$$(AB)^2 = 0 = \tfrac{1}{4}\left(\Sigma \frac{\partial^2}{\partial p_{ij}\partial q_{kl}}\right)^2 (Ap)^2(Bq)^2 = 0 \qquad \text{(xv)}$$

$$(i, j, k, l = 1, 2, 3, 4\,;\ i \neq j,\ k \neq l).$$

Now the quadric (xii), $(xABCx) = 0$,

whose line equation is given by (xiv), or by

$$(BC)^2(Ap)^2+(CA)^2)(Bp)^2+(AB)^2(Cp)^2-2(BC)(CA)(Bp)(Ap)$$
$$-2(CA)(AB)(Bp)(Cp)-2(AB)(BC)(Ap)(Cp)=0,$$

will be such that a tetrahedron whose edges touch it is conjugate to

$$(xBCDx)=0, \qquad \text{(xvi)}$$

if $$4(BC)^2\{(AB)^2(CD)^2+(AC)^2(BD)^2+(BC)^2(AD)^2-2(BC)(CD)(DA)(AB)$$
$$-2(BC)(CA)(BD)(DA)+10(BD)(CD)(AB)(AC)\}=0, \qquad \text{(xvii)}$$

a condition obtained by using the differential operator on the two line equations of the quadrics.

Now the H invariant of the (2, 2) form*

$$(bb')(cc')\,a_x a'_x d_t d'_t = 0,$$

or, adapted to the form of (v),

$$hgx^2+wvt^2+(gv+hw-uf)\,xt=0,$$

is given by $$\tfrac{1}{8}\{8hgwu+(gv+hw-uf)^2\}=0,$$

which, on substitution for the coefficients, reduces to the condition (xvii), since $(BC)\neq 0$.

Similar results hold for the other (2, 2) forms and the other quadrics of the types (xiv) and (xvi).

For instance, the H invariant of the (2, 2) form in (x, y), (vii),

$$gux^2+fvy^2+(fu+gv-hw)\,xy=0$$

corresponds to the apolar condition between the quadrics

$$(xABCx)=0 \quad \text{and} \quad (xABDx)=0.$$

It is well to state* that the H invariant of a (2, 2) form, say

$$p_x^2 q_y^2 = 0,$$

when this (2, 2) form is regarded as the curve of intersection of two quadrics, is no other than the apolar relationship of the type (xv) between the two quadrics which cut in this (2, 2) curve.

A further application of the differential operator (xv) leads to a curious result connecting the three quadrics

$$\left.\begin{aligned} p_x^2 &= (xABCx)=(xaa'.bb'.cc'x)=0\\ q_x^2 &= (xab'.bc'.ca'x)=0\\ r_x^2 &= (xac'.ba'.cb'x)=0 \end{aligned}\right\}. \qquad \text{(xviii)}$$

* Turnbull, *Proc. Royal Soc. Edinburgh, loc. cit.*

These quadrics are, as is well known, the extensionals of the invariant relations in a plane of two triply perspective triangles. They have the alternate sides of a skew hexagon as generators and have a common curve of intersection.

By forming the apolar relationship, analogous to (xv), for

$$p_x^2 = p_x'^2 = 0 \quad \text{and} \quad q_x^2 = q_x'^2 = 0,$$

namely
$$(pp'qq')^2 = (PQ)^2 = 0,$$

or
$$(aa'bb')(bc'ca')(cc'ab') + (bb'cc')(ca'ab')(aa'bc') + (cc'aa')(ab'bc')(bb'ca') = 0,$$

which is symmetrical in abc and in $a'b'c'$, this condition is also that given by

$$(pp'rr')^2 = (PR)^2 = 0 \quad \text{and} \quad (qq'rr')^2 = (QR)^2 = 0.$$

Hence if a tetrahedron be such that its edges touch one of the quadrics and be self-conjugate to a second quadric of (xviii), *any pair of the quadrics will satisfy the same relationship.*

The three quadrics of (xviii) have a common curve of intersection. Consider the analogous quadrics in four dimensions built up as extensionals of the three-dimensional invariant

$$(ABCD) = \underset{aa'}{\overset{cc'}{\Omega}} (aBc)(c'Da').$$

This, for instance, gives the quadric

$$(xABCDx) = 0 = \underset{aa'}{\overset{cc'}{\Omega}} (xaBc)(c'Da'x) = 0 = S_1.$$

The lines aa', bb', cc', dd' are generators. Similarly form

$$(xab'.bc'.cd'.da'x) = \underset{ab'}{\overset{cd'}{\Omega}} (xabc'c)(d'da'b'x) = 0 = S_2,$$

$$(xac'.bd'.ca'.db'x) = \underset{ac'}{\overset{ca'}{\Omega}} (xabd'c)(a'db'c'x) = 0 = S_3,$$

$$(xad'.ba'.cb'.dc'x) = \underset{ad'}{\overset{cb'}{\Omega}} (xaba'c)(b'dc'd'x) = 0 = S_4.$$

It is found that

$$S_2 - S_3 + S_4 - S_1 = (xac.b'd'.bd.a'c'x).$$

Thus there is a linear relationship between these five quadrics in this case, with analogous results for the other invariants of the type $(ABCD)$.

A QUARTIC FORM IN FOUR DIMENSIONS

By L. ROTH.

[Received 7 November, 1928.—Read 8 November, 1928.]

1.1. In Steiner's well known generation of the cubic surface, a correspondence is set up between the points of space and the intersections of their polars with regard to three arbitrary quadrics; then if a point describes an arbitrary plane, its corresponding point or "conjugate" describes a general cubic surface. Consideration of the Jacobian of the three quadrics leads directly to the plane representation of the surface, and its three sets of lines a_r, b_s, c_{rs} are obtained, corresponding to points, conics, and lines in the given plane.

By a precisely similar method we obtain from a polarity in $[n]$ a rational form V^n_{n-1}, the points of which correspond to those of a given prime. It may easily be shown that the form contains n systems of ∞^{n-3} lines corresponding to rational normal curves c^r $(0 \leqslant r < n)$ in the prime representation. The equation of the form thus obtained is $\Delta = 0$, where Δ is a determinant of order n, whose elements are linear in the coordinates. We thus observe that there are two conjugate generations of V^n_{n-1}, obtained by interchanging rows and columns of Δ; consideration of the n systems of lines leads to the same conclusion.

The present discussion is restricted to the four-dimensional case, as the general problem appears to require results which are not yet available.

1.2. Consider in $[4]$ a set of four arbitrary quadric forms. With each point is associated a second point or "conjugate", which is the intersection of its four polars with regard to the quadrics. If a point P describes any locus, its conjugate P' describes the "polo-locus". Evidently the polo-curve of an arbitrary line is a rational normal c^4, the polo-surface of a plane is a rational sextic surface*, and the polo-form of a prime is a rational V^4_3.

* This surface has been discussed by F. P. White, *Proc. Camb. Phil. Soc.*, 21 (1922), 216–227. The results stated follow at once from the generation of loci by collinear systems of primes.

1.3. Let Ω, Ω' be the polo-forms of any two primes ω, ω'. Their intersection V_2^{16} consists of the polo-surface of the plane π common to ω, ω', which is of order 6, and a residual surface J of order 10. Now all points of the polo-surface have a unique conjugate in π; all other points of V_2^{16}, which lie on J, must have a conjugate in ω and ω', and therefore a line of conjugates; hence J is the Jacobian of the four base quadrics; it is the locus of all points whose conjugate is not a point but a line. It contains 20 lines, which are the axes of the line cones contained in the system of quadrics*.

1.4. The polo-curve of a line l which meets J at P_1 breaks up into a twisted cubic c^3 and the conjugate line of P_1. Similarly, the polo-curve of a chord P_1P_2 of J consists of a conic c^2 and the conjugates corresponding to P_1 and P_2. The polo-curve of a trisecant $P_1P_2P_3$ consists of a line λ and the conjugates of P_1, P_2, P_3. Moreover, since the polo-curve of λ is a line, λ itself must be a trisecant of J. Finally, the polo-curve of a quadrisecant l consists of a point L and four conjugates. Since the conjugate of L is a line l, L must lie on J. Further, by the reciprocal theorem, since the conjugate of L passes through all points of l, the four conjugate lines corresponding to the intersections of l with J must pass through L. Thus four quadrisecants of J can be drawn through each point of J, and there is a reciprocity between these quadrisecants and the conjugate of the point, which is also a quadrisecant of J.

2.1. *Representation of* Ω. To a point of a prime ω corresponds in general a unique point of the polo-form Ω, and *vice versa*. Hence Ω may be represented birationally upon ω. But since J meets ω in a curve ψ of order 10, to each point of ψ corresponds a line of Ω; thus ψ is the fundamental curve in the representation.

The polo-locus of a curve or surface in ω will, of course, lie on Ω. In particular, a prime section of Ω corresponds to a quartic surface in ω, which contains ψ, since the prime meets all the corresponding lines on Ω. A plane section of Ω corresponds to the free intersection of two such quartic surfaces, namely a sextic, which must be of genus 3. Any such sextic must meet ψ in 20 points, since the intersection of three arbitrary primes meets Ω in 4 points. It follows that ψ is a decimic of genus 11†, since the residual intersection of two quartic surfaces containing it is a sextic of genus 3.

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* See Segre, *Encyklopädie der Math. Wiss.*, III C 7, 856.

† This curve has been discussed by Reye, *Journal für Math.*, 108 (1891), 89-124.

2. 2. We have already seen that Ω contains one system of lines, corresponding to points of ψ; we shall now obtain three other systems. It has been shown that to a $(4-r)$-secant of $J(0 \leqslant r \leqslant 4)$ corresponds a rational normal curve c^r. If this line c_r lies on Ω, the corresponding c^r is in ω; and since an arbitrary prime meets c_r in one point, the corresponding quartic surface, which contains ψ, must meet c_r in one free point only; hence c^r intersects ψ in $4r-1$ points. Now a conic and twisted cubic in [3] depend on 8 and 12 parameters respectively. Thus there are four systems of ∞^1 lines on Ω, namely,

c_0 lines corresponding to points of ψ,

c_1 lines corresponding to trisecants of ψ,

c_2 lines corresponding to conics meeting ψ in 7 points,

c_3 lines corresponding to twisted cubics meeting ψ in 11 points.

2 . 3. We denote by C_r the ruled surface on Ω formed by the c_r lines. It is at once seen that C_0 is of order 20; for the number of c_0 lines meeting an arbitrary plane section of Ω is equal to the number of points in which the corresponding sextic meets ψ, and this has been shown to be 20.

From the usual formula it is found that the trisecants of ψ generate a surface of order 80, on which ψ is a multiple curve of order 17. This is the surface corresponding to C_1; the order of C_1 is thus the number of free points in which the surface in ω is met by the sextic corresponding to an arbitrary plane section of Ω, that is $6 \cdot 80 - 20 \cdot 17 = 140$.

2 . 4. *The nodes of* Ω. There are 20 quadrisecants of ψ, corresponding to points of Ω. To any plane section through such a point corresponds a degenerate sextic consisting of a quintic of genus 2, having the corresponding quadrisecant for chord; this may easily be shown by the formulae for the partial intersections of two surfaces*. Hence a point corresponding to a quadrisecant is a node of Ω. This is otherwise proved by the fact that any line through the node meets Ω elsewhere in two points only; for take three quartic surfaces containing ψ and the corresponding quadrisecant, corresponding to three primes through the point. Regarding ψ and the quadrisecant as a degenerate c^{11} of genus 14 and applying the equivalence formula†, we find the number of free intersections to be two.

Any plane in ω meets each quadrisecant, so that the corresponding surface passes through all the nodes of Ω. It follows that the 20 nodes

* Cf. James, *Proc. Camb. Phil. Soc.*, 21 (1923), 444.

† Salmon-Rogers, *Analytic geometry of three dimensions*, 1 (1915), 371.

are the residual intersection of two rational sextic surfaces having in common a rational normal c^4.

3 . 1. *The conjugate systems.* As stated previously (§ 1 . 1) we know that there exist two conjugate generations of Ω. It follows that there are two systems of conjugate curves and surfaces on Ω, of the same order, with like properties. In particular, the four systems of lines correspond as shown :

$$\left.\begin{matrix} c_0 & c_1 & c_2 & c_3 \\ c_3 & c_2 & c_1 & c_0 \end{matrix}\right\}. \tag{1}$$

Suppose now that we denote any surface in ω of order λ, containing ψ as a μ-ple curve, by the symbol* (λ, μ); then this may equally well represent the corresponding surface on Ω. In particular, any plane in ω is denoted by (1, 0) and any prime section of Ω corresponds to (4, 1). We shall now show that with any surface (λ, μ) is associated a conjugate (λ', μ') or $(\lambda, \mu)'$, of the same order, such that the two have reciprocal properties with regard to the scheme (1).

The order of the surface on Ω corresponding to (λ, μ) is the number of points in which it meets an arbitrary plane section, that is, the number of free intersections of (λ, μ) with a sextic through 20 points of ψ, which is $6\lambda - 20\mu$. Since the surfaces on Ω are to be of the same order we must have

$$6\lambda - 20\mu = 6\lambda' - 20\mu'. \tag{2}$$

If the number of points in which (λ, μ) meets any line c_r is the same as the number of points in which its conjugate meets c_{3-r} we require

$$r\lambda - (4r-1)\mu = (3-r)\lambda' - \{4(3-r)-1\}\mu', \tag{3}$$

where $r = 0, 1, 2, 3$. The equations (3) for $r = 0$, $r = 3$ are

$$\left.\begin{aligned} \mu &= 3\lambda' - 11\mu' \quad (4') \\ \mu' &= 3\lambda - 11\mu \quad (4) \end{aligned}\right\}.$$

Hence
$$\lambda' = 11\lambda - 40\mu. \tag{5}$$

These equations take the symmetrical forms†

$$3(\lambda' - \lambda) = 10(\mu' - \mu), \tag{6}$$

$$\lambda + \lambda' = 4(\mu + \mu'). \tag{7}$$

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* Cf. Bath, *Proc. Camb. Phil. Soc.*, 24 (1927), 196.

† In the n-dimensional case, the relations between conjugate forms, similarly defined, are

$$\lambda' = (n^2-n-1)\lambda - (n-2)n(n+1)\mu, \quad \mu' = (n-1)\lambda - (n^2-n-1)\mu.$$

It is verified that the values of λ' and μ' given in (4) and (5) satisfy (2) and (3), since the latter are equivalent to (6) and (4′). This establishes the result.

3.2. Since a prime section of Ω is represented by (4, 1), the surface in which Ω is intersected by any V_3^m may be represented by $m(4, 1)$. It follows that any two conjugate surfaces are residual to one another*, for from (4) and (5) we have

$$(\lambda, \mu)+(\lambda', \mu') = (3\lambda-10\mu)(4, 1). \tag{8}$$

From (7) it follows that the only self-conjugate surfaces are those which are the complete intersection of Ω with some form V_3^m. Note that (7) may be written as $\lambda-4\mu = -(\lambda'-4\mu') = \rho$, say. Hence, if any surface has a ρ-ple point at a node, so has its conjugate.

3.3. *Some special cases.* We know, from § 1.3, that the Jacobian J is residual to any surface (1, 0), so that, by (8), its symbol is (15, 4).

Hence, by (4), (5), any Jacobian of the conjugate system is (5, 1). To the lines on J correspond 20 twisted cubics on (15, 4), and to the lines on a conjugate Jacobian correspond 20 nodes of (5, 1) lying on ψ.

In the same way we see that the sextic surfaces conjugate to (1, 0) are represented by (11, 3). Hence a surface of the latter type is rational. It is easy to work out the details of its representation on the plane (1, 0), using the results of the next section.

3.4. *Conjugate curves.* We denote by $(\alpha; \beta)$ any curve of order α which meets ψ in β (not necessarily distinct) points. The corresponding curve on Ω is thus of order $4\alpha-\beta$. The conjugate $(\alpha'; \beta')$ or $(\alpha; \beta)'$ is defined as meeting any surface (λ, μ) in the same number of free points as $(\alpha; \beta)$ meets $(\lambda, \mu)'$; so that we have the identity

$$\begin{aligned}\lambda\alpha'-\mu\beta' &\equiv \lambda'\alpha-\mu'\beta,\\ &\equiv (11\lambda-40\mu)\alpha-(3\lambda-11\mu)\beta, \text{ true for all } \lambda \text{ and } \mu.\end{aligned}$$

From this we obtain the relations

$$\alpha' = 11\alpha-3\beta, \tag{9}$$

$$\beta' = 40\alpha-11\beta. \tag{10}$$

We obviously have $4\alpha'-\beta' = 4\alpha-\beta$.

* Cf. the known result for the cubic surface: Baker, *Proc. London Math. Soc.* (2), 11 (1912), 285-301.

It now follows that the free intersection of any two surfaces (λ_1, μ_1), (λ_2, μ_2) is conjugate to the intersection of their conjugates $(\lambda_1, \mu_1)'$, $(\lambda_2, \mu_2)'$. For the number of points in which any other surface (λ_3, μ_3) meets the intersection of (λ_1, μ_1), (λ_2, μ_2) must be the same as the number of points in which the three conjugate surfaces intersect.

Self-conjugate curves are such that $10\alpha = 3\beta$, since (9) and (10) are equivalent for $\alpha = \alpha'$, $\beta = \beta'$. (Note that one of these conditions implies the other.) The simplest types of such curves are evidently (i) conics corresponding to (3 ; 10), (ii) plane sections of Ω corresponding to (6 ; 20).

3 . 5. In calculating the symbol of a conjugate curve, it may happen that the 20 quadrisecants, which form a curve (20 ; 80), occur redundantly in the result ; or the conjugate may consist of a proper curve $(\alpha' ; \beta')$ together with a certain multiple ρ of (20 ; 80). For example, the conjugate of an arbitrary line (1 ; 0) is a curve (11 ; 40). Now the former is the intersection of two planes (1, 0) ; hence the latter should be the free intersection of two surfaces (11, 3). In this case, however, we must exclude the 20 quadrisecants which (necessarily) lie on the two surfaces.

Again, the intersection (15 ; 40) of (15, 4) and (1, 0) should be conjugate to (25 ; 80), the intersection of the conjugate surfaces (5, 1) and (11, 3) ; whereas the calculation gives (45 ; 160). But the latter curve is actually (25 ; 80)+(20 ; 80). As a further example, the intersection (11 ; 30) of (11, 3) and (1, 0) should be self-conjugate ; the calculated conjugate is (31 ; 110), that is (11 ; 30)+(20 ; 80). This difficulty may, however, be avoided by defining a curve $(\alpha ; \beta)$ as self-conjugate (mod ρ) if

$$(\alpha ; \beta)' \equiv (\alpha ; \beta) + \rho(20 ; 80),$$

whence

$$10\alpha - 3\beta = \rho.$$

Then (6 ; 20) is self-conjugate (mod 0) and (11 ; 30) is self-conjugate (mod 1).

4 . 1. *The surfaces C_r.* It has been shown (§ 2 . 3) that C_0, represented by (0, 1), is of order 20, and that C_1, represented by (80, 17), is of order 140. We now apply the theory of conjugates to give the surfaces which represent C_2 and C_3. The usual calculation gives (200, 53) and (40, 11) in the two cases. Thus Ω contains a composite ruled surface of order $2 . 20 + 2 . 140 = 320$, in accordance with general theory*.

4 . 2. *The incidence scheme.* To find the number of c_r lines meeting an arbitrary c_s, we merely calculate the number of free points in which

* Schubert, *Math. Ann.*, 26 (1885), 52–73 (72) ; Segre, *Encyklopädie*, III C 7, p. 937.

the corresponding c^s meets C_r. Since 17 trisecants can be drawn through an arbitrary point of ψ, it follows that each c_0 meets 17 c_1 lines. Since each trisecant meets ψ in three points, each c_1 meets 3 c_0 lines. A c_2 line is represented by (2; 7), so that it meets 7 c_0 lines. Similarly each c_3 meets 11 c_0 lines; each c_0 meets 53 c_2 and 11 c_3 lines. No two c_0 or c_3 lines intersect. The number of c_1 lines meeting another c_1 is the number of trisecants of ψ which meet a given trisecant, that is $80-3\,.\,16-2=30$. Finally, the number of c_2 lines meeting a given c_1 is $200-3\,.\,53=41$. The rest of the results can be written down by symmetry, and form the following scheme:

	c_0	c_1	c_2	c_3
Each c_0 line meets	—	17	53	11
Each c_1 line meets	3	30	41	7
Each c_2 line meets	7	41	30	3
Each c_3 line meets	11	53	17	—

4 . 3. Since a node of Ω corresponds to a quadrisecant of ψ, four c_0 lines (and therefore four c_3 lines) pass through each node. To find how many c_1 and c_2 lines pass through a node, we require the following

LEMMA. *To find the number of trisecants which meet a given quadrisecant of a curve, at points not on the curve.*

If the given curve is of order n and rank N, we may suppose* it to consist of n lines having $\frac{1}{2}N$ simple intersections or nodes. Let l be the given quadrisecant, meeting four of the mutually skew lines a_i at points A_i, and let the remaining $n-4$ lines be b_i.

If all the lines, including l, were mutually skew, the required number of trisecants would be $2\binom{n}{3}$. But from this we must deduct the number of improper trisecants, consisting of

(i) chords from A_i to b_i, $4\left\{\binom{n-4}{2}-\frac{1}{2}N\right\}$ in number,

(ii) chords from the nodes to b_i, $\frac{1}{2}N(n-6)$ in number,

(iii) trisecants lost by the incidence of l with a_i, namely $2\binom{4}{3}$.

Hence the required number is

$$\tfrac{1}{3}n(n-1)(n-2)-\tfrac{1}{2}N(n-2)-2(n-4)(n-5)-8.$$

* Severi, *Rend. dei Lincei* (5), 24_1 (1915), 877-888.

Putting $n = 10$, $N = 40$, we see that 12 trisecants of ψ meet a given quadrisecant. Hence 12 c_1 and 12 c_2 lines pass through each node of Ω. But it should be noted that these lines are identical; for any such trisecant forms with the quadrisecant a degenerate (2; 7) and therefore lies on (200, 53), the surface corresponding to C_2. Further, no other conics (2; 7) can meet the quadrisecant, or we should have the unsymmetrical result that all the c_1 lines through a node lie on C_2, but not all the c_2 lines through the node lie on C_1.

Evidently each quadrisecant is a 12-ple line on C_2, since through each point of it pass 12 (degenerate) conics (2; 7). This may be verified as follows. Let $(\alpha\,;\,\beta)$ be the free intersection of C_2 and (11, 3), $(\alpha'\,;\,\beta')$ the intersection of C_1 and (1, 0). With this hypothesis,

$$\alpha = 11.200-3.53.10-12.20 = 370\,; \quad \alpha' = 80,\ \beta' = 170.$$

The relation $\alpha' = 11\alpha-3\beta$ gives $\beta = 1330$, and the second relation $\beta' = 40\alpha-11\beta$ is satisfied by the values thus found.

4.4. *Intersection of the surfaces C_r.* In the present state of the theory of surfaces it is impossible to write down the characteristic β of the intersection $(\alpha\,;\,\beta)$ of C_r and C_s, residual to ψ. We have therefore to proceed indirectly. Suppose that we wish to determine the order of the curve of intersection of any surface $(\lambda,\ \mu)$ with C_0; this will be the number of points in which it meets an arbitrary prime: hence if $(\alpha_0\,;\,\beta_0)$ is the free intersection of $(\lambda,\ \mu)$ with the corresponding (4, 1), β_0 is the order required. Now $4\alpha_0-\beta_0$ is the known order of any prime section of $(\lambda,\ \mu)$ on Ω, and α_0 can be written down. Hence β_0 is determined.

Thus, to find the order of the intersection of C_0 and C_3, let $(\alpha_{03}\,;\,\beta_{03})$ be the intersection of (40, 11) and (4, 1), so that $\alpha_{03} = 50$. Any prime section of C_3 is of order 20, so that $4\alpha_{03}-\beta_{03} = 20$, whence $\beta_{03} = 180$.

Similarly we find the orders β_{01}, β_{02} of the intersections of C_0, C_1 and C_0, C_2. Now the intersection $(\alpha_{23}\,;\,\beta_{23})$ of C_2 and C_3 must be of order β_{01}, whence $4\alpha_{23}-\beta_{23} = \beta_{01}$. Since α_{23} can be written down, β_{23} is determined. Similarly for $(\alpha_{13}\,;\,\beta_{13})$. Finally, to find $(\alpha_{12}\,;\,\beta_{12})$, we write down the value of α_{12} and use the condition of self-conjugacy to determine β_{12}. These results appear to be of no particular interest, so the considerable details of the calculation are not given.

5. *The symmetrical case.* We add a brief note on the case where Ω has for its equation a symmetrical determinant. The conjugate systems are now coincident; Ω contains a nodal curve δ of order 10 whose points correspond to ∞^1 quadrisecants of ψ. We have, from (4) and (5), $(\lambda,\ \mu)' = (\lambda,\ \mu)+(\lambda-4\mu)(10,\ 3)$, so that, if the conjugate system is

coincident with (λ, μ), it breaks up into (λ, μ) and a surface (10, 3), evidently of zero order; this clearly contains any quadrisecant of ψ, and has no free intersection with any (6; 20). Hence it must correspond to a curve δ on Ω. The number of points in which it meets an arbitrary prime is ten, corresponding to the quadrisecants of ψ which are common to (4, 1) and (10, 3). The (4, 1) corresponding to a prime section of Ω is now a Jacobian surface, containing ten quadrisecants of ψ.

The curves c^2 and c^3 lie entirely on (10, 3), so that C_2 and C_3 do not appear. C_0 is of the same order as before; but in calculating C_1 it must be noted that (10, 3) counts quadruply in (80, 17) and ψ must be regarded as a triple curve on (10, 3); the actual surface C_1 thus corresponds to (40, 8) and is of order 80.

Further, δ is not a general decimic in [4]; for since three quadrisecants can be drawn through any point of ψ, ψ being a triple curve on (10, 3), it follows that the c_0 lines are trisecants of δ, and that four of these trisecants pass through any point of δ. It may be added that the subject is considered in detail in a forthcoming paper.

SOME NOTES ON EXTENDED CONTINUED FRACTIONS

By F. G. MAUNSELL.

[Received 1 October, 1928.—Read 8 November, 1928.]

Introduction.

1. Jacobi* extended the idea of a simple continued fraction, which is essentially an algorithm based on the ratio of two numbers, and produced an algorithm based on the ratio of three (or more) numbers. O. Perron† considered Jacobi's algorithm and in particular proved the convergence theorem from a purely algebraical standpoint.

Bateman‡ (apparently in ignorance of the work of Jacobi and Perron) gave a proof of the convergence of such a fraction. The proof, however, was not complete.

The purpose of this paper is: (*a*) to introduce a simple notation for extended continued fractions (Chapter 1), and (*b*) to complete Bateman's proof (Chapter 2). It may be noted that, although Bateman's method leads to nothing not already proved by Perron, the standpoint taken up is essentially different.

Notation.

2. Confining ourselves to the ratio of three numbers for simplicity, we write the simple continued fraction as

$$p\,\frac{q}{r} = \cdot\,\frac{b_1}{a_1}\,a_2\,\frac{\cdot}{b_2}\,b_3\,\frac{a_3}{\cdot}\cdot\frac{b_4}{a_4}\,\ldots.$$

That is: p divided into q and r gives quotients b_1, a_1 and remainders α, β; α divided into β, p gives quotients b_2, a_3 and remainders α', β'. &c.

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* C. G. J. Jacobi, "Allgem. Theorie der kettenbruchähnlichen Algorithmen; herausg. von Heine", *Journal für Math.*, 69 (1868), 29–64. An account is also given by Bachmann, *Irrationalzählen* (Leipzig, 1892).

† O. Perron, "Grundlagen für eine Theorie des Jacobischen Kettenbruchalgorithmus *Math. Annalen*, 64 (1907), 1–76.

‡ H. Bateman, "The linear difference equation of the third order ...", *Quart. J. of Math.*, 41 (1910), 302–308.

3. It will be noted that :—

(1) The divisors are taken in each of the three possible places in cyclic order. In consequence we must allow a, b to vanish, though (*v. inf.*) we have in fact $a > 0$ always.

(2) Defining the convergents p_n, q_n, r_n by

$$p_n \begin{matrix} q_n \\ r_n \end{matrix} = \cdot \begin{matrix} b_1 \\ a_1 \end{matrix} \cdots 1 \begin{matrix} b_n \\ a_n \end{matrix},$$

we have (analogously to the ordinary case)

$$p_n = a_n p_{n-1} + b_n p_{n-2} + p_{n-3},$$
$$q_n = a_n q_{n-1} + b_n q_{n-2} + q_{n-3},$$
$$r_n = a_n r_{n-1} + b_n r_{n-2} + r_{n-3}.$$

(3) Any set of three rational numbers gives a terminating continued fraction.

4. A general continued fraction can be written as

$$p \begin{matrix} q \\ r \end{matrix} = c_1 \cdot \begin{matrix} b_1 \\ a_1 \end{matrix} a_2 \begin{matrix} c_2 \cdot \\ b_2 \end{matrix} b_3 \begin{matrix} a_3 \\ c_3 \cdot \end{matrix} \ldots,$$

and we have $p_n = a_n p_{n-1} + b_n p_{n+2} + c_n p_{n-3}$, etc.

Non-terminating continued fractions.

5. We can, if we wish, form a continued fraction starting from three numbers p, q, r whose ratios are irrational. Since there are essentially only two irrational quantities involved in the ratios, we can call these three quantities 1, θ, ϕ. It is obvious that multiplication by a constant will not alter the fraction.

It is easy to see that the continued fraction cannot terminate unless θ and ϕ are linearly dependent.

For a continued fraction cannot terminate unless at some point we have a divisor which goes exactly into its dividend.

Then we have

$$1 \begin{matrix} \theta \\ \phi \end{matrix} = \cdot \begin{matrix} b_1 \\ a_1 \end{matrix} \cdots \Theta \begin{matrix} 1 \\ 0 \end{matrix}.$$

Thus, working out the right-hand side,

$$\theta = (A + B\Theta)/(E + F\Theta),$$
$$\phi = (C + D\Theta)/(E + F\Theta),$$

where A, B, C, D are rational.

Eliminating Θ, we get a linear relation between θ and ϕ with integral coefficients.

6. Suppose we have such an infinite continued fraction formed from $1, \theta, \phi$ (we say nothing as to whether θ, ϕ are linearly dependent or not, but the following applies only if the continued fraction is infinite). The question now arises as to whether the continued fraction converges to the value $1, \theta, \phi$. Convergence of the continued fraction is defined in the obvious manner: by taking n terms only of the fraction we get a set of numbers p_n, q_n, r_n, the "convergents"; the question is whether the ratios $p_n : q_n : r_n$ tend to a limit as $n \to \infty$ and, if so, whether this limit is $1 : \theta : \phi$.

Bateman, in the paper cited above (§ 1), considered only a continued fraction as "given": not as arising naturally from the ratio of three quantities. In the special case when a_n, b_n, c_n are *constants* independent of n he proved that the ratios $p_n : q_n : r_n$ tend to definite limits. His proof is easily seen to hold provided that

$$\frac{c_n}{a_n + b_n + c_n} \leqslant \text{a constant} < 1.$$

It therefore certainly holds for the continued fraction we are considering, for here c_n is always unity and one at least of a_n, b_n is not less than 1. His proof, however, is not complete, since it leaves out an important possibility. I have been able to supply the deficiency and also to show that the fraction always converges to its generating function, the ratios $1 : \theta : \phi$.

7. First it will be convenient to prove some results relating to the possible values of the a_n's and b_n's.

THEOREM 1. $\qquad a_n \geqslant b_n, \quad a_n \geqslant 1.$

Proof. Let $[A]$ denote "the integral part of A".

Suppose that after $(n-1)$ divisions we are left with complete remainders $P \frac{Q}{R}$, where P is the next divisor and therefore R the last. Then

$$P < R, \quad Q < R.$$

But $$a_n = \left[\frac{R}{P}\right], \quad b_n = \left[\frac{Q}{P}\right].$$

Hence $$a_n \geqslant 1, \quad a_n \geqslant b_n.$$

THEOREM 2. *p_n, q_n, r_n are increasing functions of n (in the strict sense).*

Proof.
$$\begin{aligned} p_n &= a_n p_{n-1} + b_n p_{n-2} + p_{n-3} \\ &\geqslant a_n p_{n-1} + p_{n-3} \\ &\geqslant p_{n-1} + p_{n-3} \quad (\text{since } a_n \geqslant 1) \\ &> p_{n-1}. \end{aligned}$$

Similarly for q_n, r_n.

Corollary. $p_n, q_n, r_n \to \infty$ as $n \to \infty$ (since $p_n \geqslant 2p_{n-3}$).

8. Returning to the main line of argument, Bateman's proof consists in regarding the quantities p_n, q_n, r_n as the rectangular Cartesian coordinates of a point in three-dimensional space. Calling this point K_n, he notes that since $p_n = a_n p_{n-1} + b_n p_{n-2} + c_n p_{n-3}$, etc. (in our case $c_n = 1$), K_n is the centre of gravity of masses a_n, b_n, c_n at K_{n-1}, K_{n-2}, K_{n-3} respectively and a certain mass at O. He then shows that the solid angle $\overline{OK_n K_{n-1} K_{n-2}}$ tends to zero as $n \to \infty$. From this and the fact that OK_n lies within the solid angle $\overline{OK_n K_{n-1} K_{n-2}}$ he deduces that OK_n tends to a limiting position OK as $n \to \infty$.

This last deduction is not, however, valid,* since he omits to consider the possibility of the solid angle tending to a plane angle. This deficiency I propose to make good.

9. Consider the points J_n, J_{n+1}, ..., where OK_n, OK_{n+1}, ..., cut a fixed plane making positive intercepts on the axes. For simplicity we take the plane $x+y+z=1$.

Then J_n is the point

$$\left(\frac{p_n}{p_n+q_n+r_n}, \frac{q_n}{p_n+q_n+r_n}, \frac{r_n}{p_n+q_n+r_n}\right),$$

i.e. in vector notation $l_n J_n = K_n$, where $l_n = p_n + q_n + r_n$.

Note that, by Theorem 2, l_n is an increasing function of n.

Now
$$K_n = a_n K_{n-1} + b_n K_{n-2} + K_{n-3}.$$

Hence
$$l_n J_n = a_n l_{n-1} J_{n-1} + b_n l_{n-2} J_{n-2} + l_{n-3} J_{n-3},$$

or
$$J_n = \alpha_n J_{n-1} + \beta_n J_{n-2} + \gamma_n J_{n-3},$$

where

(1) $\alpha_n, \beta_n, \gamma_n \geqslant 0$,

(2) $\alpha_n + \beta_n + \gamma_n = 1$,

(3) $\alpha_n \geqslant \beta_n$,

(4) $\alpha_n \geqslant \gamma_n$.

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* This point was suggested to me by Professor J. E. Littlewood.

(1) and (2) are immediate, (3) and (4) follow thus—

$$\begin{aligned} a_n &= a_n \frac{l_{n-1}}{l_n} \\ &\geqslant b_n \frac{l_{n-1}}{l_n} \quad \text{(Theorem 1)} \\ &\geqslant b_n \frac{l_{n-2}}{l_n} \quad \text{(Theorem 2)} \\ &= \beta_n. \end{aligned}$$

$$\begin{aligned} a_n &= a_n \frac{l_{n-1}}{l_n} \\ &\geqslant \frac{l_{n-1}}{l_n} \quad \text{(Theorem 1, Cor.)} \\ &\geqslant \frac{l_{n-3}}{l_n} \quad \text{(Theorem 2)} \\ &= \gamma_n. \end{aligned}$$

10. Now let P, Q bisect $J_{n-1}J_{n-2}$ and $J_{n-1}J_{n-3}$ respectively, and let PJ_{n-2}, QJ_{n-3} intersect in R, and a paralliel through R to $J_{n-2}J_{n-3}$ cut $J_{n-1}J_{n-2}$, $J_{n-1}J_{n-3}$ in S, T respectively.

Since R is the centroid of the triangle $J_{n-1}J_{n-2}J_{n-3}$,

$$J_{n-1}S = \tfrac{2}{3} J_{n-1}J_{n-3}$$

and

$$J_{n-1}T = \tfrac{2}{3} J_{n-1}J_{n-3}.$$

But J_n lies inside $PRQJ_{n-1}$ (since $\alpha_n \geqslant \beta_n$, $a_n \geqslant \gamma_n$).

Hence $$J_{n-1}J_n \leqslant \tfrac{2}{3} \max (J_{n-1}J_{n-2}, J_{n-1}J_{n-3}).$$

Similarly
$$\begin{aligned} J_nJ_{n+1} &\leqslant \tfrac{2}{3} \max (J_nJ_{n-1}, J_nJ_{n-2}) \\ &\leqslant \tfrac{2}{3} \max (J_{n-1}J_{n-2}, J_{n-1}J_{n-3}, J_{n-2}J_{n-3}). \end{aligned}$$

So also
$$\begin{aligned} J_{n+1}J_{n+2} &\leqslant \tfrac{2}{3} \max (J_{n-1}J_n, J_{n-1}J_{n-2}, J_nJ_{n-2}) \\ &\leqslant \tfrac{2}{3} \max (J_{n-1}J_{n-2}, J_{n-1}J_{n-3}, J_{n-2}J_{n-3}) \end{aligned}$$

and
$$\begin{aligned} J_nJ_{n+2} &\leqslant \max (J_nJ_{n+1}, J_nJ_{n-1}) \\ &\leqslant \tfrac{2}{3} \max (J_{n-1}J_{n-2}, J_{n-1}J_{n-3}, J_{n-2}J_{n-3}). \end{aligned}$$

Thus the length of the greatest side of the triangle $J_nJ_{n-1}J_{n-2}$ is less than $k.(\frac{2}{3})^{n/3}$, where k is a constant, and so tends to zero as $n \to \infty$ *.

* This proof was suggested to me by Mr. H. D. Ursell. My original proof was considerably longer, as I used the unit sphere instead of the plane $x + y + z = 1$.

11. We have thus shown that the exceptional case is impossible and so completed Bateman's proof.

We can further show that the continued fraction actually converges to the ratios $1 : \theta : \phi$.

We have $$1 : \theta : \phi = p : q : r,$$

where $p\frac{q}{r}$ is the value of the continued fraction

$$\cdot \begin{matrix} b_1 \\ a_1 \end{matrix} \cdots b_{n-1} \begin{matrix} a_{n-1} \\ \cdot \end{matrix} 1 \begin{matrix} b_n+y \\ a_n+x \end{matrix},$$

b_n+y, a_n+x being the complete quotients after $n-1$ divisions and x, y being therefore less than unity.

Hence $$p = (a_n+x)\, p_{n-1}+(b_n+y)\, p_{n-2}+p_{n-3},$$

with similar formulae for q and r.

Therefore, if P be the point (p, q, r) and OP meet the plane $x+y+z = 1$ at J, P is the centre of gravity of masses a_n+x, b_n+y, 1 at K_{n-1}, K_{n-2}, K_{n-3}, and a certain mass at O; and J lies inside the triangle $J_{n-1}J_{n-2}J_{n-3}$.

J is, of course, a fixed point on the plane for all n and is therefore the point to which the angular points of the triangle converge.

That is, the fraction converges to $1\frac{\theta}{\phi}$.

In conclusion, I should like to express my thanks to Professor J. E. Littlewood for his kind interest in and criticism of this investigation, and also to Mr. H. D. Ursell, who has made many useful suggestions.

THE ISOLATED SINGULARITIES OF AN ALGEBRAIC SURFACE

By W. V. D. HODGE.

[Received 7 January, 1929.—Read 10 January, 1929.]

The theory of Puiseux for the singularities of curves has been extended to the isolated singularities of surfaces by Kobb*, in a memoir in which the following theorem is proved :—

The coordinates of any point of a surface in the neighbourhood of an isolated singularity can be expressed by means of a finite number of double series in two auxiliary variables, these series expressing the domains, on the surface, of points on certain curves into which the singularity may be decomposed.

The following paper is intended to illustrate the use which can be made of a Newton diagram in three dimensions in examining an isolated singularity, which we naturally take as the origin, on a surface, and also the singularities at infinity.

1. We construct a figure of straight lines parallel to three mutually perpendicular directions, forming a system of cubes whose edges are of unit length. Taking a corner of a cube as origin, and the three edges through it as axes $OXYZ$, we call the vertices of the cubes *unit points*, and mark in this figure a point corresponding to each term in the equation of the surface

$$f(x, y, z) = 0, \tag{1}$$

the unit point whose coordinates are (α, β, γ) representing the term in $x^\alpha y^\beta z^\gamma$. Such points we call *surface points*. Now, in the Newton diagram which we have formed, we can construct a convex polyhedron

* G. Kobb, *Journal de Math.* (4), 8 (1892), 385.

which has the properties :—

(1) there is no surface point outside it;

(2) each vertex of the polyhedron is a surface point.

Such a polyhedron is unique, and we shall refer to it as the *Newton polyhedron.*

Consider the transformation

$$\left.\begin{aligned} x &= \xi^l, \\ y &= \eta\xi^m, \\ z &= \zeta\xi^n, \end{aligned}\right\} \tag{2}$$

where l, m, n are positive integers.

$$x^\alpha y^\beta z^\gamma \quad \text{becomes} \quad \xi^{(l\alpha+m\beta+n\gamma)}\eta^\beta\zeta^\gamma,$$

$$f(x, y, z) = \xi^a[\eta^\rho\zeta^\sigma\phi(\eta, \zeta)+\xi\psi(\xi, \eta, \zeta)] = \xi^a F(\xi, \eta, \zeta). \tag{3}$$

A domain at the singularity can be presented in the form

$$z = \zeta\xi^n, \quad \zeta = g(\xi, \eta),$$

where $g(\xi, \eta)$ is a double series, and $g(0, \eta) \neq 0$ for a general value of η. The value of ζ when $\xi = 0$ is given by

$$\phi(\eta, \zeta) = 0, \tag{4}$$

which defines a curve (or curves) on the transformed surface. By expressing ζ as a function of ξ, η in the neighbourhood of any point of this curve, we obtain the development

$$\zeta = g(\xi, \eta),$$

and hence the required Puiseux series in a certain domain. The initial relation $\phi(\eta, \zeta) = 0$ imposes restrictions upon l, m, n which may be found in the following way.

The equation (4) determines ζ when $\xi = 0$, and as this is to be in general different from zero, there must be at least two terms in (4). Now the terms in (1) which give rise to the terms of (4) are of the form $x^\alpha y^\beta z^\gamma$, where

$$l\alpha+m\beta+n\gamma = a,$$

and for all other terms in (1),

$$l\alpha'+m\beta'+n\gamma' > a.$$

The terms of (1) which lead to (4) are therefore those which give rise to surface points in the Newton diagram in the plane

$$lX+mY+nZ = a,$$

which must contain at least two surface points, and be such that no surface points lie on the same side of it as the origin. The surface points concerned must lie on an edge, or in a face of the Newton polyhedron.

Consider first the case of terms on an edge. Let the two faces through the edge be

$$lX+mY+nZ = a,$$

$$pX+qY+rZ = b.$$

Then make the substitution

$$\left.\begin{aligned} x &= \xi^l \eta^p, \\ y &= \xi^m \eta^q, \\ z &= \xi^n \eta^r \zeta, \end{aligned}\right\} \tag{5}$$

The term $x^\alpha y^\beta z^\gamma$ becomes

$$\xi^{(l\alpha+m\beta+n\gamma)}\, \eta^{(p\alpha+q\beta+r\gamma)}\, \zeta^\gamma,$$

and the equation of the surface becomes

$$0 = f(x, y, z) = \xi^a \eta^b [\zeta^\rho \phi(\zeta)+\xi\psi_1+\eta\psi_2],$$

the terms of $\phi(\zeta)$ arising from the surface points on the edge considered. The equation

$$\phi(\zeta) = 0$$

determines ζ (different from zero) when $\xi = \eta = 0$, and we obtain a number of expansions of the form

$$\zeta = a+b_1\xi+b_2\eta+\ldots,$$

and hence obtain the double series required to represent the domains.

Next consider the case in which

$$lX+mY+nZ = a$$

is a face of the polyhedron separating the surface points from the origin. The equation (4)

$$\phi(\eta, \zeta) = 0$$

represents a curve for which a Newton polygon can be obtained from the polyhedron directly. Indeed, the curve points of

$$\eta^\rho \zeta^\sigma \phi(\eta, \zeta) = 0$$

in equation (3) are obtained by orthogonally projecting the surface points in the face considered on the plane $X = 0$, and its Newton polygon is

formed by the projection of the edges in the face. The removal of the facter $\eta^\rho \zeta^\sigma$ is equivalent to a translation of the polygon without rotation in order to bring a vertex on to each of the axes OY, OZ. At any point (λ, μ) of (4) we can expand ζ in the form

$$\zeta = \mu + (\eta - \lambda)^\nu [a_0 + a_1(\eta - \lambda) + \ldots],$$

and, as before, we deduce an expansion

$$z = \xi^n g(\xi, \eta).$$

We further see that the expansions valid in the neighbourhood of the origin or at infinity on the curve (4) lead to expansions in domains of the surface which are just those already obtained from the edges of the polyhedron which lie in the face considered.

The foregoing discussion enables us to see that an approximation to the shape of the surface at the origin is obtained by taking in turn the faces of the polyhedron which lie towards the origin. The terms in these give the equations of surfaces which show us the nature of the various sheets of the surface at the singularity.

To discuss the nature of the surface at infinity we use a similar method. To find the expansions corresponding to a face

$$lX + mY + nZ = a,$$

which lies away from the origin, we use the substitution

$$x = \xi^l,$$
$$y = \eta \xi^m,$$
$$z = \zeta \xi^\nu,$$

and we have

$$f(x, y, z) = \xi^a [\eta^\rho \zeta^\sigma \phi(\eta, \zeta) + \xi^{-1} \psi(\xi^{-1}, \eta, \zeta)],$$

and from this we obtain the expansion of $\zeta - \mu$ in terms of ξ^{-1}, $\eta - \lambda$ in the neighbourhood of a point (λ, μ) of $\phi(\eta, \zeta)$.

Similarly, if

$$lX + mY + nZ = a,$$
$$pX + qY + rZ = b$$

are two plane faces both lying away from the origin intersecting in an edge, by means of a substitution of the form (5) we have series representing the surface of the form

$$x = \xi^l \eta^p,$$
$$y = \xi^m \eta^q,$$
$$z = \xi^n \eta^r g(\xi^{-1}, \eta^{-1}),$$

where $g(\xi^{-1}, \eta^{-1})$ is a double series in $\xi^{-1}\eta^{-1}$ valid for large values of ξ, η, and $g(0, 0) \neq 0$.

It is to be noted that we may also have edges which are the intersections of faces, one of which lies towards the origin, the other away from it. If the former is

$$lX+mY+nZ = a,$$

and the latter

$$pX+qY+rZ = b,$$

we have in this case

$$x = \xi^l \eta^p,$$

$$y = \xi^m \eta^q,$$

$$z = \xi^n \eta^r g(\xi, \eta^{-1}),$$

where $g(0, 0) \neq 0$.

2. Professor Baker* has shown how we may make use of the Newton polygon of a curve in certain circumstances in the determination of the genus. The argument can be extended to surfaces, and, while its usefulness is not obvious in examining the case in which multiple curves exist on the surface, it is valuable in determining the effect of isolated singularities and of singularities at infinity on the geometric genus (p_g) of a surface.

Consider all the unit points which lie within the polyhedron. Each of these corresponds to a term $x^\alpha y^\beta z^\gamma$. Then

$$\sum_{\alpha\beta\gamma} A_{\alpha\beta\gamma} x^\alpha y^\beta z^\gamma \equiv xyz\psi$$

is a polynomial depending on parameters A whose number is equal to the number of surface points within the polyhedron. In general, if the surface (1)

$$f(x, y, z) = 0$$

is of order N, ψ is of order $N-4$, but in particular cases it may be of lower order. In this case $\psi = 0$ can be regarded as the equation of a surface of order $N-4$, of which the plane at infinity, counted a suitable number of times, forms part. We now show that when the coefficients of terms in (1), leading to surface points which lie in the faces of the polyhedron, are general

$$\iint \frac{\psi\, dy\, dz}{f_x} = \iint \frac{\psi\, dz\, dx}{f_y} = \iint \frac{\psi\, dx\, dy}{f_z}$$

remains finite at the origin and at infinity.

* H. F. Baker, *Trans. Camb. Phil. Soc.*, 15 (1894), 403–450.

We consider the origin, and first examine the integral in domains defined by surface points on an edge. If $t=1$ is a variable introduced to make the equation of the surface homogeneous, the integral can be written

$$-\iint \frac{xyz\psi}{f_t}\left(\frac{dy\,dz}{yz}+\frac{dz\,dx}{xy}+\frac{dx\,dy}{xy}\right).$$

Now the surface points of $f_t=0$ are just those of $f(x, y, z)=0$, other than those which arise from terms whose total order in x, y, z is N. Let the edge be the intersection of the faces

$$lX+mY+nZ=a,$$

$$pX+qY+rZ=b,$$

both of which, we suppose, separate the surface points from the origin.

Our transformation is

$$\left.\begin{aligned} x &= \xi^l \eta^p, \\ y &= \xi^m \eta^q, \\ z &= \xi^n \eta^r \zeta. \end{aligned}\right\} \tag{5}$$

Then $$\frac{xyz\psi}{f_t}=\frac{\xi^{a+s}\eta^{b+t}\psi_1}{\xi^a\eta^b\psi_2}=\xi^s\eta^t\frac{\psi_1}{\psi_2} \quad (s, t\geqslant 1),$$

where $\psi_2\neq 0$ when $\xi=\eta=0$. Also

$$\frac{dy\,dz}{yz}=\frac{d\xi\,d\eta}{\xi\eta\zeta}\begin{vmatrix} m & q \\ n\zeta+\zeta_1\xi & r\zeta+\zeta_2\eta \end{vmatrix}$$

$$\frac{dz\,dx}{zx}=\frac{d\xi\,d\eta}{\xi\eta\zeta}\begin{vmatrix} n\zeta+\zeta_1\xi & r\zeta+\zeta_2\eta \\ l & p \end{vmatrix}$$

$$\frac{dx\,dy}{xy}=\frac{d\xi\,d\eta}{\xi\eta}\begin{vmatrix} l & p \\ m & q \end{vmatrix},$$

where $\zeta_1=\dfrac{\partial\zeta}{\partial\xi}$, $\zeta_2=\dfrac{\partial\zeta}{\partial\eta}$. Also $\zeta\neq 0$ when $\xi, \eta=0$.

Hence, in the neighbourhood of $\xi=\eta=0$,

$$-\sum_{x, y, z}\frac{dy\,dz}{yz}=\frac{d\xi\,d\eta}{\xi\eta}\ (A+\text{positive powers of } \xi, \eta).$$

Hence

$$-\iint\frac{xyz\psi}{f_t}\Sigma\frac{dy\,dz}{yz}=\iint\frac{\psi_1}{\psi_2}\xi^{s-1}\eta^{t-1}\,d\xi\,d\eta\ (A+\text{positive powers}),$$

which remains finite in the neighbourhood of $\xi=\eta=0$.

To consider the behaviour of the integral in a domain defined by a face of the polyhedron which lies towards the origin, we make use of equations (2), (3), and (4) of § 1.

We make the transformation

$$\left.\begin{aligned} x &= \xi^l, \\ y &= \eta\xi^m, \\ z &= \zeta\xi^n, \end{aligned}\right\} \tag{3}$$

and now consider the integral

$$\iint \frac{\psi\, dx\, dy}{f_z}.$$

We have $$dx\, dy = l\xi^{l+m-1}\, d\xi\, d\eta.$$

Now $$f(x, y, z) = \xi^a F(\xi, \eta, \zeta), \tag{4}$$

therefore $$f_z = \xi^{a-n} F_\zeta.$$

Also $$\psi = \frac{xyz\psi}{\xi^{l+m+n}\eta\zeta} = \frac{\xi^{a+s}\eta\zeta\psi_1}{\xi^{l+m+n}\eta\zeta} \quad (s \geqslant 1).$$

And hence we have

$$\iint \frac{\psi\, dx\, dy}{f_z} = \iint \frac{\xi^{s-1}\psi_1 d\xi d\eta}{F_\zeta},$$

which is an Abelian integral on $F(\xi, \eta, \zeta) = 0$, which is finite for all finite values of ξ, η which are not multiple points of $F = 0$. The points in which we are interested are in the neighbourhood of $\xi = 0$. Hence, if this curve is simple for the surface $F = 0$, and has no multiple points other than at the origin or at infinity, this integral remains finite for all finite values of η, ζ for which ξ is small. We have already discussed the origin and infinity on $\xi = 0$ in our discussion of the domains arising from an edge. Hence we may say that if the coefficients are sufficiently general the integral remains finite in the neighbourhood of the origin.

A similar argument holds at infinity. For a branch arising from an edge we have now

$$-\iint \frac{xyz}{f_t}\Sigma\frac{dy\, dz}{yz} = \iint \frac{\psi_1}{\psi_2}\frac{d\xi\, d\eta}{\xi^{s+1}\eta^{t+1}}\,(A + \text{negative powers}),$$

and similarly for the faces, provided that l, m, n are not all equal. We must consider this case separately. For we see that in this case the surface points of (4) which lie in the face

$$X + Y + Z = a$$

lying away from the origin, are not surface points of $f_t = 0$. There is no difficulty for ordinary points (*i.e.* not at the origin or infinity) in the domain arising from this plane, and we need only consider the edges in this face. For such an edge we put

$$x = \xi\eta^p,$$
$$y = \xi\eta^q,$$
$$z = \xi\eta^r\zeta.$$

Then
$$\frac{dy\,dz}{yz} = \frac{d\xi\,d\eta}{\xi\eta\zeta}\{\zeta_2\eta + r\zeta - q\zeta_1\xi - q\zeta\},$$
$$\frac{dz\,dx}{zx} = \frac{d\xi\,d\eta}{\xi\eta\zeta}\{p\zeta + p\xi\zeta_1 - r\zeta - \eta\zeta_2\},$$
$$\frac{dx\,dy}{xy} = \frac{d\xi\,d\eta}{\xi\eta}\{q - p\}.$$

Hence
$$-\Sigma\frac{dy\,dz}{yz} = \frac{d\xi\,d\eta}{\eta\zeta}(p-q)\,\zeta_1.$$

Now
$$\zeta = a + \frac{b_1}{\xi} + \frac{b_2}{\eta} + \ldots,$$

so that
$$-\Sigma\frac{dy\,dz}{yz} = \frac{d\xi\,d\eta}{\xi^2\eta}(A + \text{negative powers}).$$

Also we now have
$$\frac{xyz\psi}{f_t} = \frac{1}{\eta^t}\frac{\psi_1(\xi^{-1}, \eta^{-1}, \zeta)}{\psi_2(\xi^{-1}, \eta^{-1}, \zeta)} \quad (t \geqslant 1),$$

where ψ_1 and ψ_2 remain finite as ξ, η tend to infinity. Again, the integral remains finite in the neighbourhood of ξ, η infinite, for it is of the form

$$\iint \frac{\psi_1}{\psi_2}\frac{d\xi\,d\eta}{\xi^2\eta^{t+1}}(A + \text{negative powers}) \quad (t \geqslant 1).$$

A similar argument applies to the case in which

$$X + Y + Z = a$$

lies away from the origin, while

$$pX + qY + rZ = b$$

separates the surface points from the origin.

Thus we have completely proved the theorem :—

Provided that the coefficients are sufficiently general, the unit points within the Newton polyhedron lead to polynomials of the form $xyz\psi$,

where $\psi = 0$ is of order $N-4$ and is adjoint to the surface at the origin and at infinity.

3. If the surface $f(x, y, z) = 0$ has no singularities other than at the origin or at infinity, and if, further, the nature of these singularities is fully represented in the Newton diagram, the number of unit points which lie within the Newton polyhedron will be exactly equal to the geometric genus p_g of the surface. More generally, if the Newton diagram fully represents an isolated singularity, we can at once determine the effect of this singularity on the genus p_g. We proceed to give some examples of this.

First consider a general r-fold point. The reduction in the genus due to this is found by calculating the number of unit points which lie within the tetrahedron formed by

$$X = 0,$$
$$Y = 0,$$
$$Z = 0,$$
$$X+Y+Z = r,$$

and the number which lie in the plane

$$X+Y+Z = r.$$

Hence we have just to find the number of positive integral solutions of

$$X+Y+Z \leqslant r,$$

which is known to be $\frac{1}{6}r(r-1)(r-2)$.

Now let us consider a few higher singularities. Consider a surface with a tacnode. If $z = 0$ is the tangent plane at this point, the equation of the surface is of the form

$$0 = z^2+zf_2+f_4+\sum_{\alpha+\beta+2\gamma>4} x^\alpha y^\beta z^\gamma,$$

where f_2 and f_4 are homogeneous functions of x and y of orders two and four respectively. The reduction in p_g due to the tacnode is equal, therefore, to the number of solutions of

$$X+Y+2Z \leqslant 4,$$

for which X, Y, and Z are positive (>0). There is one solution

$$X = Y = Z = 1,$$

and hence the tacnode reduces the genus by unity.

In the same way, if the singularity is formed by the contact of n branches (each touching $z=0$ at the origin), the equation of the surface is

$$0 = z^n + f_2 z^{n-1} + f_4 z^{n-2} + \ldots + f_{2n} + \sum_{\alpha+\beta+2\gamma > 2n} x^\alpha y^\beta z^\gamma,$$

and this singularity reduces the geometric genus by the number of solutions of

$$X + Y + 2Z \leqslant 2n$$

for which X, Y, Z are positive (>0).

When $Z=1$, there are $\frac{1}{2}(2n-2)(2n-3)$ solutions. When $Z=2$, there are $\frac{1}{2}(2n-4)(2n-5)$ solutions, and so on. Altogether, there are $\sum_{r=1}^{n}(r-1)(2r-3)$ solutions, that is

$$\tfrac{1}{6}n(n-1)(4n-5).$$

Again we can examine the behaviour of the adjoint surfaces of order $N-4$ at the singularity. In fact, we see that they have equations of the form

$$0 = z^{n-2}F_1 + z^{n-3}F_3 + \ldots + F_{2n-3} + \sum_{\alpha+\beta+2\gamma > 2n-3} x^\alpha y^\beta \gamma^\gamma.$$

As another example consider a singularity of the type which appears at infinity on the double plane

$$z^2 = f_{2N}(x, y).$$

If we take a surface with such a singularity at the origin, its equation can be written as

$$z^{2N-2} = f_{2N}(x, y, z),$$

where $f_{2N}(x, y, z)$ is a homogeneous polynomial in x, y, z of order $2N$. If there are no further singularities p_g is equal to the number of unit points between the coordinate planes and the planes

$$(N-1)(X+Y) + NZ = 2N(N-1)$$

and

$$X + Y + Z = 2N.$$

For $Z \leqslant N-1$ there are no unit points satisfying these conditions. If $Z=N$, there are $N-2$; $Z=N+1$, there are $N-3$, etc. Hence

$$p_g = \tfrac{1}{2}(N-1)(N-2),$$

and the adjoint surfaces ψ are of the form

$$z^{N-1}\phi(x, y, z) = 0,$$

where ϕ is homogeneous of degree $N-3$.

In many cases the foregoing theory enables us to determine the genus of a surface completely. We consider only one example—that of double planes. Let

$$z^2 = f_{2N}(x, y)$$

be any double plane. First suppose $f_{2N}(x, y) = 0$ to be a general curve. Constructing the Newton polyhedron, we see that all the unit points which lie within it lie in $Z = 1$, and the section of the polyhedron by this plane is the Newton polygon of a general curve of order N. Thus

$$p_g = \tfrac{1}{2}(N-1)(N-2).$$

Now suppose $f(x, y) = 0$ has multiple points. We know that it can be transformed so that these are all ordinary singularities of even order, and we suppose this is done. Consider one of order $2r$ at the origin. Then the section by $Z = 1$ of the Newton polyhedron is the Newton polygon of a curve of order N with an r-fold point at the origin. Thus the singularity reduces p_g by

$$\tfrac{1}{2}r(r-1).$$

Hence, we easily deduce that, if $f_{2N}(x, y) = 0$ is a curve of order $2N$ with a_r points of multiplicity $2r$ $(r = 1, 2, \ldots)$,

$$p_n = \tfrac{1}{2}(N-1)(N-2) - \tfrac{1}{2}\sum_r r(r-1)\, a_r,$$

while p_g is the number of linearly independent curves of order $N-3$, having $(r-1)$-fold points at the $2r$-fold multiple points of $f_{2N}(x, y) = 0$.

DISAPPEARING GIBBS'S PHENOMENA

Richard G. Cooke.

[Received 5 December, 1928.—Read 13 December, 1928.]

Introduction.

1. Gibbs's phenomenon in Cesàro sums of Fourier series and integrals has been studied by Cramér* and Carslaw† respectively.

The cardinal fact which emerges from both discussions is that when a Fourier series (or integral) is summed (C, k), there is a value of k between 0 and 1, say ϑ, such that Gibbs's phenomenon occurs at a point of ordinary discontinuity for $k < \vartheta$, and does not occur for $k \geqslant \vartheta$.

Both discussions, however, leave open the following important questions:

(1) When the Gibbs's phenomenon does occur, what is the Gibbs ratio?

(2) What is the numerical value of ϑ?

With regard to (2), Carslaw remarks‡ that from some work by Mr. R. J. Lyons he believes the value of ϑ to be about 0·4. This conclusion is borne out by the results of the present paper, the chief object of which is to treat the above two questions.

In addition, we shall prove that a somewhat similar "disappearing" Gibbs's phenomenon occurs in connexion with Schlömilch series of Struve functions, and also in connexion with Titchmarsh's integral§

$$\int_0^\infty \mathbf{H}_\nu(xu)\, u\, du \int_0^\infty Y_\nu(tu)\, t f(t)\, dt,$$

in each case at $x = 0$.

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* H. Cramér, "Études sur la sommation des séries de Fourier", *Arkiv för Math.*, 13 (1919), 20, §§ 5, 6, and 7.

† H. S. Carslaw, "Gibbs's phenomenon in the sum (C, r), for $r > 0$, of Fourier's integral", *Journal London Math. Soc.*, 1 (1926), 201–204.

‡ *Ibid.*, 204.

§ E. C. Titchmarsh, "Extensions of Fourier's integral formula to formulae involving Bessel functions", *Proc. London Math. Soc.* (2), 23 (1924), *Records* for 17 January, 1924.

Incidentally, various properties of functions associated with Bessel functions are obtained which do not appear to have been given before.

Gibbs's phenomenon in Cesàro sums of Fourier series and integrals.

2. From the papers by Cramér and Carslaw cited above, it is seen that the Gibbs ratio at a point of ordinary discontinuity in either the Fourier series or the Fourier integral when summed (C, k) will be obtained by evaluating the expression

$$G = \operatorname*{Max}_{\xi>0} \frac{2}{\pi} \int_0^{\xi} \left(1-\frac{s}{\xi}\right)^k \frac{\sin s}{s}\, ds \quad (k \geqslant 0).$$

This may be written in the form

$$G = \operatorname*{Max}_{\xi>0} \frac{2}{\pi} \Gamma(k+1) \int_0^{\xi} t^{-k-1} C_{k+1}(t)\, dt, \tag{2.1}$$

where $C_{k+1}(t)$ is Young's function*.

Let $c_1, c_2, c_3, \ldots, c_p, \ldots$ be the positive zeros, in ascending order, of $C_{k+1}(t)$.

THEOREM. *The Fourier series (or integral), when summed (C, k), has, at a point of ordinary discontinuity, a Gibbs ratio equal to*

$$\frac{2}{\pi} \Gamma(k+1) \int_0^{c_1} t^{-k-1} C_{k+1}(t)\, dt$$

if $0 \leqslant k \leqslant 0{\cdot}34$; *there is a Gibbs's phenomenon for* $0 \leqslant k \leqslant 0{\cdot}40$, *and none if* $k \geqslant 0{\cdot}48$, *so that* $0{\cdot}40 < \vartheta < 0{\cdot}48$.

3. *We first evaluate the integral*

$$\int_{c_p}^{\infty} t^{-k-1} C_{k+1}(t)\, dt$$

in terms of Lommel functions of the second kind.

We have†

$$f_1(\nu) \equiv \frac{d}{dz}\left\{C_{k+1}(z) \cdot z^{1-\nu} S_{\mu-1,\,\nu-1}(z)\right\}$$

$$= z^{1-\nu} C_k(z)\, S_{\mu-1,\,\nu-1}(z) + (\mu-\nu-1)\, z^{1-\nu} C_{k+1}(z)\, S_{\mu-2,\,\nu}(z),$$

$$f_2(\nu) \equiv \frac{d}{dz}\left\{C_k(z) \cdot z^{\nu} S_{\mu,\,\nu}(z)\right\}$$

$$= z^{\nu} C_{k-1}(z)\, S_{\mu,\,\nu}(z) + (\mu+\nu-1)\, z^{\nu} C_k(z)\, S_{\mu-1,\,\nu-1}(z).$$

* W. H. Young, "On infinite integrals involving a generalization of the sine and cosine functions", *Quart. J. of Math.*, 43 (1911), 161–177.

† G. N. Watson, *Theory of Bessel functions* (1922), 348 and 350.

Then, putting $\nu = \frac{1}{2}$, we have*

$$(\mu-\tfrac{1}{2})f_1(\tfrac{1}{2})-f_2(\tfrac{1}{2})$$
$$= (\mu-\tfrac{1}{2})(\mu-\tfrac{3}{2})z^{\frac{1}{2}}C_{k+1}(z)\,S_{\mu-2,\frac{1}{2}}(z)-z^{\frac{1}{2}}\left\{\frac{z^{k-1}}{\Gamma(k)}-C_{k+1}(z)\right\}S_{\mu,\frac{1}{2}}(z).$$

But† $$S_{\mu-2,\frac{1}{2}}(z) = \frac{z^{\mu-1}-S_{\mu,\frac{1}{2}}(z)}{(\mu-1)^2-\frac{1}{4}};$$

hence $$(\mu-\tfrac{1}{2})f_1(\tfrac{1}{2})-f_2(\tfrac{1}{2}) = z^{\mu-\frac{1}{2}}C_{k+1}(z)-\frac{z^{k-\frac{1}{2}}}{\Gamma(k)}S_{\mu,\frac{1}{2}}(z).$$

Now put $\mu = -k-\frac{1}{2}$, and we obtain the result

$$\int_{c_p}^{\infty} t^{-k-1}C_{k+1}(t)\,dt = c_p^{\frac{1}{2}}C_k(c_p)\,S_{-k-\frac{1}{2},\frac{1}{2}}(c_p)+\frac{1}{\Gamma(k)}\int_{c_p}^{\infty} t^{k-\frac{1}{2}}S_{-k-\frac{1}{2},\frac{1}{2}}(t)\,dt. \quad (3.1)$$

4. We next obtain some information concerning *the zeros of the function* $C_{k+1}(z)$, $(0 \leqslant k < 1)$.

Since‡ $$C_{k+1}(z) = \frac{z^k}{\Gamma(k)}\int_0^1 (1-t)^{k-1}\sin tz\,dt, \quad (4.1)$$

it follows that $c_1 > \pi$, since the integrand is positive for $z < \pi$.

But we have§

$$\Gamma(k-1)\,C_k(z) = z^{\frac{1}{2}}s_{k-\frac{3}{2},\frac{1}{2}}(z)$$
$$= z^{\frac{1}{2}}S_{k-\frac{3}{2},\frac{1}{2}}(z)+2^{k-\frac{5}{2}}\cdot z^{\frac{1}{2}}\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{k-1}{2}\right)\{\cos\tfrac{1}{2}k\pi\,J_{-\frac{1}{2}}(z)+\sin\tfrac{1}{2}k\pi\,J_{\frac{1}{2}}(z)\}$$
$$= z^{\frac{1}{2}}S_{k-\frac{3}{2},\frac{1}{2}}(z)+\Gamma(k-1)\cos(z-\tfrac{1}{2}k\pi), \quad (4.2)$$

a formula which will be subsequently employed; here $s_{\mu,\nu}(z)$ is Lommel's function of the first kind. For our immediate purpose, change k into $k+1$, and we have

$$C_{k+1}(z) = \sin(z-\tfrac{1}{2}k\pi)+z^{\frac{1}{2}}S_{k-\frac{1}{2},\frac{1}{2}}(z)/\Gamma(k). \quad (4.3)$$

This supplies an asymptotic expansion of the function $C_{k+1}(z)$, since that of $S_{k-\frac{1}{2},\frac{1}{2}}(z)$ is known‖. Hence

$$c_p = p\pi+\tfrac{1}{2}k\pi\pm\chi_p \quad (\text{since } c_1 > \pi),$$

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* See W. H. Young, *loc. cit.*, 163.
† G. N. Watson, *loc. cit.*, 348 (6).
‡ W. H. Young, *loc. cit.*, 163 (5).
§ G. N. Watson, *loc. cit.*, 540 (8) and (9), and 347 (2).
‖ *Ibid.*, 351.

where
$$\chi_p = \arcsin\left\{\frac{c_\eta^{\frac{1}{2}}}{\Gamma(k)}\, S_{k-\frac{1}{2},\,\frac{1}{2}}(c_p)\right\}, \tag{4.4}$$

the plus or minus sign being taken according as p is odd or even.

Also it is easy to see that χ_p decreases steadily as p increases; for*

$$\frac{d}{dz}\left\{z^{\frac{1}{2}} S_{\mu,\,\nu}(z)\right\} = (\mu-\tfrac{1}{2}) z^{\mu-\frac{3}{2}}$$
$$-\frac{1}{2\pi i}\int_{-\infty i-\frac{1}{2}}^{\infty i-\frac{1}{2}} \frac{\Gamma(\frac{1}{2}-\frac{1}{2}\mu+\frac{1}{2}\nu-s)\,\Gamma(\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu-s)}{\Gamma(\frac{1}{2}-\frac{1}{2}\mu+\frac{1}{2}\nu)\,\Gamma(\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu)} \cdot \frac{\pi(2s+\mu-\frac{1}{2})\, z^{2s+\mu-\frac{3}{2}}\, ds}{2^{2s}\sin s\pi}$$
$$\equiv (\mu-\tfrac{1}{2}) z^{\mu-\frac{3}{2}} + \theta(z), \quad \text{say.}$$

Then, since $|2s+\mu-\frac{1}{2}| \leqslant 2|t|+|\mu-\frac{3}{2}|$, where $s = it-\frac{1}{2}$, and

$$\int_{-\infty}^{\infty} \frac{|t|\,dt}{\cosh \pi t} = \frac{4}{\pi^2}\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{(2m-1)^2} < \frac{4}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2} = \tfrac{1}{2},$$

while
$$\int_{-\infty}^{\infty} \frac{dt}{\cosh \pi t} = 1,$$

we have
$$|\theta(z)| \leqslant (\tfrac{5}{2}-\mu) z^{\mu-\frac{3}{2}} \cdot \frac{\Gamma(1-\frac{1}{2}\mu+\frac{1}{2}\nu)\,\Gamma(1-\frac{1}{2}\mu-\frac{1}{2}\nu)}{\Gamma(\frac{1}{2}-\frac{1}{2}\mu+\frac{1}{2}\nu)\,\Gamma(\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu)}. \tag{4.5}$$

Hence, putting $\mu = k-\frac{1}{2}$, $\nu = \frac{1}{2}$, we get, since the quotient of Gamma functions in (4.5) reduces to $\frac{1}{2}(1-k)$,

$$\frac{d}{dz}\left\{z^{\frac{1}{2}} S_{k-\frac{1}{2},\,\frac{1}{2}}(z)\right\} \leqslant (1-k)\left\{-z^{k-2}+\tfrac{1}{2}(3-k)\, z^{k-3}\right\} < 0, \tag{4.6}$$

when $z \geqslant \pi$, $0 \leqslant k < 1$.

Hence, since $c_1 > \pi$, χ_p decreases steadily as p increases.

Also $\chi_p > 0$; for†

$$S_{\mu,\,\nu}(z) = z^{\mu-1} - \left\{(\mu-1)^2-\nu^2\right\} z^{\mu-3}$$
$$-\frac{z^{\mu-1}}{2\pi i}\int_{-\infty i-\frac{3}{2}}^{\infty i-\frac{3}{2}} \frac{\Gamma(\frac{1}{2}-\frac{1}{2}\mu+\frac{1}{2}\nu-s)}{\Gamma(\frac{1}{2}-\frac{1}{2}\mu+\frac{1}{2}\nu)}\,\frac{\Gamma(\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu-s)}{\Gamma(\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu)} \cdot \frac{\pi(\frac{1}{2}z)^{2s}\, ds}{\sin s\pi}$$
$$\equiv z^{\mu-1} - \left\{(\mu-1)^2-\nu^2\right\} z^{\mu-3} + \psi(z), \quad \text{say.}$$

Putting $s = it-\frac{3}{2}$, we deduce, as before, that

$$|\psi(z)| \leqslant 4 z^{\mu-4} \cdot \frac{\Gamma(2-\frac{1}{2}\mu+\frac{1}{2}\nu)\,\Gamma(2-\frac{1}{2}\mu-\frac{1}{2}\nu)}{\Gamma(\frac{1}{2}-\frac{1}{2}\mu+\frac{1}{2}\nu)\,\Gamma(\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu)}. \tag{4.7}$$

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* G. N. Watson, *loc. cit.*, 351.

† *Ibid.*, 351.

Put $\mu = k-\frac{1}{2}$, $\nu = \frac{1}{2}$, and we have, when $z \geqslant \pi$, $0 \leqslant k < 1$,

$$S_{k-\frac{1}{2},\frac{1}{2}}(z) = z^{k-\frac{3}{2}} - (1-k)(2-k)\, z^{k-\frac{7}{2}} + \psi(z),$$

where $$|\psi(z)| \leqslant \tfrac{1}{2}(1-k)(2-k)(3-k)\, z^{k-\frac{7}{2}},$$

so that $$0 < S_{k-\frac{1}{2},\frac{1}{2}}(z) < z^{k-\frac{3}{2}}. \tag{4.8}$$

Thus, since $S_{k-\frac{1}{2},\frac{1}{2}}(c_p) > 0$, it follows from (4.4) that $\chi_p > 0$; and consequently, as a special case of (4.4),

$$c_1 > (1+\tfrac{1}{2}k)\,\pi. \tag{4.9}$$

5. *We now prove that when* $0 \leqslant k \leqslant 0{\cdot}40$, *then*

$$\int_{c_1}^{\infty} t^{-k-1} C_{k+1}(t)\, dt < 0.$$

In (4.7), put $\mu = -k-\frac{1}{2}$, $\nu = \frac{1}{2}$, and we have, as for (4.8), when $z \geqslant \pi$, $0 \leqslant k < 1$,

$$0 < S_{-k-\frac{1}{2},\frac{1}{2}}(z) < z^{-k-\frac{3}{2}}. \tag{5.1}$$

Thus $$\int_{c_1}^{\infty} t^{k-\frac{1}{2}} S_{-k-\frac{1}{2},\frac{1}{2}}(t)\, dt < \int_{c_1}^{\infty} t^{-2}\, dt < c_1^{-1}.$$

Then, by (3.1), (4.2), (4.4), and (5.1), we have

$$\begin{aligned}\int_{c_1}^{\infty} t^{-k-1} C_{k+1}(t)\, dt &< \frac{c_1^{-1}}{\Gamma(k)} - \cos\chi_1 \cdot c_1^{\frac{1}{2}} S_{-k-\frac{1}{2},\frac{1}{2}}(c_1) - \frac{(1-k)}{\Gamma(k)}\, c_1 S_{k-\frac{3}{2},\frac{1}{2}}(c_1)\, S_{-k-\frac{1}{2},\frac{1}{2}}(c_1) \\ &< \frac{c_1^{-1}}{\Gamma(k)} - \cos\chi_1 \cdot \{c_1^{-k-1} - \tfrac{1}{2}(1+k) c_1^{-k-2}\}\,; \end{aligned} \tag{5.2}$$

for, (i) put $\mu = k-\frac{3}{2}$, $\nu = \frac{1}{2}$, in (4.7); then

$$S_{k-\frac{3}{2},\frac{1}{2}}(z) = z^{k-\frac{5}{2}} - (k-2)(k-3)\, z^{k-\frac{9}{2}} + \psi(z),$$

where $$|\psi(z)| \leqslant \tfrac{1}{2}(2-k)(3-k)(4-k) z^{k-\frac{11}{2}},$$

so that, when $z \geqslant \pi$ and $0 \leqslant k < 1$,

$$0 < S_{k-\frac{3}{2},\frac{1}{2}}(z) < z^{k-\frac{5}{2}}; \tag{5.3}$$

and (ii), we have*

$$\begin{aligned} S_{\mu,\nu}(z) &= z^{\mu-1} - \frac{z^{\mu-1}}{2\pi i} \int_{-\infty i-\frac{1}{2}}^{\infty i-\frac{1}{2}} \frac{\Gamma(\frac{1}{2}-\frac{1}{2}\mu+\frac{1}{2}\nu-s)\,\Gamma(\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu-s)}{\Gamma(\frac{1}{2}-\frac{1}{2}\mu+\frac{1}{2}\nu)\,\Gamma(\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu)} \cdot \frac{\pi(\frac{1}{2}z)^{2s}\, ds}{\sin s\pi} \\ &\equiv z^{\mu-1} - \psi(z), \end{aligned}$$

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* G. N. Watson, *loc. cit.*, 351.

where $$|\psi(z)| \leqslant z^{\mu-2} \cdot \frac{\Gamma(1-\frac{1}{2}\mu+\frac{1}{2}\nu)\,\Gamma(1-\frac{1}{2}\mu-\frac{1}{2}\nu)}{\Gamma(\frac{1}{2}-\frac{1}{2}\mu+\frac{1}{2}\nu)\,\Gamma(\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu)}, \tag{5.4}$$

precisely as before; whence, putting $\mu = -k-\frac{1}{2}$, $\nu = \frac{1}{2}$,

$$S_{-k-\frac{1}{2},\frac{1}{2}}(z) \geqslant z^{-k-\frac{3}{2}} - \tfrac{1}{2}(1+k)\, z^{-k-\frac{5}{2}}. \tag{5.5}$$

(5.2) is an immediate deduction from (5.1), (5.3), and (5.5).

But when $0 \leqslant k \leqslant 0{\cdot}40$ we have, by (4.4) and (4.8),

$$\chi_1 < \text{arc sin}\left\{\frac{1}{\pi^{0{\cdot}6}\,\Gamma(0{\cdot}4)}\right\} < 0{\cdot}229. \tag{5.6}$$

Thus $$c_1 = (1+\tfrac{1}{2}k)\,\pi + \chi_1 < (1+\tfrac{1}{2}k)\,\pi + 0{\cdot}229. \tag{5.7}$$

Also $$\cos\chi_1 > 0{\cdot}9739. \tag{5.8}$$

Hence, by (5.2),

$$\int_{c_1}^{\infty} t^{-k-1} C_{k+1}(t)\,dt < c_1^{-1}\left[\frac{1}{\Gamma(k)} - 0{\cdot}9739\left\{\frac{1}{(\pi+\frac{1}{2}k\pi+0{\cdot}229)^k} - \frac{\frac{1}{2}(1+k)}{(\pi+\frac{1}{2}k\pi+0{\cdot}229)^{k+1}}\right\}\right],$$

and consequently $$\int_{c_1}^{\infty} t^{-k-1} C_{k+1}(t)\,dt < 0,$$

provided that

$$\frac{1}{\Gamma(k)} < 0{\cdot}9739\left[\frac{1}{\{(1+\frac{1}{2}k)\,\pi+0{\cdot}229\}^k} - \frac{\frac{1}{2}(1+k)}{\{(1+\frac{1}{2}k)\,\pi+0{\cdot}229\}^{k+1}}\right], \tag{5.9}$$

and this is satisfied by $k = 0{\cdot}40$, and hence for $0 \leqslant k \leqslant 0{\cdot}40$, since the left-hand side of (5.9) increases as k increases from 0 to 0·40, whilst the right-hand side decreases.

This proves the result stated, which shows that *there is a Gibbs's phenomenon for* $0 \leqslant k \leqslant 0{\cdot}40$; for there is certainly a Gibbs's phenomenon when

$$\int_0^{c_1} t^{-k-1}\, C_{k+1}(t)\,dt > \int_0^{\infty} t^{-k-1}\, C_{k+1}(t)\,dt,$$

since* $$\frac{2}{\pi}\,\Gamma(k+1)\int_0^{\infty} t^{-k-1}\, C_{k+1}(t)\,dt = 1.$$

6. *We next prove that when* $k \geqslant 0{\cdot}48$ *and* $p \geqslant 1$, *then*

$$\int_{c_p}^{\infty} t^{-k-1} C_{k+1}(t)\,dt > 0,$$

so that there will be no Gibbs's phenomenon for $k \geqslant 0{\cdot}48$.

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* W. H. Young, *loc. cit.*, 162, last formula.

To effect this, we first evaluate an indefinite integral which is useful, not only in this connexion but also subsequently, in connexion with Struve functions. Let

$$\int^z z^{\nu+1} f(z)\, s_{\mu,\nu}(z)\, dz \equiv z^{\nu+1}\{F(z)\, s_{\mu,\nu}(z)+G(z)\, s_{\mu+1,\nu+1}(z)\} - E(z),$$

where $s_{\mu,\nu}(z)$ is Lommel's function of the first kind. Then, differentiating, we have*

$$\begin{aligned} z^{\nu+1} f(z)\, s_{\mu,\nu}(z) \equiv z^{\nu+1}\Big[& F'(z)\, s_{\mu,\nu}(z)+G'(z)\, s_{\mu+1,\nu+1}(z) \\ & +F(z)\Big\{(\mu-\nu-1)\, s_{\mu-1,\nu+1}(z)+\frac{\nu}{z}\, s_{\mu,\nu}(z)\Big\} \\ & +G(z)\Big\{(\mu+\nu+1)\, s_{\mu,\nu}(z)-\frac{\nu+1}{z}\, s_{\mu+1,\nu+1}(z)\Big\} \\ & +\frac{\nu+1}{z}\{F(z)\, s_{\mu,\nu}(z)+G(z)\, s_{\mu+1,\nu+1}(z)\}\Big]-E'(z). \end{aligned}$$

Choose $$F(z) \equiv (\mu+\nu+1)\, G'(z);$$

then, in virtue of the relation†

$$s_{\mu+1,\nu+1}(z) = z^{\mu} - \{\mu^2-(\nu+1)^2\}\, s_{\mu-1,\nu+1}(z),$$

the terms involving $s_{\mu+1,\nu+1}(z)$ disappear.

Again, choose $$E'(z) \equiv z^{\mu+\nu+1} F(z)/(\mu+\nu+1),$$

and the terms independent of Lommel functions then cancel.

We then have

$$f(z) = F'(z)+\frac{2\nu+1}{z}\, F(z)+(\mu+\nu+1)\, G(z).$$

Hence we obtain the formula

$$\begin{aligned} & (\mu+\nu+1)\int^z z^{\nu+1}\Big\{G''(z)+\frac{2\nu+1}{z}\, G'(z)+G(z)\Big\}\, s_{\mu,\nu}(z)\, dz \\ & = z^{\nu+1}\{(\mu+\nu+1)\, G'(z)\, s_{\mu,\nu}(z)+G(z)\, s_{\mu+1,\nu+1}(z)\} - \int^z z^{\mu+\nu+1} G'(z)\, dz. \quad (6.1) \end{aligned}$$

We give two special cases of (6.1):

(i) Since‡ $$s_{\nu,\nu}(z) = 2^{\nu-1}\Gamma(\tfrac{1}{2})\,\Gamma(\nu+\tfrac{1}{2})\,\mathbf{H}_{\nu}(z),$$

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* G. N. Watson, *loc. cit.*, 348 (2) and (3).

† *Ibid.*, 348 (1).

‡ *Ibid.*, 345 and 328.

where $\mathbf{H}_\nu(z)$ is Struve's function, we have

$$\int^z z^{\nu+1}\left\{G''(z)+\frac{2\nu+1}{z}G'(z)+G(z)\right\}\mathbf{H}_\nu(z)\,dz$$

$$=z^{\nu+1}\{G'(z)\,\mathbf{H}_\nu(z)+G(z)\,\mathbf{H}_{\nu+1}(z)\}-\frac{1}{2^\nu\Gamma(\frac{1}{2})\,\Gamma(\nu+\frac{3}{2})}\int^z z^{2\nu+1}G'(z)\,dz. \quad (6.2)$$

(ii) Since $$\Gamma(k)\,C_{k+1}(z)=z^{\frac{1}{2}}s_{k-\frac{1}{2},\frac{1}{2}}(z),$$

we have

$$(k+1)\,\Gamma(k)\int^z z\left\{G''(z)+\frac{2}{z}G'(z)+G(z)\right\}C_{k+1}(z)\,dz$$

$$=z\{(k+1)\,G'(z)\,\Gamma(k)\,C_{k+1}(z)+G(z)\,z^{\frac{1}{2}}s_{k+\frac{1}{2},\frac{3}{2}}(z)\}-\int^z z^{k+1}G'(z)\,dz.$$

But $$C_{k+2}(z)=\int^z C_{k+1}(z)\,dz=\int^z\frac{z^{\frac{1}{2}}}{\Gamma(k)}s_{k-\frac{1}{2},\frac{1}{2}}(z)\,dz=\frac{z^{\frac{1}{2}}}{\Gamma(k+1)}s_{k+\frac{1}{2},-\frac{1}{2}}(z)^*$$

$$=\frac{z^{\frac{1}{2}}}{\Gamma(k+1)}\left\{\frac{k}{k+1}s_{k+\frac{1}{2},\frac{3}{2}}(z)+\frac{1}{z(k+1)}s_{k+\frac{3}{2},\frac{1}{2}}(z)\right\}^\dagger,$$

whence $$z^{\frac{1}{2}}s_{k+\frac{1}{2},\frac{3}{2}}(z)=(k+1)\,\Gamma(k)\,C_{k+2}(z)-\frac{\Gamma(k+2)}{kz}C_{k+3}(z),$$

so that

$$\int^z z\left\{G''(z)+\frac{2}{z}G'(z)+G(z)\right\}C_{k+1}(z)\,dz$$

$$=zG(z)\,C_{k+2}(z)+\{zG'(z)+G(z)\}\,C_{k+1}(z)$$

$$-\frac{1}{(k+1)\,\Gamma(k)}\left\{\frac{z^{k+1}}{k}G(z)+\int^z z^{k+1}G'(z)\,dz\right\}, \quad (6.3)$$

since‡ $$C_{k+3}(z)=\frac{z^{k+1}}{\Gamma(k+2)}-C_{k+1}(z). \quad (6.4)$$

In (6.3) put $G(z)\equiv z^{-k-2}$; then we obtain the result

$$\int_{c_p}^\infty t^{-k-3}\{t^2+(k+1)(k+2)\}\,C_{k+1}(t)\,dt=c_p^{-k-1}\left\{C_k(c_p)+\frac{c_p^k}{\Gamma(k)}\right\}, \quad (6.5)$$

by (6.4) with $k-1$ in place of k.

But, by (4.2) and (5.3),

$$C_k(z)>-1-\frac{(1-k)\,z^{k-2}}{\Gamma(k)};$$

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* G. N. Watson, *loc. cit.*, 350.

† *Ibid.*, 348 (4).

‡ W. H. Young, *loc. cit.*, 163.

hence, by (6 . 5),

$$\int_{c_p}^{\infty} t^{-k-1}\left\{1+\frac{(k+1)(k+2)}{t^2}\right\} C_{k+1}(t)\,dt > c_p^{-k-1}\left\{\frac{c_p^k-(1-k)\,c_p^{k-2}}{\Gamma(k)}-1\right\} \quad (6.6)$$

$$> c_p^{-k-1}\left[\frac{\{(1+\frac{1}{2}k)\,\pi\}^k-(1-k)\{(1+\frac{1}{2}k)\,\pi\}^{k-2}}{\Gamma(k)}-1\right],$$

by (4 . 9), since $c_p^k-(1-k)\,c_p^{k-2}$ increases as p increases.

This is positive when $k = 0{\cdot}48$, and so is positive for $0{\cdot}48 \leqslant k \leqslant 1$, since $\Gamma(k)$ decreases as k increases up to unity, while

$$\{(1+\tfrac{1}{2}k)\,\pi\}^k-(1-k)\{(1+\tfrac{1}{2}k)\,\pi\}^{k-2}$$

increases.

Hence, by the second mean-value theorem,

$$\int_{c_p}^{\infty} t^{-k-1}\,C_{k+1}(t)\,dt$$

$$=\int_{c_p}^{\infty}\left\{1+\frac{(k+1)(k+2)}{t^2}\right\}^{-1} t^{-k-1}\left\{1+\frac{(k+1)(k+2)}{t^2}\right\} C_{k+1}(t)\,dt$$

$$=\int_{\xi}^{\infty} t^{-k-1}\left\{1+\frac{(k+1)(k+2)}{t^2}\right\} C_{k+1}(t)\,dt \quad (\xi > c_p),$$

and this is positive; for, if it were negative for any value of ξ, it would be so when $\xi = c_{2q-1}$ (an odd zero); and this is not the case, as we have proved above.

This proves the result stated.

7. In this section we prove that, *if p satisfies the inequality*

$$\Gamma(k) > c_{2p-1}^k\left\{1-(1-k)\,c_{2p-1}^{-2}\right\} \quad (7.1)$$

for $p \geqslant 1$ and $0 \leqslant k \leqslant 0{\cdot}34$, then

$$I_q \equiv \int_{c_{2q-1}}^{\infty} t^{-k-1}\,C_{k+1}(t)\,dt,$$

where $1 \leqslant q \leqslant p$, is a steadily increasing sequence.

By (4 . 2) and (4 . 4),

$$C_k(c_{2p-1}) = \cos\{(2p-1)\,\pi+\chi_{2p-1}\}+c_{2p-1}^{\frac{1}{2}}\,S_{k-\frac{3}{2},\frac{1}{2}}(c_{2p-1})/\Gamma(k-1)$$

$$= -\cos\chi_{2p-1}-\frac{(1-k)}{\Gamma(k)}\,c_{2p-1}^{\frac{1}{2}}\,S_{k-\frac{3}{2},\frac{1}{2}}(c_{2p-1}). \quad (7.2)$$

Also $c_1 > \pi$.

Consequently, by (3 . 1) and (4 . 4), if

$$\Theta(z) \equiv z^{\frac{1}{2}}\,S_{-k-\frac{1}{2},\frac{1}{2}}(z)\left[-\left\{1-\frac{z}{[\Gamma(k)]^2}\,S^2_{k-\frac{1}{2},\frac{1}{2}}(z)\right\}^{\frac{1}{2}}-\frac{(1-k)}{\Gamma(k)}\,z^{\frac{1}{2}}\,S_{k-\frac{3}{2},\frac{1}{2}}(z)\right]$$

$$+\frac{1}{\Gamma(k)}\int_z^{\infty} t^{k-\frac{1}{2}}\,S_{-k-\frac{1}{2},\frac{1}{2}}(t)\,dt,$$

it is sufficient to prove that $\Theta'(z) > 0$ when $z \geqslant \pi$, $0 \leqslant k \leqslant 0{\cdot}34$, and (7 . 1) is satisfied.

Now we have*

$$\Theta'(z) > (k+1)\, z^{\frac{1}{2}} S_{-k-\frac{3}{2},\frac{1}{2}}(z)\left[\left\{1-\frac{z}{[\Gamma(k)]^2} S^2_{k-\frac{1}{2},\frac{1}{2}}(z)\right\}^{\frac{1}{2}} + \frac{(1-k)}{\Gamma(k)} z^{\frac{1}{2}} S_{k-\frac{3}{2},\frac{1}{2}}(z)\right]$$

$$+ z^{\frac{1}{2}} S_{-k-\frac{1}{2},\frac{1}{2}}(z)\left[\frac{(1-k)(2-k)}{\Gamma(k)} z^{\frac{1}{2}} S_{k-\frac{5}{2},-\frac{1}{2}}(z)\right.$$

$$\left. + \frac{(k-1)z}{\{\Gamma(k)\}^2} S_{k-\frac{1}{2},\frac{1}{2}}(z)\, S_{k-\frac{3}{2},\frac{1}{2}}(z) \Big/ \left\{1-\frac{z}{[\Gamma(k)]^2} S^2_{k-\frac{1}{2},\frac{1}{2}}(z)\right\}^{\frac{1}{2}}\right]$$

$$- \frac{z^{k-\frac{1}{2}}}{\Gamma(k)} S_{-k-\frac{1}{2},\frac{1}{2}}(z). \qquad (7 . 3)$$

For the sake of clearness, the argument is divided into four parts; in all of them, (7 . 1) is assumed to hold.

(i) $$f(z) \equiv (k+1)\, z^{\frac{1}{2}} S_{-k-\frac{3}{2},\frac{1}{2}}(z) - \frac{z^{k-\frac{1}{2}}}{\Gamma(k)} S_{-k-\frac{1}{2},\frac{1}{2}}(z)$$

$$> \frac{z^k}{\Gamma(k)}\left[\left\{1-(1-k)\, z^{-2}\right\}(k+1)\, z^{\frac{1}{2}} S_{-k-\frac{3}{2},\frac{1}{2}}(z) - z^{-\frac{1}{2}} S_{-k-\frac{1}{2},\frac{1}{2}}(z)\right],$$

by (7 . 1);

$$> \frac{z^k}{\Gamma(k)}\left[kz^{-k-2} - (1+k)(1+\tfrac{1}{2}k)\, z^{-k-3} - (1-k^2)\, z^{-k-4}\right], \qquad (7 . 4)$$

by putting $\mu = -k-\frac{3}{2}$, $\nu = \frac{1}{2}$, in (5 . 4), and using (4 . 8) with $-k$ in place of k.

Hence $$f(z) > \frac{z^{-2}}{\Gamma(k)}\left[k-(1+k)(1+\tfrac{1}{2}k)\, z^{-1} - (1-k^2)\, z^{-2}\right]$$

$$> 0$$

when $z > c_3 > (3+\frac{1}{2}k)\pi$ [as for (4 . 9)], and $k > \frac{1}{8}$.

Consider next the first square bracket in (7 . 3).

We have, when $0 < x < 1$,

$$(1-x)^{\frac{1}{2}} = 1 - \tfrac{1}{2}x - \tfrac{1}{4}x^2 \int_0^1 (1-t)(1-xt)^{-\frac{3}{2}}\, dt.$$

Hence $$(1-\tfrac{1}{2}x) - (1-x)^{\frac{1}{2}} < \tfrac{1}{8}x^2(1-x)^{-\frac{3}{2}} < \tfrac{1}{64}x,$$

provided that $$x^3 + 61x^2 + 3x < 1. \qquad (7 . 5)$$

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* G. N. Watson, *loc. cit.*, 348.

This is satisfied by

$$x = \frac{z}{\{\Gamma(k)\}^2} S^2_{k-\frac{1}{2},\frac{1}{2}}(z) < \frac{z^{2k-2}}{\{\Gamma(k)\}^2},$$

when $z > c_3 > 3\pi$, by (4.8); *i.e.*, by

$$x < (3\pi)^{-2}\{1-(1-k)(3\pi)^{-2}\}^{-2} < 0{\cdot}114\pi^{-2}, \text{ by (7.1)}.$$

Also, by (7.1), $$\frac{1}{\Gamma(k)} < z^{-k}\{1+\tfrac{5}{4}(1-k)z^{-2}\}, \qquad (7.6)$$

since $$\tfrac{1}{4}(1-k)z^{-2} - \tfrac{5}{4}(1-k)^2 z^{-4} > 0 \quad (z \geqslant \pi).$$

Again, in (5.4) put $\mu = k-\frac{3}{2}$, $\nu = \frac{1}{2}$; then

$$S_{k-\frac{3}{2},\frac{1}{2}}(z) \geqslant z^{k-\frac{3}{2}} - (1-\tfrac{1}{2}k)\, z^{k-\frac{5}{2}}.$$

Hence the first square bracket in (7.3) is greater than

$$1 - \tfrac{3}{6}\tfrac{3}{4}\cdot\frac{z}{\{\Gamma(k)\}^2} S^2_{k-\frac{1}{2},\frac{1}{2}}(z) + \frac{(1-k)}{\Gamma(k)}\{z^{k-2} - (1-\tfrac{1}{2}k)z^{k-3}\}$$

$$> 1 - \tfrac{3}{6}\tfrac{3}{4}\cdot\frac{z^{k-2}}{\Gamma(k)}\{1+\tfrac{5}{4}(1-k)z^{-2}\} + \frac{z^{k-2}}{\Gamma(k)}(1-k)\{1-(1-\tfrac{1}{2}k)z^{-1}\}, \text{ by (7.6)};$$

$$> 1 + \frac{z^{k-2}}{\Gamma(k)}\left[\tfrac{3}{6}\tfrac{1}{4} - k - \frac{1-\frac{3}{2}k+\frac{1}{2}k^2}{z} - \tfrac{1}{2}\tfrac{6}{5}\tfrac{5}{6}\cdot\frac{(1-k)}{z^2}\right] \qquad (7.7)$$

> 1 when $z > c_3$, $\frac{1}{8} \leqslant k \leqslant 0{\cdot}40$.

Now consider the second square bracket in (7.3).
By (7.6), this is greater than

$$\frac{z^{k-2}}{\Gamma(k)}\left[(1-k)(2-k)\left(1-\frac{3-k}{2}z^{-1}\right) - (1-k)\{1+\tfrac{5}{4}(1-k)z^{-2}\}\left(1-\frac{0{\cdot}114}{\pi^2}\right)^{-\frac{1}{2}}\right] \qquad (7.8)$$

$$> \frac{z^{k-2}}{\Gamma(k)}[(1-k)(0{\cdot}9942-k) - \tfrac{1}{2}(1-k)(2-k)(3-k)z^{-1} - 1{\cdot}0058(1-k)^2\cdot\tfrac{5}{4}z^{-2}]$$

> 0 when $z > c_3$, $\frac{1}{8} \leqslant k \leqslant 0{\cdot}40$.

Also the factor $z^{\frac{1}{2}}S_{-k-1,\frac{1}{2}}(z)$, which multiplies this bracket in (7.3), is positive, by (4.8) with $-k$ in place of k. Hence the result is proved for $2 \leqslant q \leqslant p$, $\frac{1}{8} \leqslant k \leqslant 0{\cdot}40$, since $\Theta'(z)$ is then positive.

(ii) Let $k > 0$ and $c_1 \leqslant z \leqslant c_3$, and suppose that

$$\Gamma(k) > c_3^k\{1+(k+2)c_3^{-1}\}, \qquad (7.9)$$

which is true for $0 < k \leqslant 0{\cdot}30$, since $c_3 < (3+\frac{1}{2}k)\pi+\chi_3$. Then, by (7 . 7), (which holds for $z \geqslant \pi$), the first square bracket in (7 . 3) is greater than

$$1-\tfrac{1}{10}\cdot\frac{z^{k-2}}{\Gamma(k)} > 1-\tfrac{1}{10}z^{-2}\{1+\tfrac{5}{4}(1-k)z^{-2}\}, \text{ by (7 . 6);}$$

$$> 1-0{\cdot}13z^{-2}.$$

In (7 . 8), the factor $\{1-(0{\cdot}114/\pi^2)\}^{-\frac{1}{2}}$, which could be inserted only when $z \geqslant 3\pi$, must be replaced by $\{1-(1{\cdot}24/\pi^2)\}^{-\frac{1}{2}} < 1{\cdot}07$, the corresponding factor when it is only assumed that $z \geqslant \pi$. Consequently the second square bracket in (7 . 3) is here greater than $-0{\cdot}14\,.\,z^{k-2}/\Gamma(k)$.

Hence, from the first three lines of (i), we see by this and (7 . 9) that

$$\Theta'(z) > \frac{z^{-2}}{\Gamma(k)}\left[(k+1)\{1+(k+2)z^{-1}\}(1-0{\cdot}13z^{-2})\{1-(1+\tfrac{1}{2}k)z^{-1}\}\right.$$
$$\left.-1-0{\cdot}14z^{-1}\right]$$

$$> \frac{z^{-2}}{\Gamma(k)}\left[(k+1)\{1+(k+1{\cdot}92)z^{-1}\}\{1-(1+\tfrac{1}{2}k)z^{-1}\}-1-0{\cdot}14z^{-1}\right]$$

$$> \frac{z^{-2}}{\Gamma(k)}\left[k+(0{\cdot}78+1{\cdot}42k+\tfrac{1}{2}k^2)z^{-1}-(1+k)(1+\tfrac{1}{2}k)(1{\cdot}92+k)z^{-2}\right]$$

> 0 when $0 < k \leqslant 0{\cdot}30$, since $z > c_1 > \pi$.

(iii) Now still suppose $c_1 \leqslant z \leqslant c_3$, but let $0{\cdot}30 < k \leqslant 0{\cdot}34$; we now replace (7 . 9) by

$$\Gamma(k) > c_3^k\{1+(k+\tfrac{1}{4})c_3^{-1}\},$$

which holds for $0 < k \leqslant 0{\cdot}35$.

Then, as before, the first square bracket in (7 . 3) is greater than $1-0{\cdot}043z^{-2}$, and the second square bracket in (7 . 3) is greater than $-0{\cdot}086z^{k-2}/\Gamma(k)$. Hence, precisely as before,

$$\Theta'(z) > \frac{z^{-2}}{\Gamma(k)}\left[k+\frac{\frac{1}{2}k^2-0{\cdot}29k-0{\cdot}704}{1{\cdot}15\pi}-\frac{(1+k)(1+\frac{1}{2}k)(0{\cdot}21+k)}{(1{\cdot}15\pi)^2}\right],$$

since $c_1 > (1+\frac{1}{2}k)\pi$, *i.e.* $c_1 > 1{\cdot}15\pi$.

Therefore, when $0{\cdot}30 < k \leqslant 0{\cdot}34$ and $c_1 \leqslant z \leqslant c_3$, $\Theta'(z) > 0$.

Moreover, (7 . 1) is satisfied when $p = 3$, if $0 < k \leqslant 0{\cdot}34$; hence by the results of (i), (ii), and (iii), for the range of values of k, $\frac{1}{8} \leqslant k \leqslant 0{\cdot}34$,

$$\int_0^{c_1} t^{-k-1}C_{k+1}(t)\,dt > \int_0^{c_3} t^{-k-1}C_{k+1}(t)\,dt > \int_0^{c_5} t^{-k-1}C_{k+1}(t)\,dt,$$

the inequality (7 . 1) only being required; the proposition of this section is now proved for $\frac{1}{8} \leqslant k \leqslant 0{\cdot}34$, $1 \leqslant q \leqslant p$.

(iv) We finally show that the result is true for $0 < k < \frac{1}{8}$, $1 \leqslant q \leqslant p$, assuming only (7 . 1).

Choose an odd zero c_{2r-1}, such that

$$k > \frac{(1+k)(1+\frac{1}{2}k)}{c_{2r-1}} + \frac{1-k^2}{c_{2r-1}^2};$$

thus it is sufficient that c_{2r-1} is the first odd zero greater than

$$\frac{(1+k)(1+\frac{1}{2}k)+1/5\pi}{k},$$

since r is at least 3 by the result of (i).

Then $f(z) > 0$ for $z \geqslant c_{2r-1}$, by (7 . 4).

Also the first square bracket in (7 . 3) is greater than 1, and the second square bracket in (7 . 3) is greater than 0, by (7 . 7) and (7 . 8) respectively.

Thus $\Theta'(z) > 0$ for $z \geqslant c_{2r-1}$, (7 . 1) being satisfied.

By (4 . 4), $c_{2r-1} - c_{2r-3} < 2\pi$, so that

$$c_{2r-1} < 2\pi + \frac{(1+k)(1+\frac{1}{2}k)+1/5\pi}{k}.$$

This satisfies (7 . 9) with c_{2r-1} in place of c_3; for we have to prove that

$$\Gamma(k) > \frac{[(1+k)(1+\frac{1}{2}k)+(1/5\pi)+2\pi k]^k}{k^k}\left[1+\frac{k(k+2)}{(1+k)(1+\frac{1}{2}k)+(1/5\pi)+2\pi k}\right].$$

It is therefore sufficient to show that

$$\Gamma(k+1) > k^{1-k}[(1+k)(1+\tfrac{1}{2}k)+(1/5\pi)+2\pi k]^k\left[1+\frac{k(k+2)}{(1+k)(1+\frac{1}{2}k)}\right],$$

which is true, since the right-hand side is less than

$$(\tfrac{1}{8})^{\frac{7}{8}}\left[\tfrac{9}{8}\cdot\tfrac{17}{16}+(1/5\pi)+\tfrac{1}{4}\pi\right]^{\frac{1}{8}}\left[1+\tfrac{2}{9}\right] < (\tfrac{1}{8})^{\frac{7}{8}}\cdot\tfrac{11}{10}\cdot\tfrac{11}{9},$$

which is less than the minimum value of $\Gamma(k+1)$ for $0 < k \leqslant \frac{1}{8}$, *i.e.* is less than $\Gamma(\frac{9}{8})$; it is, in fact, less than $\frac{1}{4}\Gamma(\frac{9}{8})$. But the argument of (ii) holds when c_3 is replaced by c_{2r-1} ($r \geqslant 3$).

Hence the result will follow if only we can prove that $r < p$, p being defined by (7 . 1).

Since c_{2p-1} is the largest positive zero satisfying (7.1), we have, by (4.4),

$$\Gamma(k) < (c_{2p-1}+2\pi)^k[1-(1-k)(c_{2p-1}+2\pi)^{-2}];$$

but we have shown above that

$$\tfrac{1}{4}\Gamma(k) > (c_{2r-1}+2\pi)^k[1+(k+2)(c_{2r-1}+2\pi)^{-1}].$$

Hence

$$4(c_{2r-1}+2\pi)^k[1+(k+2)(c_{2r-1}+2\pi)^{-1}]$$
$$< (c_{2p-1}+2\pi)^k[1-(1-k)(c_{2p-1}+2\pi)^{-2}],$$

which clearly involves $c_{2r-1} < c_{2p-1}$, *i.e.* $r < p$.

Consequently $\Theta'(z) > 0$ when $0 < k \leqslant \frac{1}{8}$, $1 \leqslant q \leqslant p$, (7.1) being satisfied.

The proposition stated at the beginning of this section is now completely proved.

8. It follows from § 6 that, for $0{\cdot}48 \leqslant k \leqslant 1$, the greatest maximum of the integral

$$\int_0^\lambda t^{-k-1} C_{k+1}(t)\, dt$$

occurs as $\lambda \to \infty$.

The same is true when $k > 1$, for we may then apply the second mean-value theorem to the integral (4.1), and we deduce that, for $k > 1$,

$$C_{k+1}(z) = \frac{z^{k-1}}{\Gamma(k)}(1-\cos\eta z) \quad (0 \leqslant \eta \leqslant 1),$$
$$\geqslant 0.$$

Hence there is no Gibbs's phenomenon for $k \geqslant 0{\cdot}48$; the formula (2.1) then becomes

$$G = \frac{2}{\pi}\Gamma(k+1)\int_0^\infty t^{-k-1} C_{k+1}(t)\, dt = 1.$$

9. By (6.6) it is clear that the integral

$$\int_{c_{2p-1}}^\infty t^{-k-1} C_{k+1}(t)\, dt, \tag{9.1}$$

where $0 < k < \frac{1}{2}$, even when negative for $p = 1$, is always positive for p

greater than some fixed number, since the inequality

$$c_{2p-1}^{k}-(1-k)\,c_{2p-1}^{k-2} > \Gamma(k) \qquad (9\cdot 2)$$

is always satisfied by some such value of p^*.

For that value of p, then, the integral in (6 . 6) (with $2p-1$ in place of p), and hence the integral (9 . 1), will be positive.

It follows that the sequence

$$\int_{c_{2q-1}}^{\infty} t^{-k-1} C_{k+1}(t)\,dt \qquad (q = 1, 2, 3, \ldots)$$

is *not* a steadily increasing sequence when $k > 0$. But, if $0 \leqslant k \leqslant 0{\cdot}34$, it does steadily increase so long as $q \leqslant p$, where p satisfies the inequality

$$c_{2p-1}^{k}\{1-(1-k)\,c_{2p-1}^{-2}\} < \Gamma(k), \quad \text{by § 7.}$$

Also, if $$c_{2p-1}^{k}\{1-(1-k)\,c_{2p-1}^{-2}\} \geqslant \Gamma(k),$$

then $$\int_{c_{2p-1}}^{\infty} t^{-k-1} C_{k+1}(t)\,dt \geqslant 0, \quad \text{by § 6.}$$

Moreover, when $0 \leqslant k \leqslant 0{\cdot}40$,

$$\int_{c_1}^{\infty} t^{-k-1} C_{k+1}(t)\,dt < 0, \quad \text{by § 5.}$$

Hence, when $0 \leqslant k \leqslant 0{\cdot}34$, the greatest maximum of the integral

$$\int_{0}^{\lambda} t^{-k-1} C_{k+1}(t)\,dt$$

occurs when $\lambda = c_1$, and there is a Gibbs's phenomenon when $0 \leqslant k \leqslant 0{\cdot}40$.

This proves the theorem.

Gibbs's phenomenon in Schlömilch series of Struve functions and in Titchmarsh's integral.

10. It would seem to be of interest to give another example of a "disappearing" Gibbs's phenomenon. It appeared probable that this characteristic of "disappearing" was associated with series and integrals (representing an arbitrary function) whose Gibbs ratio contained a

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* It should be noted that the restriction $k \geqslant 0{\cdot}48$ in § 6 is only required for $p = 1$.

Lommel function, as distinct from a circular or Bessel function, under the sign of integration.

For, if $f(x)$ denote $\sin x/x^p$ or $J_\nu(x)/x^p$ $(p>0)$, the graph of $y=f(x)$ consists of waves alternately above and below the axis of x, whose areas form a monotone decreasing sequence*; but this does not appear to be the case for a Lommel function, as we have seen in the case of Young's function, $C_{k+1}(x)$, and as we shall see presently is so in the case of Struve's function, $\mathbf{H}_\nu(x)$.

Accordingly, the Schlömilch series† of Struve functions

$$\sum_{m=1}^{\infty} \frac{b_m \mathbf{H}_\nu(mx)}{(\frac{1}{2}mx)^\nu} \quad (|\nu|<\tfrac{1}{2}),$$

where
$$b_m = -\frac{2m}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}-\nu)}\int_0^\pi t\phi(t)\cos mt\,dt,$$

and
$$\phi(t) = \int_0^{\frac{1}{2}\pi} f(t\sin\theta)\sin^{2\nu+1}\theta\cos^{-2\nu}\theta\,d\theta,$$

is chosen for investigation.

At a point of ordinary discontinuity other than the origin, it is known‡ that the Gibbs ratio is given by

$$G = \frac{2}{\pi}\int_0^\pi \frac{\sin t}{t}\,dt.$$

It remains to investigate the phenomenon at the origin.

The Schlömilch series of Bessel functions

$$\sum_{m=1}^{\infty} \frac{a_m J_\nu(mx)}{(\frac{1}{2}mx)^\nu}$$

has already been investigated§.

11. Let $\quad f(t)=1 \ (0<t\leqslant\pi), \quad =0 \ (-\pi\leqslant t<0).$

Then
$$\phi(t) = \frac{\Gamma(\nu+1)\,\Gamma(\frac{1}{2}-\nu)}{\Gamma(\frac{1}{2})} \quad (0<t\leqslant\pi),$$
$$=0 \quad (-\pi\leqslant t<0).$$

---

* This is, of course, obvious in the case of $\sin x/x^p$; and for a proof in the case of $J_\nu(x)/x^p$, see R. G. Cooke, "Gibbs's phenomenon in Fourier-Bessel series and integrals", *Proc. London Math. Soc.* (2), 27 (1927), 171–192.

† See R. G. Cooke, "On the theory of Schlömilch series", *Proc. London Math. Soc.* (2), 28 (1928), 207–241.

‡ *Ibid.*, Theorem III.

§ R. G. Cooke, *ibid.*, § 5.2, and "Gibbs's phenomenon in Schlömilch series", *Journal London Math. Soc.*, 4 (1929), 18–21.

Hence $$b_m = -\frac{2m}{\pi}\Gamma(\nu+1)\int_0^\pi t\cos mt\,dt = \frac{2\Gamma(\nu+1)}{m\pi}\{1-(-1)^m\}.$$

Consequently

$$S_n(x) \equiv \sum_{m=1}^{n}\frac{b_m \mathbf{H}_\nu(mx)}{(\frac{1}{2}mx)^\nu} = \frac{2\Gamma(\nu+1)}{\pi}\sum_{m=1}^{n}\frac{\{1-(-1)^m\}}{m}\cdot\frac{\mathbf{H}_\nu(mx)}{(\frac{1}{2}mx)^\nu}.$$

The series $$\sum_{m=1}^{n}\frac{(-1)^m}{m}\cdot\frac{\mathbf{H}_\nu(mx)}{(\frac{1}{2}mx)^\nu}$$

converges uniformly to zero as $n\to\infty$, in the neighbourhood of $x=0$. Also, by Wilton's transformation of series*,

$$\sum_{m=n}^{\infty}\frac{1}{m}\cdot\frac{\mathbf{H}_\nu(mx)}{(\frac{1}{2}mx)^\nu} = 2^\nu\int_{nx}^{\infty} t^{-\nu-1}\mathbf{H}_\nu(t)\,dt + R,$$

where $$R = 2^{\nu+1}x^{-\nu}\sum_{m=1}^{\infty}\int_n^\infty \mathbf{H}_\nu(xt)\,t^{-\nu-1}\cos 2m\pi t\,dt.$$

It can be shown that the double limit of R, as $x\to 0$ and $n\to\infty$, is zero, along exactly similar lines to those given by Wilton† in the corresponding case where $\mathbf{H}_\nu(t)$ is replaced by $J_\nu(t)$.

Hence $$\lim_{n\to\infty}\operatorname{Max}_{x>0} S_n(x) = \frac{2^{\nu+1}}{\pi}\Gamma(\nu+1).\operatorname{Max}_{\lambda>0}\int_0^\lambda t^{-\nu-1}\mathbf{H}_\nu(t)\,dt. \qquad (11.1)$$

It can be shown without difficulty that the Titchmarsh integral

$$\int_0^\infty \mathbf{H}_\nu(xu)\,u\,du\int_0^\infty Y_\nu(tu)\,tf(t)\,dt$$

has, at $x=0$, a Gibbs ratio also given by (11.1).

The associated integral

$$\int_0^\infty Y_\nu(xu)\,u\,du\int_0^\infty \mathbf{H}_\nu(tu)\,tf(t)\,dt$$

has already been considered‡.

12. I attempted to prove that, under certain conditions, the greatest maximum of the integral (11.1) occurred at $\lambda = h_1$ [the smallest positive

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* J. R. Wilton, "Some applications of a transformation of series", *Proc. London Math. Soc.* (2), 27 (1927), 81–104, Theorem A.

† J. R. Wilton, "The Gibbs's phenomenon in series of Schlömilch type", *Messenger of Math.*, 56 (1926–7), 175–181 (179).

‡ R. G. Cooke, "A case of Gibbs's phenomenon", *Journal London Math. Soc.*, 3 (1923), 92–98.

zero of $\mathbf{H}_\nu(t)$] along the lines of § 7; but the difficulties of manipulation became overwhelming, and the interest of the result scarcely merited the labour involved.

It is merely shown, therefore, that *there is no Gibbs's phenomenon for the series and integral under consideration for* $\nu \geqslant -0{\cdot}10$, *and that there is one for* $-\frac{1}{2} \leqslant \nu \leqslant -0{\cdot}20$.

(i) We first show by a crude method that

$$\int_{h_p}^{\infty} t^{-\nu-1} \mathbf{H}_\nu(t)\, dt > 0 \qquad (12.1)$$

for $p \geqslant 1$, if $\nu > 0$; h_p being the p-th positive zero of $\mathbf{H}_\nu(t)$.

In the formula (6.2), put $G(z) \equiv z^{-2(\nu+1)}$; then

$$I_p \equiv \int_{h_p}^{\infty} t^{-\nu-1}\left\{1+\frac{4(\nu+1)}{t^2}\right\} \mathbf{H}_\nu(t)\, dt$$

$$= h_p^{-1}\left\{\frac{\nu+1}{2^{\nu-1}\,\Gamma(\frac{1}{2})\,\Gamma(\nu+\frac{3}{2})} - h_p^{-\nu}\,\mathbf{H}_{\nu+1}(h_p)\right\}. \qquad (12.2)$$

But we have*

$$\mathbf{H}_{\nu+1}(z) = \frac{2(\frac{1}{2}z)^{\nu+1}}{\Gamma(\frac{1}{2})\,\Gamma(\nu+\frac{3}{2})}\int_0^1 (1-t^2)^{\nu+\frac{1}{2}} \sin zt\, dt, \qquad (12.3)$$

so that, by the second mean-value theorem,

$$h_p^{-\nu}\,\mathbf{H}_{\nu+1}(h_p) = \mu(1-\cos \xi h_p) \quad (0 \leqslant \xi \leqslant 1),$$

$$\leqslant 2\mu\dagger,$$

where $$\mu = 1/\{2^\nu\,\Gamma(\tfrac{1}{2})\,\Gamma(\nu+\tfrac{3}{2})\}.$$

Hence $$I_p \geqslant 2\mu\nu h_p^{-1}, \quad \text{by (12.2)},$$

$$> 0 \quad \text{if } \nu > 0.$$

(ii) A little more intricate analysis shows that (12.1) is satisfied by $\nu \geqslant -0{\cdot}10$. For we have‡

$$2^{\nu-1}\,\Gamma(\tfrac{1}{2})\,\Gamma(\nu+\tfrac{1}{2})\,\mathbf{H}_\nu(z) = s_{\nu,\nu}(z) = S_{\nu,\nu}(z) + 2^{\nu-1}\,\Gamma(\tfrac{1}{2})\,\Gamma(\nu+\tfrac{1}{2})\,Y_\nu(z),$$

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* G. N. Watson, *loc. cit.*, 328 (i).

† It should be observed that a similar application of the second mean-value theorem to the integral (12.3), when $\nu+1$ is replaced by ν, affords a very brief proof of the well known result that $\mathbf{H}_\nu(x) \geqslant 0$ when $x > 0$ and $\nu \geqslant \frac{1}{2}$; cf. G. N. Watson, *loc. cit.*, 337 and 417.

‡ *Ibid.*, 347 (3).

whence* $$\mathbf{H}_\nu(z) = Y_\nu(z) + \frac{S_{\nu,\nu}(z)}{2^{\nu-1}\Gamma(\frac{1}{2})\Gamma(\nu+\frac{1}{2})}. \qquad (12.4)$$

Consequently

$$h_p^{-\nu}\mathbf{H}_{\nu+1}(h_p) > h_p^{-\nu}Y_{\nu+1}(h_p) + \mu\left(1+\frac{2\nu+1}{h_p^2}\right),$$

since the remainder of the expansion of $S_{\nu,\nu}(h_p)$ is negative†.

Hence, by (12.2), employing the asymptotic form‡ for $Y_{\nu+1}(h_p)$,

$$I_p > \sqrt{\left(\frac{2}{\pi}\right)}\, h_p^{-1}\left[\frac{1}{2^{\nu-\frac{1}{2}}\Gamma(\nu+\frac{1}{2})} - h_p^{-\nu-\frac{1}{2}}\left\{1+\frac{\frac{1}{4}-\nu^2}{2h_p}+\frac{\mu(2\nu+1)}{h_p^{3-\nu}}\sqrt{\left(\frac{\pi}{2}\right)}\right\}\right]$$

> 0, provided that

$$2^{\nu-\frac{1}{2}}\Gamma(\nu+\tfrac{1}{2}) < h_p^{\nu+\frac{1}{2}}\left(1+\frac{\frac{1}{4}-\nu^2}{2h_p}\right)^{-1}(1-h_p^{-2}). \qquad (12.5)$$

By employing the asymptotic form for $Y_\nu(z)$ in (12.4), we have

$$\mathbf{H}_\nu(z) = \sqrt{\left(\frac{2}{\pi z}\right)}\sin(z-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi)+R_\nu(z)+\frac{S_{\nu,\nu}(z)}{2^{\nu-1}\Gamma(\frac{1}{2})\Gamma(\nu+\frac{1}{2})},$$

where
$$R_\nu(z) \equiv \sqrt{\left(\frac{2}{\pi z}\right)}\left[\sin(z-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi)\sum_{m=1}^{\infty}\frac{(-1)^m(\nu,\,2m)}{(2z)^{2m}}\right.$$
$$\left.+\cos(z-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi)\sum_{m=1}^{\infty}\frac{(-1)^m(\nu,\,2m+1)}{(2z)^{2m+1}}\right]$$

and
$$(\nu,\,m) \equiv \frac{\Gamma(\nu+m+\frac{1}{2})}{m!\,\Gamma(\nu-m+\frac{1}{2})}.$$

Hence§ $$h_p = p\pi+\tfrac{1}{2}\nu\pi+\tfrac{1}{4}\pi\pm\chi_p, \qquad (12.6)$$

where $$\chi_p = \arcsin\left[\sqrt{\left(\frac{\pi h_p}{2}\right)}\left\{R_\nu(h_p)+\frac{S_{\nu,\nu}(h_p)}{2^{\nu-1}\Gamma(\frac{1}{2})\Gamma(\nu+\frac{1}{2})}\right\}\right], \qquad (12.7)$$

and where the plus or minus sign in (12.6) is to be taken according as p is odd or even.

It can be shown without difficulty that $\chi_p > 0$.

---

* It should be noted that the asymptotic form for $\mathbf{H}_\nu(z)$ [Watson, *loc. cit.*, 333 (1) and (2)] could be deduced at once from (12.4) by means of the known asymptotic form for $S_{\nu,\nu}(z)$, *ibid.*, 351. The formula (12.4) is equivalent to formulae given by Watson, *loc. cit.*, 331 (3), when $\omega = 0$, and 331 (4), when $\beta = 0$.

† G. N. Watson, *loc. cit.*, 351; this is seen from the Barnes integral; see also *ibid.*, 333, seventh line.

‡ *Ibid.*, 199 (2); see also 205–6 and 209.

§ We see that anyhow $h_1 > \pi$, by the integral (12.3), with $\nu+1$ replaced by ν; for the integrand is positive for $z < \pi$.

Thus $$h_1 > (1+\tfrac{1}{2}\nu+\tfrac{1}{4})\,\pi.$$

We now see that (12.5) is satisfied by $-0{\cdot}10 \leqslant \nu < \frac{1}{2}$, since in that case $h_1 > 1{\cdot}175\pi$.

This proves the result stated; for in either case (i) or case (ii) we have, by the second mean-value theorem,

$$\int_{h_p}^{\infty} t^{-\nu-1}\mathbf{H}_\nu(t)\,dt = \int_{h_p}^{\infty}\left\{1+\frac{4(\nu+1)}{t^2}\right\}^{-1} t^{-\nu-1}\left\{1+\frac{4(\nu+1)}{t^2}\right\}\mathbf{H}_\nu(t)\,dt$$

$$= \int_{\xi}^{\infty} t^{-\nu-1}\left\{1+\frac{4(\nu+1)}{t^2}\right\}\mathbf{H}_\nu(t)\,dt \quad (\xi > h_p),$$

> 0; since, if the integral were negative for any value of ξ, it would be so when $\xi = h_{2q-1}$ (an odd zero); and this is not the case.

Thus, when $\nu \geqslant -0{\cdot}10$, the greatest maximum of the integral

$$\int_0^{\lambda} t^{-\nu-1}\mathbf{H}_\nu(t)\,dt$$

occurs as $\lambda \to \infty$.

Hence there is no Gibbs's phenomenon for $\nu \geqslant -0{\cdot}10$; the formula (11.1) then becomes*

$$G = \frac{2^{\nu+1}}{\pi}\Gamma(\nu+1)\int_0^{\infty} t^{-\nu-1}\mathbf{H}_\nu(t)\,dt = 1.$$

Again, by (12.4), we have, from the series for $S_{\nu+1,\,\nu+1}(h_p)$,

$$h_p^{-\nu}\mathbf{H}_{\nu+1}(h_p) > h_p^{-\nu}Y_{\nu+1}(h_p)+\mu\left[1+\frac{2\nu+1}{h_p^2}-\frac{3(1-4\nu^2)}{h_p^4}\right],$$

since the remainder of its expansion is positive.

Also† $$Y_{\nu+1}(h_1) > \sqrt{\left(\frac{2}{\pi h_1}\right)}\cos\chi_1\left\{1-\frac{(\nu,\,2)}{(2h_1)^2}\right\}, \quad \text{by (12.6).}$$

Hence, by (12.2),

$$I_1 < \sqrt{\left(\frac{2}{\pi}\right)}\,h_1^{-1}$$

$$\times\left[\frac{1}{2^{\nu-\frac{1}{2}}\Gamma(\nu+\frac{1}{2})}-h_1^{-\nu-\frac{1}{2}}\left\{\tfrac{139}{140}\cos\chi_1+\mu\sqrt{\left(\frac{\pi}{2}\right)}\left(\frac{2\nu+1}{h_1^{\frac{3}{2}-\nu}}-\frac{3(1-4\nu^2)}{h_1^{\frac{7}{2}-\nu}}\right)\right\}\right]$$

< 0, provided that

$$2^{\nu-\frac{1}{2}}\Gamma(\nu+\tfrac{1}{2}) > h_1^{\nu+\frac{1}{2}}(\tfrac{139}{140}\cos\chi_1)^{-1}\{1-h_1^{-2}+3(1-2\nu)\,h_1^{-4}\},$$

---

* G. N. Watson, *loc. cit.*, 392 (2).

† *Ibid.*, 199 (2), 205–6, and 209.

and this is seen to be satisfied when

$$-\tfrac{1}{2} \leqslant \nu \leqslant -0{\cdot}20\,;$$

for $h_1 > \pi$, and so

$$R_\nu(h_1) < -\frac{(\nu,\,1)}{2h_1} + \frac{(\nu,\,2)}{4h_1^2},$$

since $(\nu,\,1) = \nu^2 - \tfrac{1}{4} < 0, \quad (\nu,\,2) = \tfrac{1}{2}(\nu^2 - \tfrac{1}{4})(\nu^2 - \tfrac{9}{4}) > 0\,;$

so that*, by (12.7),

$$\chi_1 < \arcsin\left[-\frac{(\nu,\,1)}{2h_1} + \frac{(\nu,\,2)}{4h_1^2} + \frac{h_1^{\nu-\frac{1}{2}}}{2^{\nu-\frac{1}{2}}\,\Gamma(\nu+\frac{1}{2})}\right]$$

$$< 0{\cdot}287, \text{ when } \nu \leqslant -0{\cdot}20.$$

Hence, by (12.6), $h_1 < 1{\cdot}15\pi + \chi_1 < 3{\cdot}876,$

and the result stated follows.

This proves that there is a Gibbs's phenomenon for

$$-\tfrac{1}{2} \leqslant \nu \leqslant -0{\cdot}20.$$

[*Added October* 14, 1929 :—Prof. Carslaw has informed me that after Mr. R. J. Lyons had investigated the value of $\mathfrak{H}$, the critical value of k, its value was computed by Prof. T. H. Gronwall, who has shown that Gibbs's Phenomenon disappears if $k > 0{\cdot}4395516$, the last digit possibly being in error by a unit. Gronwall's paper is to appear in the *Math. Zeitschrift*; see also *Bull Amer. Math. Soc.*, 31 (1925), 302, and 32 (1926), 316.]

* The inequality for $S_{\mu,\nu}(h_1)$ follows at once by putting $\mu = \nu$ in (4.7).

A NEW METHOD FOR SOLVING TWO-DIMENSIONAL PROBLEMS OF PHYSICAL TYPES.

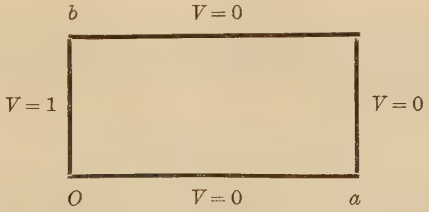
By T. J. I'A. BROMWICH.

[Received 5 May, 1928.—Read 10 May, 1928.]

Since the publication of my treatment of the general theory of dynamical systems* and of its relation to Heaviside's operational methods, I have been often asked to consider the question of adapting my method to other problems; and, in particular, to the solution of Laplace's equation in two dimensions.

The solutions given in the present paper are the outcome of these inquiries; they have been developed rather slowly on account of other work, but they have been in my possession for some seven years. As it seems unlikely that I shall have leisure to work much further in this direction at present, I have decided to publish the solutions of three known problems, in the hope that the method may some time prove of use in obtaining solutions of less familiar examples.

1. Consider *the problem of solving Laplace's equation inside a rectangle, subject to the boundary conditions indicated.*



* *Proc. London Math. Soc.* (2), 15 (1916), 401–448; for a convenient summary (including later papers) see Dr. H. Jeffreys' *Tract* [No. 23 of the *Cambridge Tracts in Mathematics and Mathematical Physics* (1927)].

The most familiar method of solution is indicated in Byerly's book*. Thus, using Fourier series, we obtain the result

$$V = \frac{4}{\pi} \sum_m \left\{ \frac{1}{m} \sin \frac{m\pi y}{b} \sinh \frac{m\pi(a-x)}{b} \operatorname{cosech} \frac{m\pi a}{b} \right\} \tag{1}$$

$$(m = 1, 3, 5, 7, \ldots).$$

It is found also that, by means of a somewhat different method, we can obtain the solution

$$V = \frac{a-x}{a} - \frac{2}{\pi} \sum_n \left\{ \frac{1}{n} \sin \frac{n\pi x}{a} \cosh \frac{n\pi(\frac{1}{2}b-y)}{a} \operatorname{sech} \frac{n\pi b}{2a} \right\} \tag{2}$$

$$(n = 1, 2, 3, 4, \ldots).$$

And the identity between the two solutions (1) and (2) can be established by methods due to Stokes†.

It is possible to solve the problem also by a conformal transformation, which introduces elliptic functions—and the identity of (1) and (2) is then a known result in the theta-function development of elliptic-function theory. The observation of this fact at once suggested to my mind the use of contour integrals‡, and I was led to consider the whole problem from this point of view *ab initio*.

The solution is in some respects very similar to that introduced in my treatment§ of Heaviside's operational solutions. But it is rather different in certain details, and thus a complete discussion may be of use to others.

We attempt then to solve our problem by writing

$$V = \frac{1}{2\pi i} \int Y \frac{\sin \lambda(a-x)}{\sin \lambda a} \, d\lambda, \tag{3}$$

where Y is independent of x (but involves y and λ), and the path of integration consists of *two* straight lines parallel to the imaginary axis in the λ-plane, not so far apart that any root of $\sin \lambda a = 0$ (except $\lambda = 0$) can fall between them‖.

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* W. E. Byerly, *Fourier series and spherical harmonics* (Boston, U.S.A., 1893), § 59.

† *Math. and Phys. Papers*, 1 (1880), 291–295.

‡ For the necessary theta-function identity finally depends on contour integration.

§ Quoted above, p. 165.

‖ It is in the use of the *pair* of lines (instead of *one* line) that the present solution differs from that devised for dynamical problems.

Then, on substitution in Laplace's equation, it is evident that the function

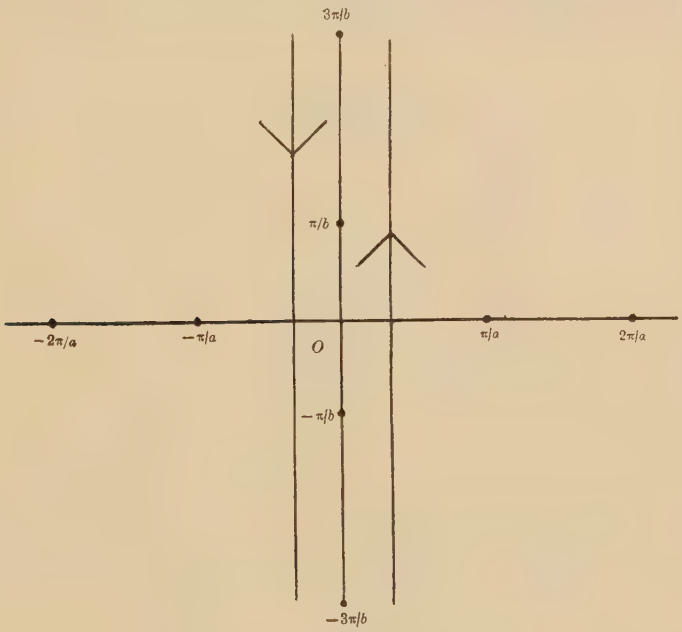
$$\frac{\partial^2 Y}{\partial y^2} - \lambda^2 Y$$

can have no poles inside the path of integration. Then, on taking $x = 0$, we see that

$$\frac{1}{2\pi i}\int Y\, d\lambda = 1 \quad \text{if} \quad 0 < y < b.$$

Also $Y = 0$ at $y = 0$ and at $y = b$. These conditions are all satisfied by the function

$$Y = \frac{1}{\lambda}\{1 - \cosh\lambda(\tfrac{1}{2}b - y)\ \operatorname{sech}\tfrac{1}{2}\lambda b\}, \tag{4}$$



and it is not difficult to see that the solution is unique. From (3) and (4) we have the solution of the original problem by means of the integral

$$V = \frac{1}{2\pi i}\int \frac{d\lambda}{\lambda}\{1 - \cosh\lambda(\tfrac{1}{2}b - y)\ \operatorname{sech}\tfrac{1}{2}\lambda b\}\ \frac{\sin\lambda(a-x)}{\sin\lambda a}. \tag{5}$$

If the path of integration is contracted to the imaginary axis, the only poles are given by

$$\lambda = \pm\frac{\pi i}{b}, \quad \pm\frac{3\pi i}{b}, \quad \pm\frac{5\pi i}{b}, \quad \ldots,$$

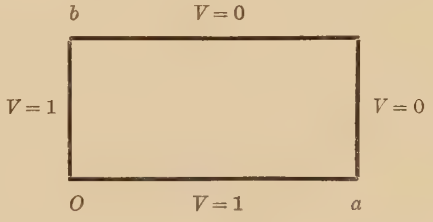
and there is no difficulty in obtaining the solution (1) by evaluating the residues of (5) as usual; it may be noted that there is no pole at $\lambda = 0$.

A similar method can be applied to obtain (2) by bending the path round so as to enclose the poles on the real axis; the only novelty is that the term 1 in the { } of (5) should be left with the path in its original form, and so this term contributes $(a-x)/a$ to the answer from the residue at $\lambda = 0$. The remaining terms lead to the series in (2) from the poles

$$\lambda = \pm\frac{\pi}{a}, \quad \pm\frac{2\pi}{a}, \quad \pm\frac{3\pi}{a}, \quad \ldots,$$

which all lie along the real axis.

As a further example, the reader may verify the solution given by Byerly (*loc. cit.*, 104, Ex. 8) for the problem indicated in the diagram, namely



$$(6) \quad V = \frac{b-y}{b} + \frac{2}{\pi}\left\{S_1 \sin\frac{\pi y}{b} - C_2 \sin\frac{2\pi y}{b} + S_3 \sin\frac{3\pi y}{b} - C_4 \sin\frac{4\pi y}{b} + \ldots\right\},$$

where

$$(7) \quad \begin{cases} S_1 = \sinh\frac{\pi}{b}(\tfrac{1}{2}a-x)\ \operatorname{cosech}\frac{\pi a}{2b}, \\ C_2 = \tfrac{1}{2}\cosh\frac{2\pi}{b}(\tfrac{1}{2}a-x)\ \operatorname{sech}\frac{2\pi a}{2b}, \\ S_3 = \tfrac{1}{3}\sinh\frac{3\pi}{b}(\tfrac{1}{2}a-x)\ \operatorname{cosech}\frac{3\pi a}{2b}, \\ C_4 = \tfrac{1}{4}\cosh\frac{4\pi}{b}(\tfrac{1}{2}a-x)\ \operatorname{sech}\frac{4\pi a}{2b}, \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \end{cases}$$

By considering the special case of a square ($b = a$) and the value of V

at the centre, we can prove that

$$\frac{1}{\cosh \frac{1}{2}\pi} - \frac{1}{3\cosh \frac{3}{2}\pi} + \frac{1}{5\cosh \frac{5}{2}\pi} - \ldots = \frac{\pi}{8}, \tag{8}$$

which is a known result from the q-series for theta functions*.

2. A second example (proposed by Stokes as an illustration of his methods of handling Fourier series†) is *to find the velocity-potential when a rectangular prism (containing fluid) is made to rotate with angular velocity ω about the line of centres.*

The boundary conditions are then

$$\frac{\partial \phi}{\partial x} = -\omega(y - \tfrac{1}{2}b) \text{ at } x = 0,\ a; \quad \frac{\partial \phi}{\partial y} = +\omega(x - \tfrac{1}{2}a) \text{ at } y = 0,\ b.$$

Write, then, $$\phi = \phi_1 - \omega(x - \tfrac{1}{2}a)(y - \tfrac{1}{2}b)$$

and we have the conditions

$$\frac{\partial \phi_1}{\partial x} = 0 \text{ at } x = 0,\ a; \quad \frac{\partial \phi_1}{\partial y} = \omega(2x - a) \text{ at } y = 0,\ b.$$

These suggest the solution

$$\phi_1 = \frac{1}{2\pi i}\int Y \left\{\frac{\cos \lambda(a-x)}{\sin \lambda a} - \frac{\cos \lambda x}{\sin \lambda a}\right\} d\lambda,$$

for then $$\frac{\partial \phi_1}{\partial x} = \frac{1}{2\pi i}\int Y\, d\lambda \text{ at } x = 0,\ a.$$

Thus (the path being taken as in § 1) the conditions at $x = 0,\ a$ will be satisfied if Y has no poles except on the imaginary axis.

Then we have

$$\phi_1 = \frac{1}{2\pi i}\int Y \frac{\sin \lambda(x - \frac{1}{2}a)}{\cos \frac{1}{2}\lambda a}\, d\lambda. \tag{10}$$

Proceeding as before, we take Y of the form

$$\frac{1}{\lambda^2}\{A + B\lambda + C_1 \cosh \lambda(b-y) + C_2 \cosh \lambda y\}$$

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* It has, for example, been discovered by Ramanujan. Cf. *Journal London Math. Soc.*, 3 (1928), 221.

† Stokes, *Math. and Phys. Papers*, 1 (1880), 288, 191; a brief summary will be found on p. 378 of my book on *Infinite series* (2nd ed., 1926).

and at $y = 0$, $$\frac{\partial Y}{\partial y} = -\frac{C_1}{\lambda}\sinh\lambda b,$$

at $y = b$, $$\frac{\partial Y}{\partial y} = +\frac{C_2}{\lambda}\sinh\lambda b.$$

Thus we can satisfy the necessary conditions by taking

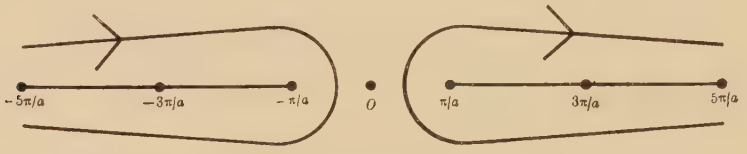
$$C_1 = -\frac{2\omega}{\lambda\sinh\lambda b}, \quad C_2 = +\frac{2\omega}{\lambda\sinh\lambda b}, \quad A = 0, \quad B = 0.$$

Then $$Y = \frac{2\omega}{\lambda^3}\frac{\sinh\lambda(y-\frac{1}{2}b)}{\cosh\frac{1}{2}\lambda b},$$

and so

$$\phi_1 = \frac{1}{2\pi i}\int\frac{2\omega\,d\lambda}{\lambda^3}\frac{\sin\lambda(x-\frac{1}{2}a)}{\cos\frac{1}{2}\lambda a}\frac{\sinh\lambda(y-\frac{1}{2}b)}{\cosh\frac{1}{2}\lambda b}. \tag{11}$$

To obtain the solution given by Stokes, we bend the path round so as to enclose the real axis in the manner indicated by the sketch.



The result is easily found to be

$$\phi_1 = \frac{8\omega a^2}{\pi^3}\Sigma\frac{1}{n^3}\frac{\sinh\{n\pi(\frac{1}{2}b-y)/a\}}{\cosh(\frac{1}{2}n\pi b/a)}\cos\frac{n\pi x}{a} \quad (n = 1, 3, 5, 7, \ldots). \tag{12}$$

It is not difficult to deduce Stokes' formulae for the kinetic energy of the fluid, by integrating under the integral sign (for λ), and then taking the residues subsequently; but there is very little saving in labour, although the reader may find the problem an interesting exercise when carried out on lines similar to the integration given in § 3 below.

The above calculation is based on the hypothesis that $b > a$, so that the series for ϕ_1 converges very rapidly; when $a > b$, we may proceed by contracting the path so as to surround the imaginary axis (and the effect of the residue at $\lambda = 0$ must not be overlooked in the calculation*). But, in effect, the quickest method is to interchange a, b and x, y (at the same time changing the sign of ω).

* This simply adds on $2\omega(x-\frac{1}{2}a)(y-\frac{1}{2}b)$ and so changes the sign of the term not included in the summation.

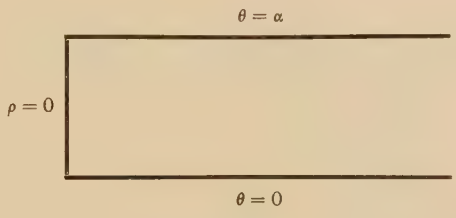
The equivalent problem for the torsion of a rectangular bar is worked out by Thomson and Tait* (apparently without knowing of the results found by Stokes); unfortunately, their final formulae contain several slips.

3. A third problem (which is also due to Stokes†) is that of finding *the velocity-potential representing fluid motion inside a rotating sector*; the boundaries may be taken to be $\theta = 0$, $\theta = a$, and $r = a$.

Then, if we write $\rho = \log(a/r)$, the problem is reduced to the solution of

$$\frac{\partial^2 \phi}{\partial \rho^2} + \frac{\partial^2 \phi}{\partial \theta^2} = 0$$

inside the infinite rectangle, subject to the conditions



$$\frac{\partial \phi}{\partial \rho} = 0 \text{ at } \rho = 0,$$

and

$$\frac{\partial \phi}{\partial \theta} = \omega r^2 = \omega a^2 e^{-2\rho} \text{ at } \theta = 0,\ \theta = a.$$

The previous examples at once suggest the form of solution (the path now being restricted to *one* line‡ on the *positive* side of $\lambda = 0$)

$$\phi = \frac{1}{2\pi i} \int F(\lambda) \frac{\sin \lambda(\theta - \frac{1}{2}\alpha)}{\cos \frac{1}{2}\lambda a} e^{-\lambda \rho} d\lambda, \tag{13}$$

and all the conditions are satisfied by taking $F(\lambda)$ so that

$$\frac{1}{2\pi i} \int \lambda F(\lambda) e^{-\lambda \rho} d\lambda = a^2 \omega e^{-2\rho}.$$

---

* *Natural philosophy*, 2 (1883), 248–250.

† *Loc. cit.*, 306.

‡ Just as in the solution given in my paper for the determination of *Normal coordinates in dynamical systems* quoted above.

This indicates that

$$\lambda F(\lambda) = a^2\omega\left(\frac{1}{\lambda-2}-\frac{1}{\lambda+2}\right) = \frac{4a^2\omega}{\lambda^2-4},$$

where the second term is added so as to make $F(\lambda)$ an odd function in λ.

Then

$$\phi = \frac{1}{2\pi i}\int \frac{4a^2\omega\, d\lambda}{\lambda(\lambda^2-4)}\,\frac{\sin\lambda(\theta-\frac{1}{2}a)}{\cos\frac{1}{2}\lambda a}\,e^{-\lambda\rho}, \tag{14}$$

and on contracting the path round the positive half of the real axis we obtain Stokes' result

$$\frac{\phi}{a^2\omega} = \frac{\sin(2\theta-a)}{2\cos a}\,e^{-2\rho}+\frac{8a^2}{\pi}\sum_n \frac{\cos(n\pi\theta/a)\,e^{-n\pi\rho/a}}{n(n^2\pi^2-4a^2)} \tag{15}$$

$(n = 1, 3, 5, 7, \ldots).$

In this problem the calculation of the kinetic energy seems distinctly easier if we work with the contour integral rather than the series.

In fact, we have only to evaluate

$$\int_0^a \phi\left(\frac{\partial\phi}{r\partial\theta}\right)dr \quad \text{for} \quad \theta = a,$$

for the contributions from $\theta = 0$ and $\theta = a$ will be equal and that from $r = a$ is zero. Thus we have to find

$$\omega\int_0^a \phi_a r\, dr = a^2\omega\int_0^\infty \phi_a e^{-2\rho}\, d\rho,$$

which is the same as

$$\frac{4(a^2\omega)^2}{2\pi i}\int \frac{d\lambda\,\tan\frac{1}{2}\lambda a}{\lambda(\lambda-2)(\lambda+2)^2}, \tag{16}$$

and from this (by calculating the residues for the positive roots $\lambda = 2$, $\lambda = \pi/a$, $3\pi/a$, $5\pi/a$, ...) it is easy to deduce the series given by Stokes.

Without giving full details for the general problem it may be worth while to carry out the work for the special case of the semicircle*, for which $a = \pi$. We then have to calculate

$$\frac{4(a^2\omega)^2}{2\pi i}\int \frac{d\lambda\,\tan\frac{1}{2}\lambda\pi}{(\lambda-2)\,\lambda(\lambda+2)^2} \tag{17}$$

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* See also Lamb, *Hydrodynamics* (1906), § 72; the notation is slightly different. Lamb uses $2a$ in place of the a used by Stokes, and measures θ from the bisector of the angle.

and the special point $\lambda = 2$ is no longer a pole; thus the only residues are given by $\lambda = 1, 3, 5, \ldots$, and at each of these the residue of $\tan \frac{1}{2}\lambda\pi$ is $-2/\pi$.

Also

$$\frac{32}{(\lambda-2)\lambda(\lambda+2)^2} = \frac{1}{\lambda-2} - \frac{4}{\lambda} + \frac{3}{\lambda+2} + \frac{4}{(\lambda+2)^2}.$$

Thus the series becomes

$$\begin{aligned} -\frac{(a^2\omega)^2}{4\pi} \Big\{ & \frac{1}{-1} - \frac{4}{1} + \frac{3}{3} + \frac{4}{3^2} \\ & + \frac{1}{1} - \frac{4}{3} + \frac{3}{5} + \frac{4}{5^2} \\ & + \frac{1}{3} - \frac{4}{5} + \frac{3}{7} + \frac{4}{7^2} \\ & + \frac{1}{5} - \frac{4}{7} + \frac{3}{9} + \frac{4}{9^2} \\ & + \ldots \quad \ldots \quad \ldots \Big\} \\ = -\frac{(a^2\omega)^2}{4\pi} & \left\{ -4 + 4\left(\frac{\pi^2}{8} - 1\right) \right\} \\ = \frac{a^4\omega^2}{\pi} & \left(2 - \frac{\pi^2}{8}\right) = \tfrac{1}{2} M k^2 \omega^2, \end{aligned}$$

where

(18) $$M = \tfrac{1}{2}\pi a^2, \quad k^2 = \left(\frac{8}{\pi^2} - \tfrac{1}{2}\right) a^2 = (\cdot 3106)\, a^2,$$

and so $$k/a = \cdot 56 \text{ nearly.}$$

DIRICHLET'S PRINCIPLE FOR A FLAT RING

(Second Paper)

By E. G. C. Poole.

[Received 28 November, 1928.—Read 13 December, 1928.]

As the infinite space around a ring is doubly connected, the harmonic functions required in physics for questions associated with a ring need not be single-valued, but may admit an additive cyclic constant for the irreducible circuit threading the ring. Such potentials may, however, be made up by adding the most general uniform potential to a multiple of the simplest potential having a cyclic constant.

The problem of the uniform potentials for a flat ring was attacked in the previous paper*, and we consider in the present one a special problem having a cyclic potential. If the ring is at rest in a liquid extending to infinity, we shall suppose the liquid to be at rest at infinity and to be circulating in steady irrotational motion, along closed stream-lines which thread the ring and are symmetrical with respect to its axis of revolution.

The method of solution in series is found to be applicable, not indeed to the velocity potential, but to Stokes's current function for this motion†. On removing a trivial factor, we require a simply infinite system of Lamé functions chosen from the doubly infinite set already introduced.

The restriction of the range of the analysis gives an opportunity for a more detailed study of the practical calculations. The results, however incomplete, appear to be an advance on those already obtained,

* *Proc. London Math. Soc.* (2), 29 (1929), 342-354.

† Lamb, *Hydrodynamics*, 5th ed. (1924), 118.

particularly as regards the second Lamé function in each term of the series.

I.

Let (ϖ, z, ϕ) be cylindrical coordinates and let the ring be in the region $z=0$, $a(1-k) \leqslant k'\varpi \leqslant a(1+k)$, where $k^2+k'^2=1$; and take V, ψ as the velocity potential and current function of the motion. The axis of z will be identified with the stream line $\psi=0$, and the surface of the ring with the locus of stream lines $\psi=\psi_1$. At any point, the velocity is given by

$$\left.\begin{aligned} v &= -\frac{\partial V}{\partial \varpi} = \frac{1}{\varpi}\frac{\partial \psi}{\partial z}, \text{ perpendicular to } 0z, \\ w &= -\frac{\partial V}{\partial z} = -\frac{1}{\varpi}\frac{\partial \psi}{\partial \varpi}, \text{ parallel to } 0z. \end{aligned}\right\} \tag{1}$$

If ψ is known, V is given by integrating

$$dV = \frac{1}{\varpi}\left(\frac{\partial \psi}{\partial \varpi}dz - \frac{\partial \psi}{\partial z}d\varpi\right), \tag{2}$$

and the condition for a perfect differential gives the equation satisfied by Stokes's current function

$$\frac{\partial^2 \psi}{\partial \varpi^2} + \frac{\partial^2 \psi}{\partial z^2} - \frac{1}{\varpi}\frac{\partial \psi}{\partial \varpi} = 0. \tag{3}$$

As Basset remarks*, this equation shows that $\varpi^{-1}\psi\cos\phi$ satisfies Laplace's equation. If u, v are a system of curvilinear orthogonal coordinates in the meridian planes, for which Laplace's equation can be satisfied by "normal products" in the variables (u, v, ϕ), we are led to treat Stokes's equation (3) by similar methods.

As in the former paper, we write

$$k'(\varpi+iz) \equiv a\{\mathrm{dn}(u+iv) - k\,\mathrm{cn}(u+iv)\}. \tag{4}$$

By the properties of conjugate functions, (2) becomes

$$dV = \frac{1}{\varpi}\left\{\frac{\partial \psi}{\partial u}dv - \frac{\partial \psi}{\partial v}du\right\}, \tag{5}$$

and (3) is equivalent to

$$\frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2} - \frac{1}{\varpi}\left\{\frac{\partial \varpi}{\partial u}\frac{\partial \psi}{\partial u} + \frac{\partial \varpi}{\partial v}\frac{\partial \psi}{\partial v}\right\} = 0. \tag{6}$$

* Basset, *Hydrodynamics*, 1 (1889), 155.

If we write $$\psi \equiv \sqrt{\varpi}.\Psi(u, v), \tag{7}$$

we have $$\frac{\partial^2\Psi}{\partial u^2}+\frac{\partial^2\Psi}{\partial v^2}-\tfrac{3}{4}\frac{\Psi}{\varpi^2}\left\{\left(\frac{\partial\varpi}{\partial u}\right)^2+\left(\frac{\partial\varpi}{\partial v}\right)^2\right\}=0, \tag{8 a}$$

or, in virtue of (4),

$$\frac{\partial^2\Psi}{\partial u^2}+\frac{\partial^2\Psi}{\partial v^2}-\tfrac{3}{4}k^2(\mathrm{sn}^2 u-\mathrm{sn}^2 iv)\,\Psi=0. \tag{8 b}$$

For a normal solution $\Psi \equiv F(u)\,G(iv)$ we find

$$\left.\begin{aligned} F''(u) &= (\tfrac{3}{4}k^2\,\mathrm{sn}^2 u+B)\,F(u),\\ G''(iv) &= (\tfrac{3}{4}k^2\,\mathrm{sn}^2 iv+B)\,G(iv),\end{aligned}\right\} \tag{9}$$

two Lamé equations of order $\frac{1}{2}$. The value of the parameter B must be chosen so that ψ shall be a single-valued function of position, or $F(u+4K)=F(u)$. This breaks up into four cases

$$F(u+2K)=\pm F(u), \quad F(-u)=\pm F(u), \tag{10}$$

as was proved in the previous paper.

Since ψ, Ψ, $\sqrt{\varpi}$ all reduce to even functions of u on the ring, where $v=0$, only the even functions $F(u)$ require to be calculated for our present purpose. Assuming the principle of the method employed in the previous paper, some details of this calculation will be given in § II.

In § III we discuss the expansion of ψ or Ψ at the surface of the ring as a series of these functions $F(u)$.

Since $\psi=0$ on the axis of revolution, we must choose each $G(iv)$ so that it shall vanish when $iv=iK'$ and take the value unity when $iv=0$. This part of the problem is attacked in § IV.

The consideration of the hydrodynamical problem will then be resumed in § V.

II.

We transform (9) by writing

$$\pi u = 2Kx, \quad F(u)=f(x), \tag{11}$$

and then

$$f''(x)+\left[b+\sum_{n=1}^{\infty}\frac{6nq^n}{1-q^{2n}}\cos 2nx\right]f(x)=0, \tag{12}$$

where b is defined by

$$\pi^2 b+4K^2B+3K(K-E)=0. \tag{13}$$

The even functions of the required system are constructed by assuming, for any integer s,

$$\left.\begin{aligned} b &= \sum_{n=0}^{\infty} b_n q^n, \quad f(x) = \sum_{n=0}^{\infty} q^n f_n(x), \\ b_0 &= s^2, \qquad\quad f_0(x) = \cos sx. \end{aligned}\right\} \tag{14}$$

We can evaluate $b_n, f_n(x)$ pair by pair; at any stage, b_n is given by the hypothesis that $f_n(x)$ has only periodic terms, and $f_n(x)$ is uniquely given by assuming that it does not contain $\cos sx$ or $\sin sx$, for $n \geqslant 1$.

The calculation of a large number of terms of (14) would be extremely laborious; but for a ring of moderate breadth the value of q is so small that such calculation is not likely to be necessary. We know that, if $k^2 \leqslant 1$, q is given by a series of positive terms

$$q = \frac{k^2}{16} + \frac{k^4}{32} + \frac{21k^6}{1024} + \frac{31k^8}{2048} + \ldots, \tag{15}$$

which takes the value 1 when $k^2 = 1$. If the breadth of the ring is equal to its inner diameter ($k = \frac{1}{2}$) already $q < 1/55$. If the breadth is only a tenth of the inner diameter, so that $k = 1/11$, then $q < 1/1920$.

The reader will be in a position to judge how far the approximation is sufficient, for any particular case which he may have in mind.

By analogy with the Mathieu functions, on the one hand, and with tesseral harmonics on the other, we may introduce a provisional notation for the functions considered in this and the former paper.

The Lamé cosine series which occurs in the solution of Laplace's equation with the azimuthal factor $\cos m\phi$, which therefore satisfies a Lamé equation of order $m - \frac{1}{2}$ and which reduces to $\cos sx$ for $q = 0$, will be written

$$f(x) = cl_s^{(m)}(x). \tag{16a}$$

The corresponding odd function, which reduces to $\sin sx$, may be written

$$f(x) = sl_s^{(m)}(x). \tag{16b}$$

The functions employed in the present paper are all of the form $cl_s^{(1)}(x)$.

We may now tabulate the leading terms of these functions, and the corresponding forms of the parameter b, for $s = 0, 1, \ldots, 6$. We shall neglect q^4, and a proof will presently be given that, even in the most unfavourable cases, the error involved by keeping only the first n terms

is $O(11^n q^n)$. There is every reason to believe that the situation is in reality much more favourable than this.

s	b	$f_0(x)$	$f_1(x)$	$f_2(x)$
0	$-\frac{9}{2}q^2$	1	$\frac{3}{2}\cos 2x$	$\frac{33}{32}\cos 4x$
1	$1-3q-\frac{9}{8}q^2-\frac{453}{64}q^3$	$\cos x$	$\frac{3}{8}\cos 3x$	$\frac{39}{84}\cos 3x+\frac{1}{4}\cos 5x$
2	$4-\frac{9}{4}q^2$	$\cos 2x$	$-\frac{3}{4}+\frac{1}{4}\cos 4x$	$\frac{27}{128}\cos 6x$
3	$9+\frac{9}{16}q^2-\frac{315}{64}q^3$	$\cos 3x$	$-\frac{3}{8}\cos x+\frac{3}{16}\cos 5x$	$-\frac{39}{64}\cos x+\frac{21}{128}\cos 7x$
4	$16+\frac{3}{10}q^2$	$\cos 4x$	$-\frac{1}{4}\cos 2x+\frac{3}{20}\cos 6x$	$-\frac{21}{24}+\frac{43}{320}\cos 8x$
5	$25+\frac{9}{16}q^2$	$\cos 5x$	$-\frac{3}{16}\cos 3x+\frac{1}{8}\cos 7x$	$-\frac{29}{128}\cos x+\frac{51}{448}\cos 9x$
6	$36+\frac{9}{70}q^2$	$\cos 6x$	$-\frac{3}{20}\cos 4x+\frac{3}{28}\cos 8x$	$-\frac{111}{640}\cos 2x+\frac{177}{1792}\cos 10x$

s	$f_3(x)$
0	$\frac{363}{128}\cos 2x+\frac{107}{128}\cos 6x$
1	$\frac{51}{512}\cos 3x+\frac{215}{512}\cos 5x+\frac{1}{4}\cos 7x$
2	$-\frac{99}{64}+\frac{131}{512}\cos 4x+\frac{95}{512}\cos 8x$
3	$-\frac{267}{1024}\cos x+\frac{345}{4096}\cos 5x+\frac{151}{1024}\cos 9x$
4	$-\frac{499}{640}\cos 2x+\frac{3117}{32000}\cos 6x+\frac{157}{1280}\cos 10x$
5	$-\frac{307}{1024}\cos x-\frac{777}{4096}\cos 3x+\frac{669}{7168}\cos 7x+\frac{1507}{14336}\cos 11x$
6	$-\frac{539}{2560}-\frac{69513}{448000}\cos 4x+\frac{151161}{1756160}\cos 8x+\frac{1979}{21504}\cos 12x$

The convergence of our series may be roughly examined by the method of *fonctions majorantes* previously employed. We write

$$\left.\begin{aligned} A_n &= \Sigma|a_r|, \quad \text{where} \quad f_n(x) \equiv \Sigma a_r \cos rx, \\ \alpha(q) \equiv \sum_{n=1}^{\infty} A_n q^n, &\quad \beta(q) \equiv \sum_{n=1}^{\infty} |b_n| q^n, \quad \phi(q) \equiv \sum_{n=1}^{\infty} \frac{nq^n}{1-q^{2n}}. \end{aligned}\right\} \tag{17}$$

Then it was shown that

$$\beta(q) \ll 6\phi(q)\{1+\alpha(q)\}. \quad 4s'\alpha(q) \ll 6\phi(q)\{2+\alpha(q)\}\{1+\alpha(q)\}, \tag{18}$$

where $s'=s+1$, if $s=0$, 1, and $s'=s-1$, if $s>1$. Then a *fonction majorante* for $\alpha(q)$ is the power series vanishing with ϕ and satisfying

$$3\phi y^2+(9\phi-2s')\,y+6\phi = 0, \tag{19}$$

or
$$6\phi y = 2s'-9\phi-\sqrt{(4s'^2-36s'\phi+9\phi^2)}.$$

Since $4s'^2-36s'\phi-9\phi^2 \equiv \{2s'-(9+6\sqrt{2})\,\phi\}\{2s'-(9-6\sqrt{2})\,\phi\}$,

we have
$$3y \ll \sum_{n=1}^{\infty}\left(\frac{9+6\sqrt{2}}{2s'}\right)^{n+1}\phi^n \ll 9\sum_{n=1}^{\infty}\left(\frac{9\phi}{s'}\right)^n,$$

so that
$$\alpha(q) \ll 3\sum_{n=1}^{\infty}\left(\frac{9\phi}{s'}\right)^n. \tag{20}$$

Thus $a(q)$ converges at least as rapidly as a geometric progression of ratio $9\phi/s'$. Using the relation

$$\phi(q) \ll \frac{q}{1-2q}, \tag{21}$$

this is found to give convergence when

$$q < s'/(9+2s'). \tag{22}$$

III.

To expand $\sqrt{(a/\varpi)}$ on the surface of the ring $v = 0$, we note that

$$\frac{a}{\varpi} \equiv \frac{\mathrm{dn}\, u + k\, \mathrm{cn}\, u}{k'} \equiv \frac{k'}{\mathrm{dn}\, u - k\, \mathrm{cn}\, u} \tag{23a}$$

is an elliptic function with poles congruent to iK', and zeros to $iK'+2K$, *modulis* $4K$, $2iK'$. From the properties of the theta functions, we have (on putting $\pi u = 2Kx$) the expressions

$$\frac{a}{\varpi} = \prod_{n=0}^{\infty}\left(\frac{1+2q^{n+\frac{1}{2}}\cos x + q^{2n+1}}{1-2q^{n+\frac{1}{2}}\cos x + q^{2n+1}}\right) = \frac{1+2\sum_{n=1}^{\infty} q^{\frac{1}{2}n^2}\cos nx}{1+2\sum_{n=1}^{\infty} q^{\frac{1}{2}n^2}\cos n(x+\pi)}. \tag{23b}$$

From this the leading terms of the square root may be computed:

$$\begin{aligned}\left(\frac{a}{\varpi}\right)^{\frac{1}{2}} = {} & 1+2q^{\frac{1}{2}}\cos x + q(1+\cos 2x) + q^{\frac{3}{2}}(3\cos x+\cos 3x) \\ & +\tfrac{3}{4}q^2(3+4\cos 2x+\cos 4x)+\tfrac{1}{4}q^{\frac{5}{2}}(22\cos x+7\cos 3x+3\cos 5x) \\ & +\tfrac{1}{8}q^3(34+43\cos 2x+14\cos 4x+5\cos 6x)+O(q^{\frac{7}{2}}).\end{aligned} \tag{24}$$

If we convert this, by successive approximations, to the form

$$\left(\frac{a}{\varpi}\right)^{\frac{1}{2}} = \sum_{n=0}^{\infty}\gamma_n\, cl_n^{(1)}(x), \tag{25}$$

then, on neglecting $q^{\frac{7}{2}}$, the leading coefficients appear to be

$$\left.\begin{aligned} &\gamma_0 = 1+q+\tfrac{15}{8}q^2+\tfrac{43}{8}q^3\ldots, && \gamma_1 = 2q^{\frac{1}{2}}+3q^{\frac{3}{2}}+\tfrac{179}{32}q^{\frac{5}{2}}\ldots, \\ &\gamma_2 = -\tfrac{1}{2}q+\tfrac{3}{8}q^2-\tfrac{5}{16}q^3\ldots, && \gamma_3 = \tfrac{1}{4}q^{\frac{3}{2}}-\tfrac{19}{32}q^{\frac{5}{2}}\ldots, \\ &\gamma_4 = -\tfrac{5}{32}q^2+\tfrac{11}{32}q^3\ldots, && \gamma_5 = \tfrac{13}{64}q^{\frac{5}{2}}\ldots, \\ &\gamma_6 = -\tfrac{21}{256}q^3\ldots. \end{aligned}\right\} \tag{26}$$

We note that the coefficients of (24) are all positive, and this holds in general because

$$\tfrac{1}{2}\log\left(\frac{a}{\varpi}\right) = 2\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} q^{2(m+\frac{1}{2})(n+\frac{1}{2})}\frac{\cos(2m+1)x}{2m+1} \tag{27}$$

has only positive coefficients, and no negative signs are introduced by the exponential. For real values of x, a *fonction majorante* is obtained by putting $x=0$, say

$$M(q) = \mathrm{Max}\left[\left(\frac{a}{\varpi}\right)^{\frac{1}{2}}\right] = \prod_{n=0}^{\infty}\left(\frac{1+q^{n+\frac{1}{2}}}{1-q^{n+\frac{1}{2}}}\right) = \sum_{n=0}^{\infty} c_n q^{\frac{1}{2}n},$$

$$\log M(q) < \frac{1}{1-q}\log\left(\frac{1+q^{\frac{1}{2}}}{1-q^{\frac{1}{2}}}\right). \tag{28}$$

Regarding this as an analytic function of $q^{\frac{1}{2}}$, we see that the interior of the unit circle is free from singularities. Evaluating c_n by Cauchy's integral along any convenient path, say the circle $|q^{\frac{1}{2}}| = 1/\sqrt{2}$, we find the inequality

$$c_n < (\sqrt{2})^n \cdot 4\log(\sqrt{2}+1) < 34\,(\sqrt{2})^n. \tag{29}$$

The expressions obtained for the *fonctions majorantes* of $\sqrt{a/\varpi}$ and of the Lamé cosine series ensure that the error involved in retaining only terms of order $q^{\frac{1}{2}n}$ in (25) is less than a certain multiple of $q^{\frac{1}{2}(n+1)}$. But the validity of an expansion of the continuous monotonic function $\sqrt{(a/\varpi)}$, as a generalized Fourier series (25), follows from the theory of systems of orthogonal functions, satisfying a differential equation and a set of Sturm-Liouville conditions at the boundaries of a fixed interval.

IV.

Instead of the treatment of the previous paper, the second factor of each normal product may be dealt with by the methods of Brioschi, Halphen, Jamet, and Crawford*. In Lamé's equation† of order $\frac{1}{2}$,

$$\frac{d^2y}{dx^2} = (\tfrac{3}{4}k^2\,\mathrm{sn}^2 x + B)\,y, \tag{30}$$

we put

$$x = 2z - iK', \quad y = w/\sqrt{F(z)}, \tag{31}$$

where

$$F(z) \equiv \mathrm{sn}\,z\ \mathrm{cn}\,z\ \mathrm{dn}\,z.$$

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* Halphen, *Fonctions elliptiques*, 2 (1888), 471; Whittaker and Watson, *Modern analysis*, chap. xxiii.

† No confusion should arise from the use of the convenient variables x, y, z, t independently of their sense in other sections.

A straightforward calculation gives the equation

$$F(z)\frac{d}{dz}\left\{\frac{1}{F(z)}\frac{dw}{dz}\right\}+4\beta=0, \tag{32a}$$

where the parameter β is defined by

$$\beta+B+\tfrac{1}{4}(1+k^2)=0. \tag{33}$$

By putting $t=\mathrm{dn}^2 z$, we obtain an equation of the Fuchsian class with four regular singularities, namely

$$t(1-t)(t-k'^2)\frac{d^2w}{dt^2}+\beta w=0. \tag{32b}$$

The singularities whose affixes are 0, 1, k'^2 have the exponents (0, 1); and that at infinity the exponents (0, -1), with the usual convention that the variable is taken to be t^{-1}. These four singularities correspond to singularities of identical character in the original equation (30), so that it is sufficient to examine one of them; we choose $t=\infty$. Now, since the exponents differ by an integer, the singularity must be either logarithmic or apparent. By substituting the trial solution

$$w=t\sum_{n=0}^{\infty}u_n t^{-n}\quad[u_0\neq 0], \tag{34}$$

we obtain the condition

$$\beta u_0=0. \tag{35}$$

(i) If $\beta=0$, the singularity is apparent and the equation (32b) is trivial. This is one of the cases where Lamé's equation can be integrated in finite terms, and the solution has been given in various forms by Jamet and Halphen. We find, in particular, the two forms

$$\left.\begin{aligned} w&=c_1+c_2\,\mathrm{dn}^2 z,\\ y&=\frac{\{c_3\,\mathrm{cn}\,\tfrac{1}{2}x\,\mathrm{dn}\,\tfrac{1}{2}x+c_4\,\mathrm{sn}\,\tfrac{1}{2}x\}}{\{1-k^2\,\mathrm{sn}^4\,\tfrac{1}{2}x\}^{\frac{1}{2}}}.\end{aligned}\right\} \tag{36}$$

(ii) If $\beta\neq 0$, a solution such as (34), belonging to the exponent -1 and free from logarithms, cannot exist. But the solution belonging to the exponent 0 is free from logarithms at $t=\infty$ and can be expanded in various ways.

Our procedure will be influenced by the fact that, in this special problem, the solution must be adapted for calculation when x lies between 0 and iK' or when $1+k\leqslant t<\infty$.

If we follow Crawford's method and expand in descending powers of t, the solution is valid for $|t|>1$, and converges at our least favourable point like $\Sigma(1+k)^{-n}$.

Better convergence is given by expanding in descending powers of $t-\frac{1}{2}$, but the coefficients are more complicated. This will behave like $\Sigma(1+2k)^{-n}$ at our least favourable point, since the solution is valid outside the circle $|t-\frac{1}{2}|=\frac{1}{2}$, which is the smallest circle enclosing the singularities 0, 1, k'^2.

We write
$$\left.\begin{aligned} w &\equiv \sum_{n=0}^{\infty} a_n t^{-n} \equiv \sum_{n=0}^{\infty} b_n (t-\tfrac{1}{2})^{-n}, \\ \tfrac{1}{2}-k^2 &= k'^2-\tfrac{1}{2} = h, \end{aligned}\right\} \tag{37}$$

and find the recurrence formulae

$$(n+1)(n+2)\,a_{n+1}-\{\beta+n(n+1)(1+k'^2)\}\,a_n+k'^2 n(n-1)\,a_{n-1}=0, \tag{38}$$

$$(n+1)(n+2)\,b_{n+1}-\{\beta+n(n+1)\,h\}\,b_n-\tfrac{1}{4}n(n-1)\,b_{n-1} +\tfrac{1}{4}(n-1)(n-2)\,h\,b_{n-2}=0, \tag{39}$$

from which we may calculate as many terms as we please. The values of a_n are given by determinants of the type occurring in continued fractions, all of whose elements are zero, except along the principal diagonal and the two adjacent diagonals. Thus we have :

$$\frac{n!\,(n+1)!\,a_n}{\beta a_0} = \begin{vmatrix} \beta+1.2k'^2, & -2.3, & 0, & \dots & \dots & \dots & \dots & \dots \\ -1.2k'^2, & \beta+2.3k'^2, & -3.4, & \dots & \dots & \dots & \dots & \dots \\ 0, & -2.3k'^2, & \beta+3.4k'^2, & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \beta+(n-1)(n-2)\,k'^2, & -n(n-1) & \\ \dots & \dots & \dots & \dots & \dots & -(n-1)(n-2)\,k'^2, & \beta+n(n-1)\,k'^2 & \end{vmatrix}. \tag{40}$$

This solution depends on s only through the parameter β We may therefore write it in the form $w(\beta,\ t^{-1})$. In order that the solution may take the value unity at the surface of the flat ring, we must divide by $w\,[\beta,\ (1+k)^{-1}]$. We shall denote $G^{(1)}(\beta,\ x)$, the solution of (30) corresponding to this and reducing to unity when $x=0$, $z=\frac{1}{2}iK'$, $t=1+k$, namely

$$G^{(1)}(\beta,\ x) \equiv \left\{\frac{\operatorname{cn}\tfrac{1}{2}x\,\operatorname{dn}\tfrac{1}{2}x+i(1+k)\operatorname{sn}\tfrac{1}{2}x}{\sqrt{(1-k^2\operatorname{sn}^4\tfrac{1}{2}x)}}\right\} \cdot \frac{w\{\beta,\ 1/\operatorname{dn}^2\tfrac{1}{2}(x+iK')\}}{w\{\beta,\ 1/(1+k)\}}. \tag{41}$$

The upper index, with the same meaning as in (16), is present because there is a corresponding reduction of Lamé's equation of order $m-\frac{1}{2}$, leading to functions of index m. As $x\to iK'$, we can verify the

approximate formula

$$G^{(1)}(\beta,\ iK'-i\epsilon)\sim g(\beta)\left[\epsilon^3 k(1+k)/8\right]^{\frac{1}{2}}, \tag{42}$$

where ϵ is small and, by definition,

$$g(\beta)=\frac{a_0}{\sum_{n=0}^{\infty}\frac{a_n}{(1+k)^n}}=\frac{b_0}{\sum_{n=0}^{\infty}\frac{b_n}{(\frac{1}{2}+k)^n}}. \tag{43}$$

The above analysis holds for any value of β. But, in the potential problem, β has a value determined by our previous choice of b; on eliminating B between (13) and (33), we have

$$\beta+\tfrac{1}{4}(1+k^2)=\frac{\pi^2}{4K^2}b+\frac{3(K-E)}{4K}. \tag{44}$$

For such a choice of β, it can be shown directly by the methods of Sturm and Liouville that the expression (41) cannot vanish again in the interior of the segment $(0,\ iK')$, a result previously inferred from the principle that the potential cannot have a maximum in free space. For, on the one hand, in the equation (12) the solution $f(x)$ cannot be periodic if the coefficient of $f(x)$ is always negative for real values of x; referring back to the form (9), we see that $B<0$, and *a fortiori* $(\frac{3}{4}k^2\,\mathrm{sn}^2\,iv+B)<0$, when v is real. But this means that $G(iv)$, which is a real function of v, is monotonic and convex to the axis of v in the region $0<v<K'$.

A further application of the method shows that $\beta_0<0<\beta_1<\beta_2\ldots.$ For β increases with the density of the real zeros of the Lamé cosine series. The value β_0 corresponds to a solution which has no real zeros, and β_1 to a solution with zeros at intervals $2K$. Now the value $\beta=0$, corresponding to the special solution (36), is not one of our critical values, because the solution admits $4K$ only as a semi-period; and, since its real zeros (if any) occur at intervals $4K$, when the constants in (36) are chosen to be real, the value $\beta=0$ must separate β_0 and β_1.

In the equation (32) increasing positive values of β correspond to increasing curvature of the graph of the solution w, so that, of two solutions tending asymptotically to the value $+1$ at $t=\infty$, the one with the larger positive value of β will be greater for $t>1$. Now, by (43), we see that $g(\beta)$ is a decreasing function of β, and this fact has an important bearing on the convergence of a series we shall obtain in the next section.

V.

The normal solutions of the equation (8) are clearly

$$\Psi_s(u,\ v)=cl_s^{(1)}(x)\,G^{(1)}(\beta_s,\ iv), \tag{45}$$

where we have restored the notation $\pi u=2Kx$.

We now expand the constant ψ_1 in a form suggested by (25), namely

$$\psi_1 = \psi_1 \left(\frac{\varpi}{a}\right)^{\frac{1}{2}} \sum_{s=0}^{\infty} \gamma_s c l_s^{(1)}(x). \tag{46}$$

This is the value of Stokes's current function on the ring, and its general value, expanded in a series of normal solutions of (6), is therefore

$$\psi = \psi_1 \left(\frac{\varpi}{a}\right)^{\frac{1}{2}} \sum_{s=0}^{\infty} \gamma_s \Psi\,(u, v). \tag{47}$$

On the axis we have the exact values

$$\left.\begin{aligned} v = K', \quad \varpi = 0, \quad z &= \frac{a(\mathrm{dn}\, u - \mathrm{cn}\, u)}{k' \,\mathrm{sn}\, u}, \\ -\frac{\partial \varpi}{\partial v} = \frac{\partial z}{\partial u} &= \frac{ak'}{\mathrm{dn}\, u + \mathrm{cn}\, u}, \end{aligned}\right\} \tag{48}$$

and the approximation, valid near the axis,

$$\varpi \sim (v - K') \frac{\partial \varpi}{\partial v} = \frac{(K' - v)\, ak'}{\mathrm{dn}\, u + \mathrm{cn}\, u} \tag{49}$$

Inserting the approximations (42) and (49) in the formulae (45) and (47), we find for ψ an approximation valid near the axis

$$\psi \sim \psi_1 \left\{\frac{kk'(1+k)}{8(\mathrm{dn}\, u + \mathrm{cn}\, u)}\right\}^{\frac{1}{2}} S(x)(v - K')^2, \tag{50}$$

where

$$S(x) \equiv \sum_{s=0}^{\infty} \gamma_s g(\beta_s)\, c l_s^{(1)}(x). \tag{51}$$

The velocity-potential is given by integrating (5) and is a single-valued function of (u, v) in the strip $0 \leqslant v \leqslant K'$. A circuit threading the ring from the side where $z < 0$ gives an increase of $4K$ in u. The corresponding circulation κ may be found by deforming this circuit into the axis of z described from $-\infty$ to ∞, and a distant return path, making no sensible contribution. Thus we have

$$V(u, v) = \int_{(u_0, v_0)}^{(u, v)} \frac{1}{\varpi} \left(\frac{\partial \psi}{\partial u}\, dv - \frac{\partial \psi}{\partial v}\, du\right), \tag{52}$$

$$\kappa = V(2K, K') - V(-2K, K') = \int_{-2K}^{2K} \frac{\partial}{\partial u} V(u, K')\, du. \tag{53}$$

Combining (5) and (50), we find

$$\frac{\partial}{\partial u} V(u, K') = -\lim_{v \to K'} \frac{1}{\varpi} \frac{\partial \psi}{\partial v} = \frac{\psi_1}{a} \left\{\frac{k(1+k)(\mathrm{dn}\, u + \mathrm{cn}\, u)}{2k'}\right\}^{\frac{1}{2}} S(x). \tag{54}$$

On substituting this value in (53), expanding $S(x)$ according to the formula (51), and integrating term by term, taking x as the variable of integration, we have an expression for κ as an infinite series, namely

$$\kappa = \frac{2K\psi_1}{a}\sqrt{\{k(1+k)\}}\,.\sum_{s=0}^{\infty}\delta_s g(\beta_s), \tag{55}$$

where
$$\delta_s = \gamma_s\int_{-\pi}^{\pi}\left\{\frac{\mathrm{dn}\,(2Kx)/\pi+\mathrm{cn}\,(2Kx)/\pi}{2k'}\right\}^{\frac{1}{2}} cl_s^{(1)}(x)\,dx. \tag{56}$$

The function $(\mathrm{dn}\,u+\mathrm{cn}\,u)$ has double zeros congruent to $2K$, and simple poles congruent to iK', $2K+iK'$, *modulis* $4K$, $2iK'$. By comparing the value at $u=K$, we have

$$\left(\frac{\mathrm{dn}\,u+\mathrm{cn}\,u}{2k'}\right)^{\frac{1}{2}}$$

$$= \frac{\cos\frac{x}{2}\prod_{m=1}^{\infty}(1+2q^m\cos x+q^{2m})(1+q^{2m-1})}{\prod_{m=1}^{\infty}(1-2q^{2m-1}\cos 2x+q^{4m-2})^{\frac{1}{2}}(1+q^{2m})}$$

$$= \frac{\left\{\sum_{m=0}^{\infty}q^{m(m+1)/2}\cos(m+\frac{1}{2})x\right\}\left\{1+\sum_{m=1}^{\infty}2q^{m^2}\right\}^{\frac{1}{2}}}{\left\{1+\sum_{m=1}^{\infty}2q^{m^2}(-)^m\cos 2mx\right\}^{\frac{1}{2}}\{1-q-q^3+q^6+q^{10}-q^{15}\ldots\}}. \tag{57a}$$

Neglecting q^4, we find

$$\left(\frac{\mathrm{dn}\,u+\mathrm{cn}\,u}{2k'}\right)^{\frac{1}{2}} = \frac{(\cos\frac{1}{2}x+q\cos\frac{3}{2}x+q^3\cos\frac{5}{2}x)(1+\frac{2q+\frac{3}{2}q^2+3q^3)}{(1-2q\cos 2x)^{\frac{1}{2}}}+O(q^4), \tag{57b}$$

and we are now enabled to evaluate δ_s with the same degree of approximation as the γ_s given in (26).

I find the values

$$\left.\begin{aligned}
&\delta_0 = 4-10q+\tfrac{205}{8}q^2+\tfrac{5739}{80}q^3\ldots, && \delta_1 = \tfrac{8}{3}q^{\frac{1}{2}}+\tfrac{47}{3}q^{\frac{3}{2}}+\tfrac{6077}{132}q^{\frac{5}{2}}\ldots,\\
&\delta_2 = \tfrac{2}{15}q-\tfrac{1}{6}q^2-\tfrac{97}{192}q^3\ldots, && \delta_3 = \tfrac{1}{35}q^{\frac{3}{2}}-\tfrac{41}{560}q^{\frac{5}{2}}\ldots,\\
&\delta_4 = \tfrac{5}{504}q^2-\tfrac{23}{1008}q^3\ldots, && \delta_5 = \tfrac{13}{1584}q^{\frac{5}{2}}\ldots,\\
&\delta_6 = \tfrac{21}{9152}q^3\ldots.
\end{aligned}\right\} \tag{58}$$

The expression (56) exhibits δ_s as a product of two generalized Fourier coefficients, corresponding to the expansions of $(\mathrm{dn}\,u+k\,\mathrm{cn}\,u)^{\frac{1}{2}}$ and $|\,\mathrm{dn}\,u+\mathrm{cn}\,u\,|^{\frac{1}{2}}$ as series of $cl_s^{(1)}(x)$. The generalized Fourier series of the former is highly convergent, since the function may be differentiated as

often as we please; the latter function has a discontinuity of the derivative at $u = \pm 2K$, and so its Fourier constants may be expected to be $O(s^{-2})$. In any case, $\Sigma\delta_s$ is absolutely convergent and $g(\beta_s)$ is monotonic decreasing, for $s \geqslant 1$, so that the convergence of (55) is assured.

The two points of the investigation which appear to the writer most deserving of further attention are the existence theorems in the earlier part of the paper and the practical calculations in the latter part. The crude *fonction majorante* obtained for the Lamé cosine series appears to give a very inadequate notion of the true rate of convergence. Trivial improvements in the proof, which would only make it a little longer, would cover the series for which s is large; but a new method seems to be needed for small values of s. On the other hand, to evaluate $g(\beta_s)$ we have only to calculate a power series at a point in the interior of its region of convergence, and the difficulty does not lie in the theory, but in the labour of the calculations. A good approximate formula for $g(\beta_s)$, not in the form of a power series, would be very useful. It is noteworthy that the calculations of the second part of the paper become more difficult as k diminishes and as those of the first part become easier.

STIELTJES' DERIVATIVES

By P. J. DANIELL.

[Received 15 November, 1928.—Read 13 December, 1928.]

Several attempts have been made to obtain the fundamental theorem for differentiation of one additive function of sets with respect to another in the case of several dimensions. Difficulty arises on account of the fact that the ratio of the masses of two concentric intervals whose edges are in a fixed ratio is not necessarily bounded. This invalidates, for our purpose, both Vitali's theorem and the methods of de la Vallée Poussin*, which were invented to suit the case of differentiation with respect to Lebesgue measure. We shall refer to the above-mentioned book as d.l.V.P.

The present writer has turned this difficulty by replacing the ordinary derivative by a net derivative† and by a core derivative‡. But neither of these is completely satisfying. In the present paper we consider the proper central derivative, but we restrict its definition in certain ways, and then show how de la Vallée Poussin's methods can be amended to suit our problem.

1.1 We consider a space of k dimensions consisting of points $x = (x_1, x_2, \ldots, x_k)$. A closed central square interval $D(x_0, \delta)$ consists of the points x such that

$$(x_0)_i - \delta \leqslant x_i \leqslant (x_0)_i + \delta \quad (i = 1, 2, \ldots, k).$$

Where there is no ambiguity we shall mean by "an interval" such a special interval, and by a "sequence of intervals" a sequence of such

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* de la Vallée Poussin, *Intégrales de Lebesgue* (Paris, 1916).

† P. J. Daniell, *Bull. Amer. Math. Soc.*, 26 (1920), 444.

‡ P. J. Daniell, *Proc. London Math. Soc.* (2), 26 (1926), 95.

intervals centred at the same point x and with δ tending to zero. All our sets of points will be understood to be measurable in the sense of Borel.

A general "mass" or additive function of sets $\mu(e)$ can be expressed in the form

$$\mu(e) = \mu(eE_1) + \mu(eE_2),$$

where E_1, E_2 are disjoint and

$$\mu(eE_1) = p(e),\ \mu(eE_2) = -n(e),$$

where $p(e)$, $n(e)$ are *positive* (that is to say never negative) masses. We can then define

$$|\mu|(e) = p(e) + n(e)$$

as the *modular mass* associated with $\mu(e)$. In this paper $m(e)$ will denote a general *positive* mass.

1.2. *Definition.* If $D(x, \delta)$ is a square interval centred at x, and if $D' = D(x, \lambda\delta)$ $(\lambda > 1)$, then

$$\begin{aligned} \rho(\lambda, D) &= mD'/mD \quad (mD' > 0), \\ &= 1 \qquad\quad (mD' = 0), \end{aligned}$$

is called the *index of irregularity* of D for m relative to λ. It is clear that, if $\lambda_1 < \lambda_2$,

$$\rho(\lambda_1, D) \leqslant \rho(\lambda_2, D).$$

1.3. *Definition.* The upper and lower irregularities of a point x relative to λ are the upper and lower limits of $\rho(\lambda, D)$ where $D = D(x, \delta)$ as δ tends to zero.

A regular point x is one for which a value of $\lambda > 1$ exists such that the upper limit of $\rho(\lambda, D)$ is finite. An irregular point x is one for which the lower limit of $\rho(\lambda, D)$ is infinite for all $\lambda > 1$.

A partially regular point is one which is not irregular. Hence regular points are also partially regular.

1.4. *Definition.* Given the additive functions of sets $\mu(e)$, $m(e)$ $(m \geqslant 0)$, we define

$$\begin{aligned} fD(x, \delta) &= \frac{\mu D(x, \delta)}{mD(x, \delta)} \quad (mD > 0), \\ &= +\infty \qquad (mD = 0,\ \mu D > 0), \\ &= -\infty \qquad (mD = 0,\ \mu D < 0), \\ &= 0 \qquad\quad (mD = 0,\ \mu D = 0). \end{aligned}$$

Given a positive R and $\lambda > 1$, we define

$$UD(R, \lambda) = \overline{\lim} fD(x, \delta) \quad (\delta \to 0),$$

taking into consideration only intervals D *for which*

$$\rho(\lambda, D) \leqslant R.$$

Similarly
$$LD(R, \lambda) = \underline{\lim} fD(x, \delta) \quad (\delta \to 0),$$

using the same intervals.

1.41. The intervals for which $\rho(\lambda_1, D) \leqslant R_1$ are included among those for which $\rho(\lambda_2, D) \leqslant R_2$ if $R_1 \leqslant R_2$ and $\lambda_1 \geqslant \lambda_2$.

Hence we can define the *central upper and lower derived numbers* at x as

$$UD\mu = \lim UD(R, \lambda) \quad (R \to \infty, \ \lambda \to 1),$$

$$LD\mu = \lim LD(R, \lambda) \quad (R \to \infty, \ \lambda \to 1).$$

At all partially regular points these can be defined. If $UD\mu = LD\mu$ we say that each equals $D\mu$ the *central derivative* of μ with respect to m.

1.5. Consider a net system $N_n(a, t)$, of which the x_i-coordinates of the knots are of the form

$$a_i + p_i t/2^n,$$

where p_i is a positive or negative integer or zero, a_i is zero or a proper rational fraction of the form r_i/q, with q an odd integer and where t is a positive number.

[If the whole space is being considered we can suppose that at stage n we only take those knots whose coordinates lie between $\pm n$.]

Following d.l.V.P. (page 64), the q^k nets obtained by taking a fixed q and varying r_i we shall call associated nets.

If, in addition, we vary t so that

$$t = 2^{j/s} \quad (j = 0, 1, 2, \ldots, s-1),$$

the whole set of nets will be called an (s, q) complex of nets.

1.6. *The set of irregular points is of zero mass in* m.

Proof. This set is included, for any $\lambda > 1$, in the set where

$$\underline{\lim} \rho(\lambda, D) = +\infty.$$

Let x be one of these points and let lD denote the Lebesgue measure of an interval D. Then

$$lD(x, \lambda D)/lD(x, D) = \lambda^k.$$

It is possible to choose $\eta > 0$ and θ $(0 < \theta < 1)$ such that, if $\delta < \eta$,

$$mD(x, \delta)/mD(x, \lambda\delta) < \theta\lambda^{-k}.$$

Let $D_0 = D(x, \eta)$, and let r be the greatest integer such that

$$\delta \leqslant \lambda^{-r}\eta.$$

Then r increases indefinitely with $1/\delta$ and

$$\frac{mD(x, \delta)}{lD(x, \delta)} < \theta^r\lambda^k \frac{mD_0}{lD_0},$$

so that $\lim mD/lD = 0$.

Suppose now that we are given an interval $D(x, \delta)$ and a set of associated nets. Choose n as the largest integer such that

$$\frac{q-1}{q} \frac{t}{2^n} \geqslant 2\delta.$$

There cannot be more than $q-1$ consecutive numbers of the form

$$r_i/q + p_i t/2^n$$

in $(x_i-\delta, x_i+\delta)$. In each coordinate we can pick out a mesh interval (α_i, β_i) of order n covering $(x_i-\delta, x_i+\delta)$ and therefore a mesh ω_n of one of the associated nets enclosing D. Now

$$\frac{q-1}{q} \frac{t}{2^{n+1}} < 2\delta,$$

so that, if $q \geqslant 3$,

$$t\,2^{-(n+3)} < \delta,$$

and there must be knots of the chosen net of order $n+3$ in $(x_i-\delta, x_i)$ and in $(x_i, x_i+\delta)$. Hence there is a mesh ω_{n+3} enclosing x and contained in D. But

$$\frac{m\omega_{n+3}}{l\omega_{n+3}} = \frac{m\omega_{n+3}}{lD} \cdot \frac{lD}{l\omega_{n+3}}$$

$$\leqslant \frac{mD}{lD} \cdot \frac{l\omega_n}{l\omega_{n+3}} \leqslant 8^k \frac{mD}{lD}.$$

As δ tends to zero, the mesh ω_{n+3} will be chosen from one or other of the associated nets. Hence, at an irregular point, the lower derived number of m with respect to l must be zero for at least one of the asso-

ciated nets. The theorem then follows from d.l.V.P. (page 97). This shows that the central derived numbers are defined nearly everywhere (m).

1.7. *Given that $\mu(e)$ is a positive mass, if its net derivative at x with respect to m is zero for all nets of all (s, q) complexes with $s = 8r$, $q = 2r+1$ $(r = 2, 3, \ldots)$, then the central derivative at x is also zero provided that x is not irregular.*

Proof. If x is not irregular it is possible to find $r > 1$ so that

$$\underline{\lim}\, \rho(\lambda, D) = \text{finite} \quad \text{with} \quad \lambda = 1+2/r.$$

Then, for this r,

$$\frac{2q}{q-1} = 2+\frac{1}{r} < 1+\lambda,$$

$$1+\frac{2}{s} < (1+\lambda)\frac{q-1}{2q}.$$

But $s > 1$, and therefore

$$\frac{1}{s} < \log_2\left(1+\frac{2}{s}\right) < \log_2\left[(1+\lambda)\frac{q-1}{2q}\right].$$

Given any sufficiently small positive δ it is possible to choose integers n and j so that

$$\log_2 \frac{2q}{q-1}+\log_2 \delta < \frac{j}{s}-n < \log_2(1+\lambda)+\log_2 \delta.$$

That is to say, it is possible to choose an integer n and a number $t = 2^{j/s}$, so that simultaneously

$$\frac{q-1}{q}\,\frac{t}{2^n} > 2\delta, \qquad \frac{t}{2^n} < (1+\lambda)\,\delta.$$

By the same reasoning as in (1.6) there exists a mesh ω_n of one of the q^k associated nets for this t enclosing $D(x, \delta)$. In the x_i-coordinate let its end points be α_i, β_i. By the second inequality there must be a knot between $x_i-\lambda\delta$ and $x_i+\delta$. This cannot be between $x_i \pm \delta$, so that it must lie between $x_i-\lambda\delta$ and $x_i-\delta$. Hence α_i must be between these values. Similarly β_i must be between $x_i+\delta$ and $x_i+\lambda\delta$. Hence ω_n is included in $D(x, \lambda\delta)$. Then

$$\frac{\mu D(x, \delta)}{m D(x, \delta)} \leqslant \frac{\mu\omega_n}{m\omega_n} \cdot \frac{m D(x, \lambda\delta)}{m D(x, \delta)}$$

$$= \frac{\mu\omega_n}{m\omega_n}\,\rho(D, \lambda).$$

By hypothesis there exists a number R and intervals $D(x, \delta)$, with δ arbitrarily small, such that $\rho(D, \lambda) \leqslant R$. Hence

$$UD(R, \lambda) = 0,$$

for all such R and for any $\lambda = 1+2/r$. Thus, finally, $UD\mu = 0$ and $0 \leqslant LD\mu \leqslant UD\mu$, so that $D\mu = 0$.

2. *Net derivatives.*

The previous theorem will enable us to obtain results for central derivatives after the facts are known about the more manageable net derivatives. In this Section 2 "derivatives" will mean net derivatives for a particular net.

2.1. If E is an interval or any set which is the sum of a countable infinity of meshes, and if, on E, $UD\mu > 0$, then $\mu E \geqslant 0$.

Proof. It is sufficient to prove the theorem for a single mesh ω.

First, if $\mu\omega < 0$, as the division proceeds there will be a sequence of meshes ω_n defining a point x, and for each of these $\mu\omega_n < 0$, so that, at x, $UD\mu \leqslant 0$, contrary to hypothesis. Secondly, if $\mu\omega = 0$, the previous paragraph shows that μ must be zero for every mesh and $UD\mu = 0$ everywhere.

2.2. If, at every point of a set E contained in an interval J, $UD\mu > 0$, then $\mu E \geqslant 0$.

Proof. μE is the limit of μF_n where F_n is a suitable sequence of closed sets contained in E. We may therefore suppose E to be *closed*. Consider

$$\nu(e) = \mu(e) + |\mu|(e \,.\, CE) + m(e \,.\, CE) + l(e \,.\, CE),$$

where l is the Lebesgue measure. Then $\mu E = \nu E$ and, on E, $UD\nu \geqslant UD\mu > 0$.

Every point of CE is in the interior of an interval Δ which is disjoint from E and, for any mesh ω in Δ,

$$\begin{aligned} \nu\omega &= \mu\omega + |\mu|\omega + m\omega + l\omega \\ &\geqslant m\omega + l\omega, \end{aligned}$$

so that, at every point of CE, $UD\nu > 0$. Hence, by (2.1), $\nu K \geqslant 0$ for every open set contained in J. But νE is the limit of νK_n, where K_n is a suitable sequence of open sets containing E. Hence $\mu E = \nu E \geqslant 0$.

2.3. If, at every point of a bounded set E, $UD\mu \geqslant 0$, then $\mu E \geqslant 0$.

To prove this choose any $\epsilon > 0$, and define

$$\nu e = \mu e + \epsilon me + \epsilon le$$

and use (2.2), and then make ϵ tend to zero.

2.4. If E is a bounded set for which $mE > 0$, and if, for every set e in E, $\mu e = 0$, then, at nearly every point of E, $D\mu = 0$.

To prove this, since $|\mu|E = 0$, it is sufficient to assume μ to be a positive mass. If the theorem is untrue, there exists an $\epsilon > 0$ and a set F contained in E for which $mF > 0$, and on which $UD\mu \geqslant \epsilon$. If we define

$$\nu e = \mu e - \epsilon me,$$

then $UD\nu \geqslant 0$ everywhere on F, and, by (2.3),

$$\mu E \geqslant \mu F = \nu F + \epsilon mF \geqslant \epsilon mF > 0.$$

2.5. The density relative to m of a set E is equal to 1 on E and 0 on CE nearly everywhere (m).

The proof follows from (2.4) and the equation

$$me = m(eE) + m(e \,.\, CE),$$

so that, *except on the desert of* m (that is, the points contained in intervals D for which $mD = 0$),

$$UDm(eE) + LDm(e \,.\, CE) = LDm(eE) + UDm(e \,.\, CE)$$

$$= 1.$$

Furthermore, the desert is of zero mass in m.

2.6. The derivative of

$$F(e) = \int_e f(x)\, dm(e)$$

is equal to $f(x)$ nearly everywhere (m) provided that $f(x)$ is integrable with respect to m.

The proof agrees with d.l.V.P. (page 72) now that the preparatory lemmata have been obtained.

2.71. *Definition.* $\mu(e)$ is said to be *based on* $m(e)$ if $m(e) = 0$ implies $\mu(e) = 0$.

2.72. If $\mu(e)$ is based on $m(e)$, and $UD\mu > 0$ nearly everywhere (m) on E where $mE > 0$, then $\mu E > 0$.

To prove this, take away from E the set E_1 for which $UD\mu \leqslant 0$. Then, since $mE_1 = 0$, $\mu E_1 = 0$ and $\mu E = \mu(E - E_1)$. By hypothesis it is possible to find an $\epsilon > 0$ and a set F contained in $E - E_1$ for which $mF > 0$ and on which $UD\mu \geqslant \epsilon$. Then, on F, $UD(\mu - \epsilon m) \geqslant 0$ and $\mu F \geqslant \epsilon m F > 0$. The set $E - E_1$ is the sum of the disjoint sets F_n for which

$$1/n \leqslant UD\mu < 1/(n-1),$$

and, for at least one of these, $mF_n > 0$, so that $\mu F_n > 0$ for at least one set F_n, and is only non-positive when $mF_n = 0$, so that $\mu F_n = 0$. Hence

$$\mu E = \mu(E - E_1) = \Sigma \mu F_n > 0.$$

The following theorems of this section, (2.73), (2.74), (2.75), and (2.8) are proved as in d.l.V.P. (pages 68, 68, 73, and 97 respectively).

2.73. If $\mu(e)$ is based on $m(e)$ and $UD\mu = 0$ nearly everywhere (m) on E, then $\mu E = 0$.

2.74. If μ_1, μ_2 are based on m, and if $UD\mu_1$, $UD\mu_2$ are finite nearly everywhere (m) in E, and if, on E,

$$UD\mu_1 \geqslant UD\mu_2$$

nearly everywhere (m), then

$$\mu_1 E \geqslant \mu_2 E.$$

2.75. If $\mu(e)$ is based on $m(e)$, it has a finite derivative nearly everywhere (m), which is integrable over any bounded set (we assume all sets to be measurable in the sense of Borel) and

$$\mu(e) = \int_e D\mu \,.\, dm(e).$$

2.8. If $\mu(e)$, $m(e)$ are additive functions of sets of which $m(e)$ is positive, the net derivative of μ, viz. $D\mu$, is defined and finite nearly everywhere (m); it is integrable with respect to m and

$$\mu(e) = \mu(eF_1) + \mu(eF_2) + \int_e D\mu \,.\, dm(e),$$

where F_1 is the set of points where $D\mu = +\infty$ and F_2 is the set where $D\mu = -\infty$.

As de la Vallée Poussin points out in the case of Lebesgue measure, so here the sets F_1, F_2 can be reduced to those on which the net deriva-

tive of μ is $+\infty$ $(-\infty)$ on every one of the countable collection of complexes of nets ($s = 8r$, $q = 2r+1$, $r = 2, 3, \ldots$) of (1.7). Also, except on F_1 and F_2, the net derivatives of these nets will all exist and be equal to the above $D\mu$, except on a set whose mass both in m and in $|\mu|$ is zero.

3. *Central derivatives.*

3.1. If E is a bounded set for which $mE > 0$ and such that, for every e in E, $\mu e = 0$, then, at nearly every point of E, $D\mu = 0$, where $D\mu$ now refers to the central derivative.

Proof. This follows directly from (1.7) if we use (2.4) for each of the countable infinity of complexes of nets ($s = 8r$, $q = 2r+1$, $r = 2, 3, \ldots$), and remember that the sum of a countable infinity of sets of zero mass in m is also of zero mass in m.

3.2. The (central) density relative to m of a set E is equal to 1 on E and to 0 on CE nearly everywhere (m). The proof is the same as that of (2.5), using (3.1) in place of (2.4). The only point requiring notice is that the irregular points for m also form a part of the exceptional set of zero mass.

3.3. The central derivative of

$$F(e) = \int_e f(x)\, dm(e)$$

is equal to $f(x)$ nearly everywhere (m) provided that $f(x)$ is integrable with respect to m.

The proof is the same as for (2.6).

3.4. *If $\mu(e)$, $m(e)$ are additive functions of sets of which $m(e)$ is positive, the central derivative of μ, $D\mu$, is defined and finite nearly everywhere (m) and is integrable with respect to m, and*

$$\mu(e) = \mu(eF_1) + \mu(eF_2) + \int_e D\mu \,.\, dm(e),$$

where F_1, F_2 are the sets of zero mass in m defined in (2.8).

Proof. On $CF_1 \,.\, CF_2$ the derivative of μ is equal to that of

$$\int_e N\, dm(e),$$

where N is the net derivative on account of (3.1). Then, by (3.3), this

central derivative is equal to N nearly everywhere (m), and the theorem is proved.

If e is a set contained in $CF_1 . CF_2$, $me = 0$ implies $\mu e = 0$. *Hence the central derivative exists and is finite, except on* F_1, F_2, *together with a set whose mass is zero both in* m *and in* $|\mu|$.

3.51. *Definition.* The positive mass M is said to be *regular* (λ) at x *relative to* m provided that, at x,

$$\underline{\lim}\, \rho(\lambda, D) = R \text{ is finite,}$$

and that wherever

$$\rho(\lambda, D) = \frac{mD(x, \lambda\delta)}{mD(x, \delta)} \leqslant R_1,$$

there exists a finite R_2 and a number $\lambda_1 > 1$ such that

$$\frac{MD(x, \lambda_1\delta)}{MD(x, \delta)} \leqslant R_2.$$

3.52. *Definition.* M is said to be regular at x relative to m if there exists a $\lambda_0 > 1$ such that M is regular (λ) for all λ such that $\lambda_0 \geqslant \lambda > 1$.

3.53. *Definition.* M is said to be regular relative to m if it is regular except possibly on a set e for which $Me = 0$.

3.61. *If* M *is based on* m, M *is regular relative to* m.

Proof. By hypothesis $me = 0$ implies $Me = 0$, and the central derivative of M with respect to m exists and is finite nearly everywhere (m). Wherever it is finite and positive,

$$\frac{MD(x, \lambda\delta)}{MD(x, \delta)} = \frac{mD(x, \lambda\delta)}{mD(x, \delta)} \frac{1+\epsilon(\lambda\delta)}{1+\epsilon(\delta)},$$

where $\epsilon(\delta)$ tends to zero with δ. On the other hand, if e_1 is the set where $DM = 0$, then $Me_1 = 0$, and the theorem is proved.

3.62. If the *positive* mass M is regular relative to m its central derivative exists, except over a set G for which $mG + MG = 0$, and

$$M(e) = M(eF) + \int_e DM \, . \, dm(e),$$

where F is the set of points at which $DM = +\infty$.

In view of Theorem (3.4) it is evident that the only part remaining to be proved is that, if F_1 is the set defined in (2.8), the central derivative

of M is $+\infty$ over a set F contained in F_1 and for which $M(F_1-F)=0$. On all of F_1, except a set for which M is zero, the central derivative of m with respect to M is zero, and that of M with respect to m is $+\infty$, since every interval used in defining the latter derivative is among the intervals used in defining the former.

3.63. *If $|\mu|$ is regular relative to m, the central derivative of μ exists, except on a set G for which $mG+|\mu|G=0$, and*

$$\mu(e)=\mu(e.F_1)+\mu(e.F_2)+\int_e D\mu\,.\,dm(e),$$

where F_1, F_2 are the sets of points at which the central derivative is $+\infty$, $-\infty$ respectively.

Proof. Let us denote by H_1, H_2 the sets which in (3.4) are called F_1, F_2.

It only remains to be proved that the central derivative of μ is $+\infty$ over a set F_1 contained in H_1 and is $-\infty$ over a set F_2 contained in H_2 where

$$|\mu|(H_1-F_1)+|\mu|(H_2-F_2)=0.$$

By (3.62),

$$|\mu|(e)=|\mu|(e.F)+\int_e D|\mu|\,.\,dm(e)$$

and $D|\mu|=+\infty$ on F. By using the theorems already established it becomes clear that

$$|\mu|(F_1.CF)+|\mu|(F_2.CF)=0,$$

$$\begin{aligned}\mu(e)-\int_e D\mu\,dm(e)&=\mu(e.F)\\&=\mu(e.FF_1)+\mu(e.F.F_2)\\&=p_1(e)-n_1(e),\end{aligned}$$

$$\begin{aligned}|\mu|(e)-\int_e D|\mu|dm(e)&=|\mu|(e.F)\\&=\mu(e.FF_1)-\mu(e.FF_2)\\&=p_1(e)+n_1(e).\end{aligned}$$

By (3.62), on $F.F_2$, except on a set for which $n_1e=0$, and therefore $p_1e+n_1e=0$, the central derivative of p_1 is zero and that of p_1+n_1 is $+\infty$, so that the derivative of p_1-n_1 is $-\infty$. Similarly on $F.F_1$, except on a set for which $p_1e=0$ and therefore $p_1e+n_1e=0$, the central de-

rivative of $p_1 - n_1$ is $+\infty$. This, combined with already proved results, gives us the required theorem.

In conclusion it may be mentioned that the writer discarded an investigation using a modification of Vitali's theorem, because for that purpose it would have been necessary to consider as intervals in terms of which the central derivative should be defined those intervals for which

$$\rho(D, 3) \leqslant R,$$

that is, for which $\lambda = 3$. The derivative in the present paper uses a wider class of intervals, and the existence in the wider class carries with it that in the narrower. Hence all our statements (with modifications in the definitions of irregular points and of relative regularity) would remain true if λ were taken to be 3 throughout. This makes no difference at regular points, but does so at only partially regular points. It would form an interesting study to discuss the nature of partial irregularities for various classes of positive mass distributions.

MOMENTS AND PRODUCT MOMENTS OF SAMPLING DISTRIBUTIONS

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[Received 5 December, 1928.—Read 13 December, 1928.]

1. *Introductory.*

If a random sample of n observations be taken from a univariate distribution, and the sample values obtained be designated by $x_1, x_2, \ldots, x_n$, then any symmetric function of these sample values of degree r may be termed a moment function of the sample of the r-th degree. If the coefficients of the symmetric function involve the sample number n in such a way that, as n tends to infinity, the value of the function tends to a finite limit, in the sense that the probability of exceeding or falling short of that limit by a positive quantity ϵ, however small, tends to zero, then the limit to which it tends is a moment function of the population sampled, and the moment function of the sample may be regarded as a statistical estimate of the corresponding moment function of the population.

If we consider the random sampling distribution of such a statistic it is evident that the moment functions of this distribution will be expressible in terms of the moment functions of the original distribution, in so far as these are finite, by means of formulae which will be independent of the nature of this distribution. For example, a moment function of degree s of the sampling distribution of a moment function of degree r will involve only symmetric functions of the observations of degree rs, and will therefore be expressible as a moment function of the population of this degree, irrespective of the moments of higher degree.

Numerous researches have been made into the moments, chiefly of the second order, of moment statistics. The algebraic method was developed by Sheppard [1], and used extensively by Pearson [2, 3] and Isserlis [4, 5]; in all these researches, however, owing to the supposition that the mean of the sample coincides with the mean of the population, or for other similar reasons, the results are only first approximations neglecting n^{-1}. In 1913 [6] Soper obtained a number of approximations as far as n^{-2}. In 1908 "Student" [7] derived an exact formula for the second moment of the variance as estimated, which corresponds in a different notation to equation (1) of this paper for the univariate case. Later, much work, by the exact algebraic method, was carried out by Tchouproff [8], who obtained in this way the first eight moments of the mean, in addition to the univariate formulae corresponding to numbers (5) and (14). Tchouproff's version of (14) in the univariate problem was subsequently corrected by Church [9]. The application of the combinatorial method developed below to the general moments of the distribution of statistics of the second degree from normal multivariate populations has already appeared in a paper by J. Wishart [11].

Apart from the last, these results are subject to two somewhat serious limitations; the great complexity of the results attained detracts largely from the possibility either of a theoretical comprehension of their meaning, or of numerical applications; it has also led to great difficulties in the detection of errors, which have had on more than one occasion to be corrected by subsequent workers. Secondly, partly no doubt in consequence of this complexity, attention has been almost solely confined to the direct moments of single statistics, and the product moments, specifying the simultaneous distribution of two or more statistics, have been largely neglected. The total number of formulae of degree no higher than 12 is large, and it is scarcely possible that the whole body should be made available, either for study or for use, unless an improved notation can be found which will greatly simplify the algebraic expressions. It will be shown that the formulae are much simplified by the use of the cumulative moment functions, or semi-invariants, in place of the crude moments.

The importance of the formulae lies in their generality; they are applicable to all distributions for which the expressions have a meaning. In the present state of our knowledge any information, however incomplete, as to sampling distributions is likely to be of frequent use, irrespective of the fact that moment functions only provide statistical estimates of high efficiency for a special type of distribution [10].

2. *The cumulative moment functions.*

If the probability that a single sample value falls in the range dx is

$$\phi(x)\,dx,$$

then the function

$$M = \int e^{tx}\,\phi(x)\,dx,$$

taken over all possible values of the variate x, may, or may not, have a meaning for real values of t. If it has a meaning we may expand the exponential term, and, writing

$$\mu_r = \int x^r\,\phi(x)\,dx,$$

we have $$M = 1+\mu_1 t+\mu_2\frac{t^2}{2!}+\mu_3\frac{t^3}{3!}+\ldots.$$

If we expand the logarithm of M in powers of t we may write

$$K = \log M = \kappa_1 t+\kappa_2\frac{t^2}{2!}+\kappa_3\frac{t^3}{3!}+\ldots,$$

where the cumulative moment functions κ are determinate functions of the moments μ, whether the series converges or not; moreover, since κ_r involves only μ_r, and lower orders, it follows that, if $\mu_1, \ldots, \mu_r$ are finite, so will $\kappa_1, \ldots, \kappa_r$ be finite.

The expression of κ_r in terms of μ will involve the term

$$\mu_{p_1}^{\pi_1}\mu_{p_2}^{\pi_2}\ldots\mu_{p_h}^{\pi_h}$$

corresponding to any partition

$$(p_1^{\pi_1}p_2^{\pi_2}\ldots p_h^{\pi_h})$$

of the integer r, with coefficient

$$\frac{(-)^{\rho-1}(\rho-1)!}{\pi_1!\,\pi_2!\ldots\pi_h!}\,\frac{r!}{(p_1!)^{\pi_1}(p_2!)^{\pi_2}\ldots(p_h!)^{\pi_h}},$$

where $\rho = \Sigma(\pi)$ is the number of parts.

Similarly, the expression μ_r in terms of κ will involve the term

$$\kappa_{p_1}^{\pi_1}\kappa_{p_2}^{\pi_2}\ldots\kappa_{p_h}^{\pi_h}$$

with coefficient

$$\frac{1}{\pi_1!\,\pi_2!\,\dots\,\pi_h!}\,\frac{r!}{(p_1!)^{\pi_1}(p_2!)^{\pi_2}\dots(p_h!)^{\pi_h}}.$$

The simplification of moment formulae obtained by referring the moments to the mean of the distribution is due to the fact that, when $\mu_1 = 0$, no subsequent μ involves κ_1, and the number of partitions required is much reduced; thus

$$\mu_2 = \kappa_2, \quad \mu_3 = \kappa_3, \quad \mu_4 = \kappa_4 + 3\kappa_2^2,$$

and so on. The advantage of this simplification may be carried to higher orders by consistently using the cumulative moment functions κ in place of the moments μ.

The cumulative moment functions supply an immediate solution of the problem of the distribution of the mean, for, using the well known cumulative property, that, if x and y are independent variates,

$$K(x+y) = K(x)+K(y),$$

where $K(x)$ stands for the K function specifying the distribution of x, we find that, if $s_1 = S(x)$ is the sum of n independent values constituting a sample from a given distribution, then

$$\begin{aligned} K(s_1) &= nK(x) \\ &= n\kappa_1 t + n\kappa_2 \frac{t^2}{2!} + nK_3 \frac{t^3}{3!} + \dots; \end{aligned}$$

but the mean is $\bar{x} = (1/n)s_1$; consequently the K function of the mean is found by substituting t/n for t in the series for $K(s_1)$, giving

$$K(\bar{x}) = \kappa_1 t + \frac{\kappa_2}{n}\frac{t^2}{2!} + \frac{\kappa_3}{n^2}\frac{t^3}{3!} + \dots.$$

The value of κ_r in the distribution of the mean is thus found from that of the sampled distribution by dividing by n^{r-1}.

3. *The appropriate moment statistics.*

In order to take the full advantage of the properties of the cumulative moment functions, it is necessary to introduce a modification also into the form of the moment statistics; it is usual to employ statistics which

may be written $$m_r = \frac{1}{n} S(x-\bar{x})^r,$$

which are called the moments of the sample about its mean, together with the mean itself, $\bar{x}$. These moments may be expressed in terms of the symmetric functions s_r, defined by

$$s_r = S(x^r),$$

by direct expansion; for example,

$$\bar{x} = n^{-1} s_1,$$

$$m_2 = n^{-1} s_2 - n^{-2} s_1^2,$$

$$m_3 = n^{-1} s_3 - 3n^{-2} s_1 s_2 + 2n^{-3} s_1^3,$$

and so on. While the coefficients n^{-1}, n^{-2}, etc., are kept simple, we here encounter the complication that the mean value of m_s is not in finite samples equal to μ_s; in order that this should be so we should multiply m_2 by $n/(n-1)$, and m_3 by $n^2/[(n-1)(n-2)]$; further, for functions of the fourth and higher degrees, κ_r is not a linear function of the moments μ, and, in consequence, a moment statistic of which the mean is κ_r will not be exactly the same function of moment statistics, of which the means are μ_r, as κ_r is of μ_r. As a preliminary step, therefore, to the simplification of the formulae to be obtained, it will be desirable to obtain, in terms of the direct summation values s_r, the moment statistics of each degree of which the sampling means shall be κ_1, κ_2, κ_3, ... They will be represented by k_1, k_2, k_3, ...

The first few statistics which fulfil this condition are

$$k_1 = m_1 = n^{-1} s_1,$$

$$k_2 = \frac{n}{n-1} m_2 = \frac{1}{n-1} (s_2 - n^{-1} s_1^2),$$

$$k_3 = \frac{n^2}{(n-1)(n-2)} m_3 = \frac{n}{(n-1)(n-2)} (s_3 - 3n^{-1} s_1 s_2 + 2n^{-2} s_1^3),$$

$$k_4 = \frac{n^2}{(n-1)(n-2)(n-3)} \{(n+1) m_4 - 3(n-1) m_2^2\}$$

$$= \frac{n}{(n-1)(n-2)(n-3)} \{(n+1) s_4 - 4n^{-1}(n+1) s_1 s_3 - 3n^{-1}(n-1) s_2^2 + 12n^{-1} s_1^2 s_2 - 6n^{-2} s_1^4\}$$

$$k_5 = \frac{n^3}{(n-1)(n-2)(n-3)(n-4)} \{(n+5)\, m_5 - 10(n-1)\, m_2 m_3\}$$

$$= \frac{n^2}{(n-1)(n-2)(n-3)(n-4)}$$

$$\times \left\{(n+5)\, s_5 - 5\,\frac{n+5}{n}\, s_1 s_4 - 10\,\frac{n-1}{n}\, s_2 s_3 \right.$$

$$\left. + 20\,\frac{n+2}{n^2}\, s_1^2 s_3 + 30\,\frac{n-1}{n^2}\, s_1 s_2^2 - \frac{60}{n^2}\, s_1^3 s_2 + \frac{24}{n^3}\, s_1^5 \right\},$$

$$k_6 = \frac{n^2}{(n-1)\ldots(n-5)} \{(n+1)(n^2+15n-4)\, m_6 - 15(n-1)^2(n+4)\, m_2 m_4$$

$$- 10(n-1)(n^2-n+4)\, m_3^2 + 30n(n-1)(n-2)\, m_2^3\}$$

$$= \frac{n}{(n-1)\ldots(n-5)} \left\{(n+1)(n^2+15n-4)\, s_6 - 6\,\frac{n+1}{n}\,(n^2+15n-4)\, s_1 s_5 \right.$$

$$- 15\,\frac{(n-1)^2}{n}\,(n+4)\, s_2 s_4 - 10\,\frac{n-1}{n}\,(n^2-n+4)\, s_3^2$$

$$+ 30\,\frac{n^2+9n+2}{n}\, s_1 s_4 + 120\,\frac{n^2-1}{n}\, s_1 s_2 s_3$$

$$+ 30\,\frac{(n-1)(n-2)}{n}\, s_2^3 - 120\,\frac{n+3}{n}\, s_1^3 s_3$$

$$\left. - 270\,\frac{n-1}{n}\, s_1^2 s_2^2 + \frac{360}{n}\, s_1^4 s_2 - \frac{120}{n^2}\, s_1^6 \right\}.$$

If these be employed we have not only the result that the r-th cumulative moment function of the mean is $n^{-(r-1)}\kappa_r$, but also that the mean of k_r is κ_r, thus reducing a second group of the required formulae to its simplest form. It is, however, the effect of their use upon the more complex formulae which is of the greater importance. The general structure of k for any degree will be elucidated in § 10.

4. *The aggregate of moment sampling formulae.*

If we consider in its full generality the simultaneous distribution in random samples of the statistics k_1, k_2, k_3, ..., it is clear that we can represent it by means of cumulative moment functions analogous to those

developed for a single variate. To any partition

$$(p_1^{\pi_1} p_2^{\pi_2} \dots p_h^{\pi_h})$$

of the number r, there will correspond a moment

$$\mu(p_1^{\pi_1} p_2^{\pi_2} \dots p_h^{\pi_h}) = \text{mean value of } k_{p_1}^{\pi_1} k_{p_2}^{\pi_2} \dots k_{p_h}^{\pi_h},$$

and, if we write

$$M = \Sigma \mu(p_1^{\pi_1} p_2^{\pi_2} \dots p_h^{\pi_h}) \frac{t_{p_1}^{\pi_1}}{\pi_1!} \frac{t_{p_2}^{\pi_2}}{\pi_2!} \dots \frac{t_{p_h}^{\pi_h}}{\pi_h!},$$

the expansion in terms of t_1, t_2, ... of $K = \log M$ assumes the form

$$K = \Sigma \kappa(p_1^{\pi_1} p_2^{\pi_2} \dots p_h^{\pi_h}) \frac{t_{p_1}^{\pi_1}}{\pi_1!} \frac{t_{p_2}^{\pi_2}}{\pi_2!} \dots \frac{t_{p_h}^{\pi_h}}{\pi_h!}.$$

There will thus be a separate formula of degree r for every partition of the number r, and for the complete specification of the distribution each must be expanded in terms of the cumulative moment functions of the sampled population. For example, the semi-invariants of the distribution of the second moment statistic k_2 will be given by the terms corresponding to the partitions (2), (2^2), (2^3), (2^4), ..., which we designate by

$$\kappa(2), \quad \kappa(2^2), \quad \kappa(2^3), \quad \kappa(2^4), \quad \text{and so on.}$$

The well known solution of the distribution of the mean, given above, may now be written

$$\kappa(1^r) = \frac{\kappa_r}{n^{r-1}}, \tag{I}$$

while from the manner in which the statistics k have been constructed we have also

$$\kappa(r) = \kappa_r. \tag{II}$$

In general, the expression for the κ corresponding to any given partition of r will include a term in κ_r together with terms of the form

$$A(q_1^{\chi_1} q_2^{\chi_2} \dots q_h^{\chi_h}) \kappa_{q_1}^{\chi_1} \kappa_{q_2}^{\chi_2} \dots \kappa_{q_h}^{\chi_h},$$

where $q_1^{\chi_1} q_2^{\chi_2} \dots q_h^{\chi_h}$ is any partition of r in which no part is unity. This restriction, which greatly diminishes the number of terms to be evaluated, flows from the consideration that κ_1, unlike all other cumulative moment functions, is altered by a change of origin, and by such a change can be given any desired value, while of the moment statistics

also k_1 is the only one affected by such a change, and that by addition of a quantity which is invariable from sample to sample; consequently, κ_1 can only appear in the single formula

$$\kappa(1) = \kappa_1,$$

expressing that the mean of the sample of n will be the mean of the population.

5. *Partitions involving unit parts.*

A relationship exists, of which a proof may be deduced from the general theory to be developed, which enables us to dispense with the separate examination and tabulation of the formulae corresponding to all those partitions which involve unit parts. The effect upon the corresponding formula of adding a new unit part to the partition is (1) to modify every term in the formula by increasing the suffix of one of its κ functions by unity in every possible way, and (2) to divide the whole by n. For example, the formula for the variance of k_2 is

$$\kappa(2^2) = \frac{1}{n}\kappa_4 + \frac{2}{n-1}\kappa_2^2,$$

whence we may deduce, by applying the above rules,

$$\kappa(2^2 1) = \frac{1}{n^2}\kappa_5 + \frac{4}{n(n-1)}\kappa_2\kappa_3,$$

and, by further applications,

$$\kappa(2^2 1^2) = \frac{1}{n^3}\kappa_6 + \frac{4}{n^2(n-1)}\kappa_2\kappa_4 + \frac{4}{n^2(n-1)}\kappa_3^2,$$

$$\kappa(2^2 1^3) = \frac{1}{n^4}\kappa_7 + \frac{4}{n^3(n-1)}\kappa_2\kappa_5 + \frac{12}{n^3(n-1)}\kappa_3\kappa_4,$$

and so on.

An immediate consequence of the same relationship is that

$$\kappa(r1^s) = \frac{\kappa_{r+s}}{n^s} = \frac{1}{n^s}\kappa_{r+s}. \qquad \text{(III)}$$

The number of formulae remaining of any degree r is the number

of partitions of r into parts of 2 or more; these are

r	4	5	6	7	8	9	10	11	12	13	14	15	16	17
partitions	1	1	3	3	6	7	11	13	20	23	33	40	54	65

Up to the 12th degree there are therefore 65 formulae, while 150 more will only reach the 16th degree. It is proposed to put on record, as a basis for discussion, the formulae up to the 10th degree, together with a few others of special interest, with an explanation of the procedure of calculation.

6. *Calculation of formulae.*

In the calculation of the formulae by the algebraic method it is desirable to proceed somewhat formally, although the results for the 4th and 5th degrees may be obtained fairly readily by writing down the algebraical expressions at length. The procedure may be illustrated by the work for the formulae of the eighth degree. There will be six of these, and corresponding to any of these, such as $\kappa(62)$, the k product, $k_6 k_2$, may be written down and expanded in the symmetric functions s. The work proceeds in three steps: (1) the mean value of the k product is expressed in terms of the population moments μ; (2) by substitution, the expression in terms of μ is condensed into its equivalent in terms of κ; (3) from the moment thus obtained, corresponding to the required partition, the corresponding cumulative moment function is found by the use of formulae of lower degree previously prepared.

The first step is carried out by means of easily verified relationships giving the mean value of such a product as $s_p s_q s_r$ in the form

$$n\mu_{p+q+r}+n(n-1)(\mu_p\mu_{q+r}+\mu_q\mu_{r+p}+\mu_r\mu_{p+q})+n(n-1)(n-2)\mu_p\mu_q\mu_r.$$

In order to apply these relationships expeditiously a table is prepared for each degree, showing the coefficients with which each μ product, ignoring μ_1, occurs in the expansion of each s product.

To evaluate the mean value of any k product, such as $k_3^2 k_2$, it is first expanded in s products as

$$\frac{n^2}{(n-1)^3(n-2)^2}\left(s_3^2 s_2-\frac{1}{n}s_3^2 s_1^2-\frac{6}{n}s_3 s_2^2 s_1+\frac{10}{n^2}s_3 s_2 s_1^3+\frac{9}{n^2}s_2^3 s_1^2-\frac{4}{n^3}s_3 s_1^5\right.$$
$$\left.-\frac{21}{n^3}s_2^2 s_1^4+\frac{16}{n^4}s_2 s_1^6-\frac{4}{n^5}s_1^8\right),$$

whence from a table of the separations of 8 the following table may at once be constructed.

TABLE 1.

Calculation of the mean value of $k_3^2 k_2$.

		$n(n-1)$	$n(n-1)$	$n(n-1)$	$n(n-1)(n-2)$	$n(n-1)(n-2)$	$n(n-1)(n-2)(n-3)$
	$n\mu_8$	$\mu_6\mu_2$	$\mu_5\mu_3$	μ_4^2	$\mu_4\mu_2^2$	$\mu_3^2\mu_2$	μ_2^4
$s_3^2 s_2$	1	1	2	—	—	1	—
$s_3^2 s_1^2$	n^{-1} −1	−1	−2	2	—	−1	—
$s_3 s_2^2 s_1$	−6	−12	−18	−8	−8	−12	—
$s_3 s_2 s_1^3$	n^{-2} 10	40	50	30	30	40	—
$s_2^3 s_1^2$	9	36	54	27	54	54	9
$s_3 s_1^5$	n^{-3} −4	−40	−44	−20	−60	−40	—
$s_2^2 s_1^4$	−21	−168	−252	−147	−336	−420	−63
$s_2 s_1^6$	n^{-4} 16	256	416	240	960	1120	240
s_1^8	n^{-5} −4	−112	−224	−140	−840	−1120	−420

Collecting like terms and cancelling the factors $n-1$ and $n-2$ whenever possible, we get

$$\mu(3^2 2) = \frac{\mu_8}{n^2} + \frac{n^2-8n+28}{n^2(n-1)}\mu_6\mu_2 + \frac{2n^3-12n^2+48n-56}{n^2(n-1)^2}\mu_5\mu_3$$
$$+ \frac{-8n^2+25n-35}{n^2(n-1)^2}\mu_4^2 + \frac{1}{n^2(n-1)^2(n-2)}$$
$$\times \{(-6n^4+84n^3-396n^2+960n-840)\mu_4\mu_2^2$$
$$+(n^5-13n^4+94n^3-460n^2+1120n+1120)\mu_3^2\mu_2$$
$$+(9n^4-90n^3+429n^2-1140n+1260)\mu_2^4\}.$$

The second step consists in substituting

$$\mu_4 = \kappa_4+3\kappa_2^2,$$
$$\mu_5 = \kappa_5+10\kappa_2\kappa_3,$$
$$\mu_6 = \kappa_6+15\kappa_4\kappa_2+10\kappa_3^2+15\kappa_2^3,$$
$$\mu_8 = \kappa_8+28\kappa_6\kappa_2+56\kappa_5\kappa_3+35\kappa_4^2+210\kappa_4\kappa_2^2+280\kappa_3^2\kappa_2+105\kappa_2^4,$$

which reduces the expression to the simpler form

$$\mu(3^2 2) = \frac{\kappa_8}{n^2} + \frac{n+20}{n(n-1)}\kappa_6\kappa_2 + \frac{2n^2+44n-64}{n(n-1)^2}\kappa_5\kappa_3 + \frac{27n-45}{n(n-1)^2}\kappa_4^2$$
$$+ \frac{9n^2+81n-180}{(n-1)^2(n-2)}\kappa_4\kappa_2^2 + \frac{n^3+17n^2+104n-320}{(n-1)^2(n-2)}\kappa_3^2\kappa_2$$
$$+ \frac{6n^2+30n}{(n-1)^2(n-2)}\kappa_2^4.$$

The third stage consists in removing from $\mu(3^2 2)$ those terms which do not belong to $\kappa(3^2 2)$; from the general relationship which connects these two groups of functions

$$\mu(3^2 2) = \kappa(3^2 2) + 2\kappa_3 \kappa(32) + \kappa_2 \kappa(3^2) + \kappa_3^2 \kappa_2,$$

and from formulae of lower degree already evaluated we know that

$$\kappa(32) = \frac{\kappa_5}{n} + \frac{6\kappa_2 \kappa_3}{n-1},$$

while

$$\kappa(3^2) = \frac{\kappa_6}{n} + \frac{9\kappa_2 \kappa_4}{n-1} + \frac{9\kappa_3^2}{n-1} + \frac{6n\kappa_2^3}{(n-1)(n-2)}.$$

Removing the superfluous terms we are left with

$$\kappa(3^2 2) = \frac{\kappa_8}{n^2} + \frac{21}{n(n-1)}\kappa_6 \kappa_2 + \frac{6(8n-11)}{n(n-1)^2}\kappa_5 \kappa_3 + \frac{9(3n-5)}{n(n-1)^2}\kappa_4^2$$

$$+ \frac{18(6n-11)}{(n-1)^2(n-2)}\kappa_4 \kappa_2^2 + \frac{18(9n-20)}{(n-1)^2(n-2)}\kappa_3^2 \kappa_2 + \frac{36n}{(n-1)^2(n-2)}\kappa_2^4,$$

an expression in which the part played by each of the characteristic coefficients of the original distribution is clearly apparent. In the normal distribution, for example, when every coefficient beyond κ_2 vanishes, only the last term remains to be evaluated.

7. *The univariate formulae.*

In addition to the partitions involving unit parts, which have already been set aside, the numbers 4 and 5 have only one partition each, 6 and 7 have three partitions each, while 8, 9, and 10 bring the total up to 32. These are given in the following Table. Since it is scarcely to be hoped that all of these, especially the heavier formulae, will be entirely free from error, it should be particularly noted that any suspected term may be evaluated separately and independently by means of the combinatorial method elaborated below. I am indebted to Dr. J. Wishart and Prof. Hotelling for checking these formulae.

In addition to these formulae, which are complete up to the tenth degree, four others of the twelfth degree may be put on record, namely those for the variance of k_6, the third moment of k_4, fourth moment of k_3,

TABLE OF FORMULAE.

The 32 univariate formulae up to the 10-th degree.

	$\frac{\kappa_4}{n}$	$\frac{\kappa_2^2}{n-1}$			
$\kappa(2^2)$	1	2			
	$\frac{\kappa_5}{n}$	$\frac{\kappa_3\kappa_2}{n-1}$			
$\kappa(32)$	1	6			
	$\frac{\kappa_6}{n}$	$\frac{\kappa_4\kappa_2}{n-1}$	$\frac{\kappa_3^2}{n-1}$	$\frac{n\kappa_2^3}{(n-1)(n-2)}$	
$\kappa(42)$	1	8	6	—	
$\kappa(3^2)$	1	9	9	6	
	$\frac{\kappa_6}{n^2}$	$\frac{\kappa_4\kappa_2}{n(n-1)}$	$\frac{\kappa_3^2}{n(n-1)^2}$	$\frac{\kappa_2^3}{(n-1)^2}$	
$\kappa(2^3)$	1	12	$4(n-2)$	8	
	$\frac{1}{n}\kappa_7$	$\frac{1}{n-1}\kappa_5\kappa_2$	$\frac{1}{n-1}\kappa_4\kappa_3$	$\frac{n}{(n-1)(n-2)}\kappa_3\kappa_2^2$	
$\kappa(52)$	1	10	20	—	
$\kappa(43)$	1	12	30	36	
	$\frac{\kappa_7}{n^2}$	$\frac{\kappa_5\kappa_2}{n(n-1)}$	$\frac{\kappa_4\kappa_3}{n(n-1)^2}$	$\frac{\kappa_2^2\kappa_3}{(n-1)^2}$	
$\kappa(32^2)$	1	16	$12(2n-3)$	48	
	$\frac{1}{n}\kappa_8$	$\frac{1}{n-1}\kappa_6\kappa_2$	$\frac{1}{n-1}\kappa_5\kappa_3$	$\frac{1}{n-1}\kappa_4^2$	$\frac{n}{(n-1)(n-2)}\kappa_4\kappa_2^2$
$\kappa(62)$	1	12	30	20	—
$\kappa(53)$	1	15	45	30	60
$\kappa(4^2)$	1	16	48	34	72
	$\frac{1}{n}\kappa_8$	$\frac{\kappa_6\kappa_2}{n(n-1)}$	$\frac{\kappa_5\kappa_3}{n(n-1)^2}$	$\frac{\kappa_4^2}{n(n-1)^2}$	$\frac{\kappa_4\kappa_2^2}{(n-1)^2(n-2)}$
$\kappa(42^2)$	1	20	$8(5n-7)$	$4(7n-10)$	$80(n-2)$
$\kappa(3^22)$	1	21	$6(8n-11)$	$9(3n-5)$	$18(6n-11)$
	$\frac{\kappa_8}{n^3}$	$\frac{\kappa_6\kappa_2}{n^2(n-1)}$	$\frac{\kappa_5\kappa_3}{n^2(n-1)^2}$	$\frac{\kappa_4^2}{n^2(n-1)3}$	$\frac{\kappa_4\kappa_2^2}{n(n-1)^2}$
$\kappa(2^4)$	1	24	32	$8(4n^2-9n+6)$	144
	$\frac{\kappa_9}{n}$	$\frac{\kappa_7\kappa_2}{n-1}$	$\frac{\kappa_6\kappa_3}{n-1}$	$\frac{\kappa_5\kappa_4}{n-1}$	$\frac{n\kappa_5\kappa_2^2}{(n-1)(n-2)}$
$\kappa(72)$	1	14	42	70	—
$\kappa(63)$	1	18	63	105	90
$\kappa(54)$	1	20	70	120	120
	$\frac{\kappa_9}{n^2}$	$\frac{\kappa_7\kappa_2}{n(n-1)}$	$\frac{\kappa_6\kappa_3}{n(n-1)^2}$	$\frac{\kappa_5\kappa_4}{n(n-1)^2}$	$\frac{\kappa_5\kappa_2^2}{(n-1)^2(n-2)}$
$\kappa(52^2)$	1	24	$20(3n-4)$	$20(5n-7)$	$120(n-2)$
$\kappa(432)$	1	26	$24(3n-4)$	$10(11n-17)$	$36(5n-9)$
$\kappa(3^3)$	1	27	$27(3n-4)$	$27(4n-7)$	$54(4n-7)$
	$\frac{\kappa_9}{n^3}$	$\frac{\kappa_7\kappa_2}{n^2(n-1)}$	$\frac{\kappa_6\kappa_3}{n^2(n-1)^2}$	$\frac{\kappa_5\kappa_4}{n^2(n-1)^3}$	$\frac{\kappa_5\kappa_2^2}{n(n-1)^2}$
$\kappa(32^3)$	1	30	$2(31n-53)$	$12(9n^2-23n+16)$	240

			(1)
			(2)
			(3)
			(4)
			(5)
			(6)
			(7)
			(8)
$\frac{n}{(n-1)(n-2)}\kappa_3^2\kappa_2$	$\frac{n(n+1)}{(n-1)(n-2)(n-3)}\kappa_2^4$		
—	—		(9)
90	—		(10)
144	24		(11)
$\frac{\kappa_3^2\kappa_2}{(n-1)^2(n-2)}$	$\frac{\kappa_2^4}{(n-1)^2(n-2)}$		
$120(n-2)$	—		(12)
$18(9n-20)$	$36n$		(13)
$\frac{\kappa_3^2\kappa_2}{2(n-1)^3}$	$\frac{\kappa_2^4}{(n-1)^3}$		
$96(n-2)$	48		(14)
$\frac{n\kappa_4\kappa_3\kappa_2}{(n-1)(n-2)}$	$\frac{n\kappa_3^3}{(n-1)(n-2)}$	$\frac{n(n+1)\kappa_3\kappa_2^3}{(n-1)(n-2)(n-3)}$	
—	—	—	(15)
360	90	—	(16)
600	180	240	(17)
$\frac{\kappa_4\kappa_3\kappa_2}{(n-1)^2(n-2)}$	$\frac{\kappa_3^3}{(n-1)^2(n-2)^2}$	$\frac{n\kappa_3\kappa_2^3}{(n-1)^2(n-2)^2}$	
$480(n-2)$	$120(n-2)^2$	—	(18)
$12(61n-128)$	$36(n-2)(5n-12)$	$360(n-2)$	(19)
$162(5n-12)$	$36(7n^2-30n+34)$	$108(5n-12)$	(20)
$\frac{\kappa_4\kappa_3\kappa_2}{n(n-1)^3}$	$\frac{\kappa_3^3}{n(n-1)^3}$	$\frac{\kappa_3\kappa_2^3}{(n-1)^3}$	
$360(2n-3)$	$24(5n-12)$	336	(21)

TABLE OF FORMULAE—*continued.*

	$\frac{\kappa_{10}}{n}$	$\frac{\kappa_8\kappa_2}{n-1}$	$\frac{\kappa_7\kappa_3}{n-1}$	$\frac{\kappa_6\kappa_4}{n-1}$	$\frac{\kappa_5^2}{n-1}$	$\frac{n\kappa_6\kappa_2^2}{(n-1)(n-2)}$	$\frac{n\kappa_5\kappa_3\kappa_2}{(n-1)(n-2)}$
$\kappa(82)$	1	16	56	112	70	—	—
$\kappa(73)$	1	21	84	168	105	126	630
$\kappa(64)$	1	24	96	194	120	180	1080
$\kappa(5^2)$	1	25	100	200	125	200	1200
	$\frac{\kappa_{10}}{n^2}$	$\frac{\kappa_8\kappa_2}{n-1}$	$\frac{\kappa_7\kappa_3}{n(n-1)^2}$	$\frac{\kappa_6\kappa_4}{n(n-1)^2}$	$\frac{\kappa_5^2}{n(n-1)^2}$	$\frac{\kappa_6\kappa_2^2}{(n-1)^2(n-1)}$	$\frac{\kappa_5\kappa_3\kappa_2}{(n-1)^2(n-2)}$
$\kappa(62^2)$	1	28	$12(7n-9)$	$4(41n-56)$	$20(5n-7)$	$168(n-2)$	$840(n-2)$
$\kappa(532)$	1	31	$101n-131$	$5(37n-55)$	$5(23n-35)$	$30(9n-16)$	$30(45n-92)$
$\kappa(4^22)$	1	32	$8(13n-17)$	$4(49n-73)$	$4(29n-46)$	$8(37n-65)$	$1536(n-2)$
$\kappa(43^2)$	1	33	$6(19n-25)$	$3(65n-107)$	$6(19n-34)$	$18(19n-33)$	$72(23n-52)$
	$\frac{\kappa_{10}}{n^3}$	$\frac{\kappa_8\kappa_2}{n^2(n-1)}$	$\frac{\kappa_7\kappa_3}{n^2(n-1)^2}$	$\frac{\kappa_6\kappa_4}{n^2(n-1)^3}$	$\frac{\kappa_5^2}{n^2(n-1)^3}$	$\frac{\kappa_6\kappa_2^2}{n(n-1)^2(n-2)}$	$\frac{\kappa_5\kappa_3\kappa_2}{n(n-1)^3(n-2)}$
$\kappa(42^3)$	1	36	$4(23n-37)$	$4(47n^2-120n+81)$	$12(9n^2-24n+17)$	$360(n-2)$	$288(5n-7)(n-$
$\kappa(3^22^2)$	1	37	$6(17n-27)$	$3(61n^2-166n+117)$	$2(59n^2-154n+113)$	$6(67n-131)$	$24(71n^2-246n+$
	$\frac{\kappa_{10}}{n^4}$	$\frac{\kappa_8\kappa_2}{n^3(n-1)}$	$\frac{\kappa_7\kappa_3}{n^3(n-1)^2}$	$\frac{\kappa_6\kappa_4}{n^3(n-1)^3}$	$\frac{\kappa_5^2}{n^3(n-1)^4}$	$\frac{\kappa_6\kappa_2^2}{n^2(n-1)^2}$	$\frac{\kappa_5\kappa_3\kappa_2}{n^2(n-1)^3}$
$\kappa(2^5)$	1	40	$80(n-2)$	$40(5n^2-12n+9)$	$16(n-2)(6n^2-12n+17)$	480	$1280(n-2)$

and the sixth moment of k_2. These are :—

$$\kappa(6^2) = \frac{1}{n}\kappa_{12} + \frac{1}{n-1}(36\kappa_{10}\kappa_2 + 180\kappa_9\kappa_3 + 465\kappa_8\kappa_4 + 780\kappa_7\kappa_5 + 461\kappa_6^2)$$
$$+ \frac{n}{(n-1)(n-2)}(450\kappa_8\kappa_2^2 + 3600\kappa_7\kappa_3\kappa_2 + 7200\kappa_6\kappa_4\kappa_2 + 6300\kappa_6\kappa_3^2$$
$$+ 4500\kappa_5^2\kappa_2 + 21600\kappa_5\kappa_4\kappa_3 + 4950\kappa_4^3)$$
$$+ \frac{n(n+1)}{(n-1)(n-2)(n-3)}(2400\kappa_6\kappa_2^3 + 21600\kappa_5\kappa_3\kappa_2^2$$
$$+ 15300\kappa_4^2\kappa_2^2 + 54000\kappa_4\kappa_3^2\kappa_2 + 8100\kappa_3^4)$$
$$+ \frac{n^2(n+5)}{(n-1)(n-2)(n-3)(n-4)}(5400\kappa_4\kappa_2^4 + 21600\kappa_3^2\kappa_2^3)$$
$$+ \frac{n(n+1)(n^2+15n-4)}{(n-1)(n-2)(n-3)(n-4)(n-5)}\,720\kappa_2^6. \qquad (50)$$

$$\kappa(4^3) = \frac{1}{n^2}\kappa_{12} + \frac{48}{n(n-1)}\kappa_{10}\kappa_2 \frac{16(13n-17)}{n(n-1)^2}\kappa_9\kappa_3 + \frac{12(41n-65)}{n(n-1)^2}\kappa_8\kappa_4$$
$$+ \frac{48(16n-29)}{n(n-1)^2}\kappa_7\kappa_5 + \frac{12(37n-70)}{n(n-1)^2}\kappa_6^2 + \frac{72(11n-19)}{(n-1)^2(n-2)}\kappa_8\kappa_2^2 + \ldots$$ (p. 213)

$n\kappa_4^2\kappa_2$	$n\kappa_4\kappa_3^2$	$n(n+1)\kappa_4\kappa_2^3$	$n(n+1)\kappa_3^2\kappa_2^2$	$n^2(n+5)\kappa_2^5$	
$(n-1)(n-2)$	$(n-1)(n-2)$	$(n-1)(n-2)(n-3)$	$(n-1)(n-2)(n-3)$	$(n-1)(n-2)(n-3)(n-4)$	
—	—	—	—	—	(22)
420	630	—	—	—	(23)
720	1260	480	1080	—	(24)
850	1200	600	1800	120	(25)
$\kappa_4^2\kappa_2$	$\kappa_4\kappa_3^2$	$\kappa_4\kappa_2$	$\kappa_3^2\kappa_2^2$	$n\kappa_2^5$	
$(n-1)^2(n-2)$	$(n-1)^2(n-2)$	$(n-1)^2(n-2)$	$(n-1)^2(n-2)$	$(n-1)^2(n-2)$	
$560(n-2)$	$840(n-2)$	—	—	—	(26)
$30(15n-31)$	$30(45n-103)$	$720n$	$1620n$	—	(27)
$144(7n-15)$	$72(21n-50)$	$\dfrac{96(10n^2-27n-1)}{n-3}$	$\dfrac{144(17n^2-53n-2)}{n-3}$	$\dfrac{192(n+1)}{n-3}$	(28)
$54(19n-48)$	$\dfrac{54(33n^2-148n+172)}{n-2}$	$\dfrac{72n(17n-40)}{n-2}$	$\dfrac{108n(27n-70)}{n-2}$	$\dfrac{216n}{n-2}$	(29)
$\kappa_4^2\kappa_2$	$\kappa_4\kappa_3^2$	$\kappa_4\kappa_2^3$	$\kappa_3^2\kappa_2^2$	$n\kappa_2^5$	
$(n-1)^3(n-2)$	$n(n-1)^3(n-2)$	$(n-1)^3(n-2)$	$(n-1)^3(n-2)$	$(n-1)^3(n-2)$	
$(7n-10)(n-2)$	$24(49n-95)(n-2)$	$768(n-2)$	$1872(n-2)$	—	(30)
$9n^2-103n+93)$	$36(38n^2-155n+160)$	$72(14n-23)$	$144(19n-44)$	288	(31)
$\kappa_4^2\kappa_2$	$\kappa_4\kappa_3^2$	$\kappa_4\kappa_2^3$	$\kappa_3^2\kappa_2^2$	κ_2^5	
$n^2(n-1)^4$	$n^2(n-1)^4$	$n(n-1)^3$	$n(n-1)^4$	$(n-1)^4$	
$0(4n^2-9n+6)$	$480(2n^2-7n+6)$	1920	$1920(n-2)$	384	(32)

$$+\frac{288(19n-41)}{(n-1)^2(n-2)}\kappa_7\kappa_3\kappa_2+\frac{48(203n-523)}{(n-1)^2(n-2)}\kappa_6\kappa_4\kappa_2$$

$$+\frac{144(56n^2-257n+302)}{(n-1)^2(n-2)^2}\kappa_6\kappa_3^2+\frac{1440(4n-11)}{(n-1)^2(n-2)}\kappa_5^2\kappa_2$$

$$+\frac{1152(22n^2-106n+133)}{(n-1)^2(n-2)^2}\kappa_5\kappa_4\kappa_3+\frac{8(709n^2-3430n+4456)}{(n-1)^2(n-2)^2}\kappa_4^3$$

$$+\frac{288(19n^3-98n^2+125n+2)}{(n-1)^2(n-2)^2(n-3)}\kappa_6\kappa_2^3+\frac{1728(21n^3-119n^2+164n+4)}{(n-2)^2(n-2)^2(n-3)}\kappa_5\kappa_3\kappa_2^2$$

$$+\frac{432(49n^3-287n^2+408n+12)}{(n-1)^2(n-2)^2(n-3)}\kappa_4^2\kappa_2^2+\frac{864(103n^5-629n^2+984n+24)}{(n-1)^2(n-2)^2(n-3)}\kappa_4\kappa_3^2\kappa_2$$

$$+\frac{288(41n^4-384n^3+1209n^2-1282n-36)}{(n-1)^2(n-2)^2(n-3)^2}\kappa_3^4+\frac{288(89n^2-323n-88)n}{(n-1)^2(n-2)^2(n-3)}\kappa_4\kappa_2^4$$

$$+\frac{1728(29n^3-196n^2+317n+62)n}{(n-1)^2(n-2)^2(n-3)^2}\kappa_3^2\kappa_2^3+\frac{1728(n^2-5n+2)(n+1)n}{(n-1)^2(n-2)^2(n-3)^2}\kappa_2^6, \quad (57)$$

$$\kappa(3^4) = \frac{1}{n^3}\kappa_{12} + \frac{54}{n^2(n-1)}\kappa_{10}\kappa_2 \frac{108(2n-3)}{n^2(n-1)^2}\kappa_9\kappa_3 + 27\frac{17n^2-49n+35}{n^2(n-1)^3}\kappa_8\kappa_4$$

$$+108\frac{7n^2-20n+16}{n^2(n-1)^3}\kappa_7\kappa_5 + 27\frac{17n^2-47n+39}{n^2(n-1)^3}\kappa_6^2 + 27\frac{37n-70}{n(n-1)^2(n-2)}\kappa_8\kappa_2^2$$

$$+324\frac{19n^2-67n+54}{n(n-1)^3(n-2)}\kappa_7\kappa_3\kappa_2 + 162\frac{65n^2-245n+234}{n(n-1)^2(n-2)}\kappa_6\kappa_4\kappa_2$$

$$+108\frac{82n^3-481n^2+958-640}{n(n-1)^3(n-2)^2}\kappa_6\kappa_3^2 + 108\frac{59n^2-220n+224}{n(n-1)^3(n-2)}\kappa_5^2\kappa_2$$

$$+324\frac{75n^3-473n^2+1016n-756}{n(n-1)^3(n-2)^2}\kappa_5\kappa_4\kappa_3$$

$$+27\frac{173n^4-1503n^3+4962n^3-7380n+4200}{n(n-1)^3(n-2)^3}\kappa_4^3$$

$$+108\frac{71n^2-263n+234}{(n-1)^3(n-2)^2}\kappa_6\kappa_2^3 + 648\frac{79n^2-343n+378}{(n-1)^3(n-2)^2}\kappa_5\kappa_3\kappa_2^2$$

$$+486\frac{63n^2-290n+352}{(n-1)^3(n-2)^3}\kappa_4^2\kappa_2^2 + 972\frac{99n^3-688n^2+1612n-1280}{(n-1)^3(n-2)^3}\kappa_4\kappa_3^2\kappa_2$$

$$+162\frac{87n^3-594n^2+1420n-1176}{(n-1)^3(n-3)^3}\kappa_3^4 + 972\frac{29n^2-121n+118}{(n-1)^3(n-2)^3}\kappa_4\kappa_2^4$$

$$+648n\frac{103n^2-510n+640}{(n-1)^3(n-2)^3}\kappa_2^2\kappa_2^3 + 648n^2\frac{5n-12}{(n-1)^3(n-2)^3}\kappa_2^6, \qquad (62)$$

$$\kappa(2^6) = \frac{1}{n^5}\kappa_{12} + \frac{60}{n^4(n-1)}\kappa_{10}\kappa_2 + \frac{160(n-2)}{n^4(n-1)^2}\kappa_9\kappa_3 + 240\frac{2n^2-5n+4}{n^4(n-1)^3}\kappa_8\kappa_4$$

$$+96(n-2)\frac{7n^2-14n+9}{n^4(n-1)^4}\kappa_7\kappa_5 + 4\frac{113n^4-520n^3+950n^2-800n+265}{n^4(n-1)^5}\kappa_6^2$$

$$+\frac{1200}{n^3(n-1)^2}\kappa_8\kappa_2^2 + 4800\frac{n-2}{n^3(n-1)^3}\kappa_7\kappa_3\kappa_2 + 2400\frac{5n^2-12n+9}{n^3(n-1)^4}\kappa_6\kappa_4\kappa_2$$

$$+160(n-2)\frac{31n-53}{n^3(n-1)^4}\kappa_6\kappa_3^2 + 960(n-2)\frac{6n^2-12n+7}{n^3(n-1)^5}\kappa_5^2\kappa_2$$

$$+1920(n-2)\frac{9n^2-23n+16}{n^3(n-1)^5}\kappa_5\kappa_4\kappa_3 + 480\frac{11n^3-41n^2+59n-31}{n^3(n-1)^5}\kappa_4^3$$

$$+\frac{9600}{n^2(n-1)^3}\kappa_6\kappa_2^3 + \frac{38400(n-2)}{n^3(n-1)^4}\kappa_5\kappa_3\kappa_2^2 + 9600\frac{4n^2-9n+6}{n^2(n-1)^5}\kappa_4^2\kappa_2^2$$

$$+28800\frac{2n^2-7n+6}{n^2(n-1)^5}\kappa_4\kappa_3^2\kappa_2 + 960(n-2)\frac{5n-12}{n^2(n-1)^5}\kappa_3^4 + \frac{28800}{n(n-1)^4}\kappa_4\kappa_2^4$$

$$+38400\frac{n-2}{n(n-1)^5}\kappa_3^2\kappa_2^3 + \frac{3840}{(n-1)^5}\kappa_2^6. \qquad (65)$$

Some idea of the advantage of using the cumulative moment functions in place of the moments will be obtained by comparing the above formula (14) with the corresponding formula as obtained by Tchouproff, and corrected by Church :

$$
\begin{aligned}
{}_2M_4 = {} & \frac{3}{n^2}(\mu_4-\mu_2^2)^2+\frac{1}{n^3}(\mu_8-4\mu_6\mu_2-24\mu_5\mu_3-15\mu_4^2 \\
& \qquad +48\mu_4\mu_2^2+96\mu_3^2\mu_2-30\mu_2^4) \\
& -\frac{1}{n^4}(4\mu_8-40\mu_6\mu_2-96\mu_5\mu_3-54\mu_4^2+336\mu_4\mu_2^2+528\mu_3^2\mu_2-306\mu_2^4) \\
& -\frac{1}{n^5}(6\mu_8-96\mu_6\mu_2-176\mu_5\mu_3-102\mu_4^2+924\mu_4\mu_2^2+1232\mu_3^2\mu_2-1044\mu_2^4) \\
& -\frac{1}{n^6}(4\mu_8-88\mu_6\mu_2-160\mu_5\mu_3-95\mu_4^2+1050\mu_4\mu_2^2+1360\mu_3^2\mu_2-1395\mu_2^4) \\
& +\frac{1}{n^7}(\mu_8-28\mu_6\mu_2-56\mu_5\mu_3-35\mu_4^3+420\mu_4\mu_2^2+560\mu_3^2\mu_2-630\mu_2^4).
\end{aligned}
$$

The term involving κ_2 only in $\kappa(2^r)$ is already known as the r-th semi-invariant of the distribution of the variance for samples from the normal curve, and is simply $2^{r-1}.(r-1)!/(n-1)^{r-1}$. The corresponding term in $\kappa(3^4)$ is of interest as showing that the distribution of k_3 in samples from a normal distribution, though necessarily symmetrical, yet tends somewhat slowly to normality. Comparing the first term of (17) with that of (4) it is evident that

$$\frac{\kappa(3^4)}{\{\kappa(3^2)\}^2} = \frac{18(5n-12)}{(n-1)(n-2)},$$

or is somewhat greater than $90/n$, a fact which indicates that the occurrence of values of k_3 greater than 2 or 3 times its standard error will, except in very large samples, be materially more frequent than one would judge from an assumed normal distribution. The effect upon tests of normality will be examined in §§ 11 and 12.

8. *Bivariate and multivariate distributions.*

The extension to bivariate and multivariate data of the methods of classification and calculation developed above is of both practical and theoretical importance. Apart from the variance the product moment of a bivariate distribution is the most important of all moment statistics.

Moreover, the multivariate formulae, by reason of their greater number, and the confusion caused by the various possible notations in

which they may be expressed, are particularly in need of orderly classification. It will be found, in addition, that the examination of the multivariate formulae in their generality throws much light on the expressions already obtained.

It will be seen that just as the univariate formulae correspond to all the possible partitions of unipartite numbers, so the multivariate formulae correspond to all the possible partitions of multipartite numbers, having multiplicities equal to the number of variates.

To make the notation clear let us consider, in the first place, two variates only, and let the frequency with which the two variates x and y fall simultaneously in the ranges dx and dy be

$$df = \phi\, dx\, dy,$$

in which ϕ is the simultaneous frequency function of x and y.

The general moment about any origin is defined as

$$\mu_{pq} = \iint x^p y^q \phi\, dx\, dy$$

over the whole range of possible values of the variates. So far as these moments have a meaning we can build up the expression

$$M = \sum_{p=0} \sum_{q=0} \mu_{pq} \frac{t_1^p}{p!} \frac{t_2^q}{q!},$$

and equally, with the same limitation, the coefficients of the expression

$$K = \log M = \sum_{p=0} \sum_{q=0} \kappa_{pq} \frac{t_1^p}{p!} \frac{t_2^q}{q!}$$

will be well defined. The general expressions connecting the cumulative moment functions, κ, with the moments, μ, of the simultaneous distribution are analogous to those given for univariate distribution; if

$$\{(p_1 p_1')^{\pi_1} (p_2 p_2')^{\pi_2} \dots\}$$

is any partition of the bipartite number r, s consisting of ρ parts,

$$\kappa_{r,s} = S\left\{\frac{(-)^{\rho-1}(\rho-1)!}{\pi_1!\,\pi_2! \dots} \frac{r!}{(p_1!)^{\pi_1}(p_2!)^{\pi_2}\dots} \frac{s!}{(p_1'!)^{\pi_1}(p_2'!)^{\pi_2}\dots} \mu_{p_1 p_1'}^{\pi_1} \mu_{p_2 p_2'}^{\pi_2} \dots\right\}$$

and

$$\mu_{r,s} = S\left\{\frac{1}{\pi_1!\,\pi_2! \dots} \frac{r!}{(p_1!)^{\pi_1}(p_2!)^{\pi_2}\dots} \frac{s!}{(p_1'!)^{\pi_1}(p_2')^{\pi_2}\dots} \kappa_{p_1 p_1'}^{\pi_1} \kappa_{p_2 p_2'}^{\pi_2} \dots\right\},$$

the summation being taken over all possible partitions. For any sample, we may define s_{pq} as the sum of the values of $x^p y^q$ for each pair of values

in the sample, and obtain, as for single variates, the statistics k_{11}, k_{21}, k_{31}, k_{22}, etc., as expressions in terms of these sums, with such coefficients that the mean value of k_{pq} shall be κ_{pq}. Thus we have

$$k_{11} = \frac{1}{n-1}\left(s_{11} - \frac{1}{n}\, s_{10}s_{01}\right),$$

$$k_{21} = \frac{n}{(n-1)(n-2)}\left(s_{21} - \frac{2}{n}\, s_{10}s_{11} - \frac{1}{n}\, s_{20}s_{01} + \frac{2}{n^2}\, s_{10}^2 s_{01}\right),$$

$$k_{31} = \frac{n}{(n-1)(n-2)(n-3)}\left\{(n+1)\, s_{31} - \frac{n+1}{n}\, s_{30}s_{01} - \frac{3(n-1)}{n}\, s_{11}s_{20}\right.$$
$$\left. - \frac{3(n+1)}{n}\, s_{21}s_{10} + \frac{6}{n}\, s_{11}s_{10}^2 + \frac{6}{n}\, s_{20}s_{10}s_{01} - \frac{6}{n^2}\, s_{01}s_{10}^3\right\},$$

$$k_{22} = \frac{n}{(n-1)(n-2)(n-3)}\left\{(n+1)\, s_{22} - 2\,\frac{n+1}{n}\, s_{21}s_{01} - \frac{n+1}{n}\, s_{12}s_{10}\right.$$
$$- \frac{n-1}{n}\, s_{20}s_{02} - 2\,\frac{n-1}{n}\, s_{11}^2 + \frac{8}{n}\, s_{11}s_{01}s_{10}$$
$$\left. + \frac{2}{n}\, s_{02}s_{10}^2 + \frac{2}{n}\, s_{20}s_{01}^2 - \frac{6}{n^2}\, s_{10}^2 s_{01}^2\right\}.$$

The mean value of any product involving such statistics, as, for example, $k_{20}k_{11}$, may be evaluated in terms of the cumulative moment functions of the bivariate distribution; such mean values may be written

$$\overline{k_{20}\, k_{11}} \equiv \mu\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix},$$

giving one line to each variate; its value is easily found to be

$$\frac{1}{n}\,\kappa_{31} + \frac{n+1}{n-1}\,\kappa_{20}\kappa_{11}.$$

Hence, subtracting the product of the mean values, $\kappa_{20}\kappa_{11}$, we have the formula

$$\kappa\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = \frac{1}{n}\,\kappa_{31} + \frac{2}{n-1}\,\kappa_{20}\kappa_{11} \qquad (1\,a)$$

in which each column represents the particular statistic entering into the product, and the marginal column found by summing each row is the multipartite number (31) representing the degree in which each variate is involved. Similarly, we may deduce the two formulae for partitions

of the bipartite number (22), namely

$$\kappa\begin{pmatrix}2&0\\0&2\end{pmatrix}=\frac{1}{n}\kappa_{22}+\frac{2}{n-1}\kappa_{11}^2 \qquad (1\,b)$$

and

$$\kappa\begin{pmatrix}1&1\\1&1\end{pmatrix}=\frac{1}{n}\kappa_{22}+\frac{1}{n-1}\kappa_{11}^2+\frac{1}{n-1}\kappa_{02}\kappa_{20}, \qquad (1\,c)$$

representing the product moment of the estimates of variance of the two correlated variates, and the variance of the estimated product moment.

It will be observed that by equating the two variates, which is carried out by summing the columns of the partition, and replacing the two suffixes of each κ by their sum, equations (1a), (1b), and (1c) are reduced to equation (1). As with univariate formulae, the partitions involving parts of the first degree may be directly derived from formulae of lower degree and therefore need receive no separate consideration.

With more than two variates the bivariate notation may be extended to the use of three or more rows in the representation of a partition of a tripartite number, and three or more suffixes to the parameters κ. The remaining formulae of the fourth degree are therefore

$$\kappa\begin{pmatrix}2&0\\0&1\\0&1\end{pmatrix}=\frac{1}{n}\kappa_{211}+\frac{2}{n-1}\kappa_{110}\kappa_{101}, \qquad (1\,d)$$

$$\kappa\begin{pmatrix}1&1\\1&0\\0&1\end{pmatrix}=\frac{1}{n}\kappa_{211}+\frac{1}{n-1}\kappa_{200}\kappa_{011}+\frac{1}{n-1}\kappa_{101}\kappa_{110}, \qquad (1\,e)$$

$$\kappa\begin{pmatrix}1&0\\1&0\\0&1\\0&1\end{pmatrix}=\frac{1}{n}\kappa_{1111}+\frac{1}{n-1}\kappa_{1010}\kappa_{0101}+\frac{1}{n-1}\kappa_{1001}\kappa_{0110}, \qquad (1^*)$$

representing the partitions of the tripartite number (211) and of the quadrupartite (1111), ignoring such as have unitary parts.

Just as equation (1) may be derived from either of equations (1a), (1b), or (1c) by identifying the variates, so, by equating appropriate variates, (1a) may be derived from (1d), or (1b) from (1d), or (1c) from (1e), and finally all can be derived from the general multivariate formula (1*).

It appears, therefore, that the formulae appropriate for both univariate and multivariate distributions may all be expressed in terms of those representing partitions of the multipartite number (1^h). Thus of the

sixth degree, a series of formulae, of which formula (3) is the final condensation, will be given by the partition of the multipartite (1^6) into parts $(1^4 0^2)$ and $(0^4 1^2)$, a series of formulae reducing to (4) by the partition into the parts $(1^3 0^3)$ and $(0^3 1^3)$, and a series of formulae reducing to (5) by the partition into parts $(1^2 0^4)$, $(0^2 1^2 0^2)$ and $(0^4 1^2)$. The presentation of formulae of the type here discussed for the case of many variates might therefore be completed by the tabulation of the general multivariate formulae (2*), (3*), etc.

The disadvantage of such a course is that such general formulae will consist of a large number of terms equal to the sum of the coefficients (of the highest powers of n) of the formulae already tabulated, and that each term will consist of a product of κ's, each having as many suffixes as the degree of the equation. The general formulae are therefore extremely cumbrous, and, as the suffixes will consist merely of repetitions in different orders of the numbers 0 and 1, it will be of more value if general rules can be found by which these particular combinations are to be selected. Such rules will then apply to the univariate and less general multivariate cases, the coefficients being merely the number of ways in which each selection can be made.

Now the suffixes of the product terms are merely other partitions of the same number, whether unipartite or multipartite, of which one particular partition specifies our formula; we are therefore concerned with the difficult question of the relations which can exist between different partitions of the same number. This question may be considered solely with respect to unipartite numbers, for if the rules can be made out which govern the coefficients in such cases, the same identical rules must apply to multipartite numbers by reason of the methods by which one formula may be condensed into another. For example, if we start with the partition (2^2) of the number 4, in conjunction with the rule that only such partitions are to be considered as in each part involve elements from both parts of the old partition, we should obtain equally the coefficient 2 of the term κ_2^2, and by applying the same rule to the partition of the multipartite number (1^4) into parts $(1^2 0^2)$ and $(0^2 1^2)$ should obtain the terms κ_{1010}, κ_{0101}, and κ_{1001}, κ_{0110}, having in both cases the same divisor $n-1$.

9. *Empirical statement of the rules for the direct evaluation of the coefficients.*

Although the rules of the combinatorial procedure were not completed before the development of the method of Section 10, yet so much

can be learned by an empirical study of the formulae that it is convenient to make a complete statement of the rules in an empirical form, prior to the demonstration of their validity.

(1) The coefficient of $\kappa_{q_1}^{\chi_1}\kappa_{q_2}^{\chi_2}\ldots$ in the expression for $\kappa(p_1^{\pi_1}p_2^{\pi_2}\ldots)$ depends on the possible partitions of the second order of which the column totals give the partition $(p_1^{\pi_1}p_2^{\pi_2}\ldots)$, and the row totals give the partition $(q_1^{\chi_1}q_2^{\chi_2}\ldots)$.

For example, the coefficient of $\kappa_6\kappa_2^2$ in the expression for $\kappa(4^2 2)$ may be obtained by inspection of the partitions of the second order

2 2 2	6	2 3 1	6	3 3 .	6
1 1 .	2	1 1 .	2	1 . 1	2
1 1 .	2	1 . 1	2	. 1 1	2
4 4 2	10	4 4 2	10	4 4 2	10

in each of which the sums of the rows constitute the partition (62^2), while the sums of the columns constitute the partition $(4^2 2)$.

(2) The numerical factor in the contribution made by any partition of the second order is the number of ways in which the totals in the lower margin may be allocated to form a partition of the type considered. The numerical factors corresponding to the three partitions set out above are 72, 192, and 32 respectively. In the first case, for example, the number may be arrived at from the consideration that the pair of units to be separated in the first four may be chosen in six ways, and that these may be assigned partners from the second pair in twelve ways. In the second case we may choose either of the two fours to be parted into (21^2), as in the first column, and, whichever is chosen, we may allocate the units in the three columns in twelve, four, and two ways respectively; while, in the third case, we may choose the units from the two fours in sixteen ways and associate them in two ways with the units of the two.

(3) Before considering the general rule for determining the function of n by which the numerical factor is to be multiplied, it is convenient to note that certain partitions of the second order make no contribution whatever to the coefficient, and so may be neglected at once. The most useful class consists of those in which any row has only one entry other than zero; for example, such partitions as

2 3 1	6
. 1 1	2
2 . .	2
4 4 2	10

are to be ignored. It is obvious for statistical reasons, as has been mentioned above, that κ_1 cannot appear in any of these formulae, and as it will be seen that the function of n involved depends only upon the configuration of the zeros of the partition of the second order, the necessity for this rule will become apparent. More generally, we may exclude any partition in which any set of rows is connected to its complementary set by a single column only.

(4) The usefulness of rule (3) for excluding superfluous partitions is extended by employing it in conjunction with the rule which holds when any column has only one entry other than zero; for in these cases we may introduce the factor n^{-1} and ignore the column concerned. For example, the partition pattern

$$\begin{array}{l} \times\times\times \\ \times\times\,. \\ \times\times\,. \end{array}$$

irrespective of its numerical coefficient, is associated with a function of n which is one n-th of that associated with

$$\begin{array}{l} \times\times \\ \times\times \\ \times\times \end{array}$$

Moreover, such a partition as

$$\begin{array}{lll} 4\,2\,. & | & 6 \\ .\,1\,1 & | & 2 \\ .\,1\,1 & | & 2 \\ \hline 4\,4\,2 & | & 10 \end{array}$$

is to be ignored (although every row has two entries) by reason of its connection with

$$\begin{array}{l} \times\,. \\ \times\times \\ \times\times \end{array}$$

in which this condition is not fulfilled.

With these criteria of rejection one may easily assure oneself that the three partitions set out above are the only ones which need be considered in that case.

(5) To find, in general, the function of n with which any pattern is associated, we consider all the possible ways in which the rows can be

separated into 1, 2, 3, ... separate groups, or separates. Thus with three rows we have one separation into one separate, with which is associated the factor n; three separations into two separates, with which is associated the factor $n(n-1)$; and one separation into three separates, with which is associated the factor $n(n-1)(n-2)$. In each of these five separations we count in how many separates each column is represented by entries other than zero. If in one separate, that column contributes a factor n^{-1}; if in 2, 3, 4, ... separates, the factors are

$$\frac{-1}{n(n-1)}, \quad \frac{2!}{n(n-1)(n-2)}, \quad \frac{-3!}{n(n-1)(n-2)(n-3)}.$$

In applying this rule all patterns which are resolvable into two parts, each confined to separable sets of rows and columns, must be ignored.

As an example, consider the five possible separations of the pattern

$$\begin{matrix} \times\times \\ \times\times \\ \times\times ; \end{matrix}$$

the first supplies the term

$$\frac{n}{n^2} = \frac{1}{n},$$

the separations into two separates supply

$$\frac{3n(n-1)}{n^2(n-1)^2} = \frac{3}{n(n-1)},$$

while the separation into three separates gives

$$\frac{4n(n-1)(n-2)}{n^2(n-1)^2(n-2)^2} = \frac{4}{n(n-1)(n-2)},$$

the total being $n/\{(n-1)(n-2)\}$, the function appropriate to this pattern.

It is equally easy to verify that the functions appropriate to the patterns

$$\begin{matrix} \times\times\times \\ \times\times\, . \\ \times\, .\, \times \end{matrix} \qquad \begin{matrix} \times\times\, . \\ \times\, .\, \times \\ .\, \times\times \end{matrix}$$

both reduce to $1/(n-1)^2$. The required coefficient is therefore

$$\frac{72}{(n-1)(n-2)}+\frac{224}{(n-1)^2}=\frac{8(37n-65)}{(n-1)^2(n-1)},$$

as appears in formula 28.

It will be obvious from the preceding section that the same rules must be applicable to multivariate problems, the only difference being that the column totals are then regarded as consisting of objects of two or more kinds. For example, to find the coefficient of $\kappa_{33}\kappa_{11}^2$ in the expression for $\kappa\begin{pmatrix}2&2&1\\2&2&1\end{pmatrix}$, it is merely necessary to note that the second order partitions of the bipartite (55) corresponding to the three partitions of 10, used above, can be allocated in 20, 48, and 8 ways respectively, yielding a coefficient

$$\frac{4(19n-33)}{(n-1)^2(n-2)}.$$

Alternatively the contributions to the coefficient of the univariate formula may be each split up among the six coefficients by which it is replaced in the bivariate formula, giving in this case

$$4(19n-33)\,\kappa_{33}\kappa_{11}^2+8(11n-20)\,\kappa_{33}\kappa_{20}\kappa_{02}+8(7n-12)\,\kappa_{42}\kappa_{11}\kappa_{02}$$
$$+8(7n-12)\,\kappa_{24}\kappa_{11}\kappa_{20}+2(5n-9)\,\kappa_{31}\kappa_{20}^2+2(5n-9)\,\kappa_{15}\kappa_{20}^2$$

in place of $$8(37n-65)\,\kappa_6\kappa_2^2.$$

In the same way the appropriate subdivision of the other bivariate and multivariate formulae may be obtained from an examination of the same set of two-way partitions, and it will evidently be sufficient for practical purposes to tabulate all the univariate formulae up to a given degree in order that all the corresponding multivariate formulae should be rapidly obtainable.

The algebraic equivalents of a number of the more commonly occurring patterns are given on pages 223–226.

Some useful patterns.

Two rows.

$$\begin{matrix}\times\times\\ \times\times\end{matrix}\ \frac{1}{n-1} \qquad \begin{matrix}\times\times\times\\ \times\times\times\end{matrix}\ \frac{n-2}{n(n-1)^2} \qquad \begin{matrix}\times\times\times\times\\ \times\times\times\times\end{matrix}\ \frac{n^2-3n+3}{n^2(n-1)^3}$$

In general, if $\alpha=-\{1/(n-1)\}$, we have $1/(n^{p-1})(1-\alpha^{p-1})$.

Three rows.

$$\begin{matrix}\times\times\\ \times\times\\ \times\times\end{matrix}\ \frac{n}{(n-1)(n-2)} \qquad \begin{matrix}\times\times\times\\ \times\times\times\\ \times\times\times\end{matrix}\ \frac{n^2-6n+10}{(n-1)^2(n-2)^2} \qquad \begin{matrix}\times\times\times\\ \times\times\times\\ \times\times\ .\end{matrix}\ \frac{n-3}{(n-1)^2(n-2)}$$

$$\begin{matrix}\times\times\times\\ \times\ .\ \times\\ \times\times\ .\end{matrix} \equiv \begin{matrix}.\ \times\times\\ \times\ .\ \times\\ \times\times\ .\end{matrix}\ \frac{1}{(n-1)^2} \qquad \begin{matrix}\times\times\times\times\\ \times\times\times\times\\ \times\times\times\times\end{matrix}\ \frac{n^4-9n^3+33n^2-60n+48}{n(n-1)^3(n-2)^3}$$

$$\begin{matrix}\times\times\times\times\\ \times\times\times\times\\ \times\times\times\ .\end{matrix}\ \frac{(n-3)(n^2-4n+6)}{n(n-1)^3(n-2)^3} \qquad \begin{matrix}\times\times\times\times\\ \times\times\times\times\\ \times\times\ .\ .\end{matrix}\ \frac{n^2-4n+5}{n(n-1)^3(n-2)}$$

$$\begin{matrix}\times\times\times\times\\ \times\times\ .\ \times\\ \times\times\times\ .\end{matrix}\ \frac{n^2-5n+7}{n(n-1)^3(n-2)} \qquad \begin{matrix}\times\ .\ \times\times\\ \times\times\ .\ \times\\ \times\times\times\ .\end{matrix}\ \frac{n-3}{n(n-1)^3}$$

$$\begin{matrix}\times\times\times\times\\ \times\ .\ \times\times\\ \times\times\ .\ .\end{matrix} \equiv \begin{matrix}.\ \times\times\times\\ \times\ .\ \times\times\\ \times\times\ .\ .\end{matrix}\ \frac{n-2}{n(n-1)^3} \qquad \begin{matrix}\times\times\times\times\\ .\ .\ \times\times\\ \times\times\ .\ .\end{matrix}\ \frac{1}{n(n-1)^2}$$

Four rows.

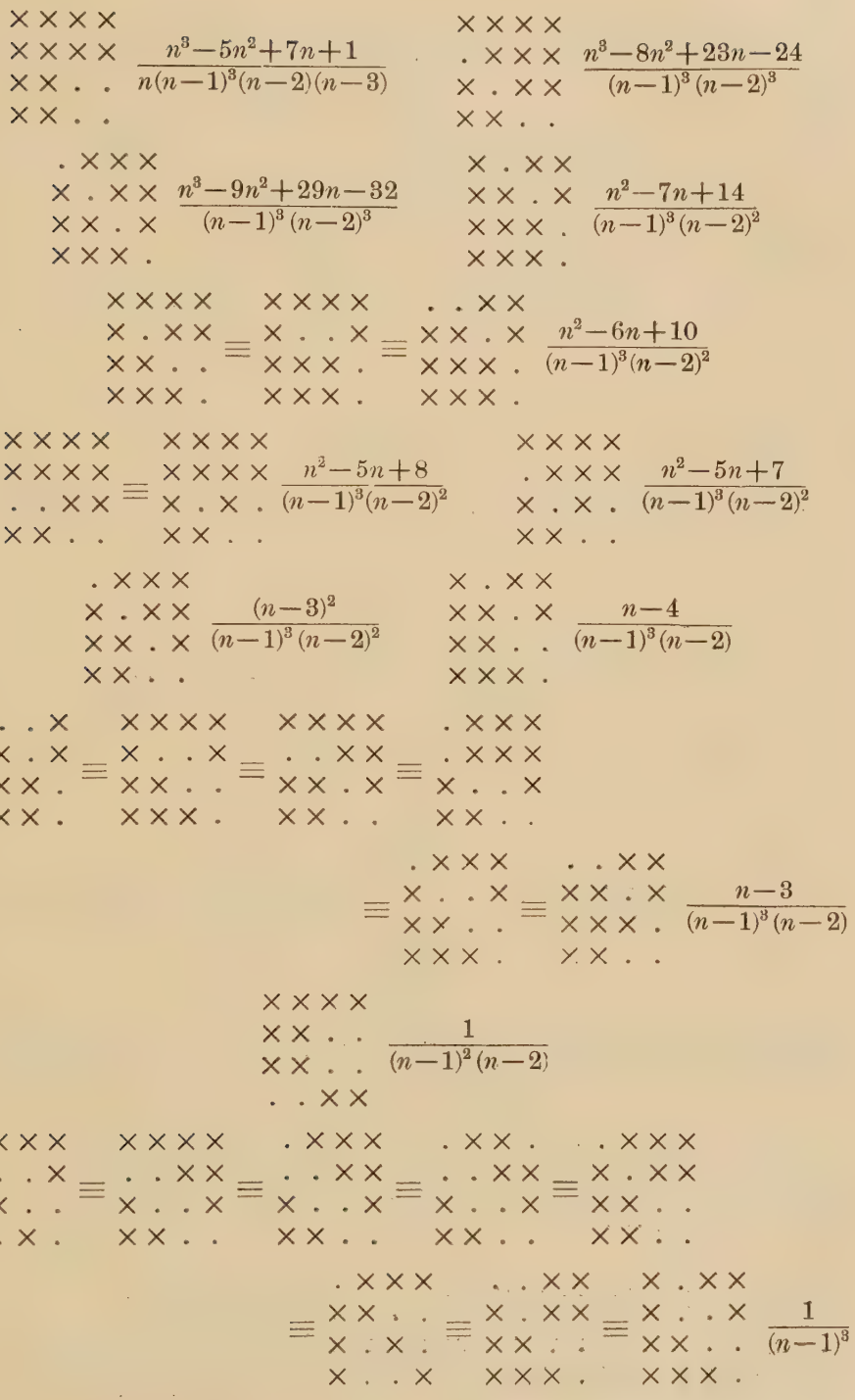
$$\begin{matrix}\times\times\\ \times\times\\ \times\times\\ \times\times\end{matrix}\ \frac{n(n+1)}{(n-1)(n-2)(n-3)} \qquad \begin{matrix}\times\times\times\\ \times\times\times\\ \times\times\times\\ \times\times\times\end{matrix}\ \frac{n^4-12n^3+51n^2-74n-18}{(n-1)^2(n-2)^2(n-3)^2}$$

$$\begin{matrix}\times\times\times\\ \times\times\times\\ \times\times\times\\ \times\times\ .\end{matrix}\ \frac{n^3-8n^2+17n+2}{(n-1)^2(n-2)^2(n-3)} \qquad \begin{matrix}\times\times\times\\ \times\times\times\\ \times\times\ .\\ \times\times\ .\end{matrix}\ \frac{n^2-4n-1}{(n-1)^2(n-2)(n-3)}$$

$$\begin{matrix}\times\times\times\\ \times\times\times\\ \times\ .\ \times\\ \times\times\ .\end{matrix}\ \frac{n(n-4)}{(n-1)^2(n-2)^2} \qquad \begin{matrix}\times\times\times\\ .\ \times\times\\ \times\ .\ \times\\ \times\times\ .\end{matrix}\ \frac{n(n-3)}{(n-1)^2(n-2)^2}$$

$$\begin{matrix}\times\times\times\\ \times\ .\ \times\\ \times\times\ .\\ \times\times\ .\end{matrix} \equiv \begin{matrix}.\ \times\times\\ \times\ .\ \times\\ \times\times\ .\\ \times\times\ .\end{matrix}\ \frac{n}{(n-1)^2(n-2)}$$

$$\begin{matrix}\times\times\ \times\\ \times\times\ \times\\ \times\times\times\ .\\ \times\times\times\ .\end{matrix} \equiv \begin{matrix}\times\times\times\times\\ \times\times\ .\ \times\\ \times\times\ .\ .\\ \times\times\times\ .\end{matrix}\ \frac{n^3-7n^2+13n+1}{n(n-1)^3(n-2)(n-3)}$$


$\frac{n^3-5n^2+7n+1}{n(n-1)^3(n-2)(n-3)}$
$\frac{n^3-8n^2+23n-24}{(n-1)^3(n-2)^3}$
$\frac{n^3-9n^2+29n-32}{(n-1)^3(n-2)^3}$
$\frac{n^2-7n+14}{(n-1)^3(n-2)^2}$
$\frac{n^2-6n+10}{(n-1)^3(n-2)^2}$
$\frac{n^2-5n+8}{(n-1)^3(n-2)^2}$
$\frac{n^2-5n+7}{(n-1)^3(n-2)^2}$
$\frac{(n-3)^2}{(n-1)^3(n-2)^2}$
$\frac{n-4}{(n-1)^3(n-2)}$
$\frac{n-3}{(n-1)^3(n-2)}$
$\frac{1}{(n-1)^2(n-2)}$
$\frac{1}{(n-1)^3}$

Five and six row patterns.

$$\begin{matrix}\times\times\,. \\ \times\times\,. \\ \times\,.\,\times \\ \times\,.\,\times \\ .\,\times\times\end{matrix} \equiv \begin{matrix}\times\times\,. \\ \times\times\,. \\ \times\,.\,\times \\ \times\,.\,\times \\ \times\times\times\end{matrix} \quad \frac{n^2}{(n-1)^3(n-2)^2}$$

$$\begin{matrix}\times\times\,. \\ \times\times\,. \\ \times\times\,. \\ \times\,.\,\times \\ .\,\times\times\end{matrix} \equiv \begin{matrix}\times\times\,. \\ \times\times\,. \\ \times\times\,. \\ \times\,.\,\times \\ \times\times\times\end{matrix} \quad \frac{n(n+1)}{(n-1)^2(n-2)(n-3)}$$

$$\begin{matrix}\times\times \\ \times\times \\ \times\times \\ \times\times \\ \times\times\end{matrix} \quad \frac{n^2(n+5)}{(n-1)(n-2)(n-3)(n-4)}$$

$$\begin{matrix}\times\times\times \\ \times\times\,. \\ \times\times\,. \\ \times\,.\,\times \\ .\,\times\times\end{matrix} \quad \frac{n(n^2-4n-1)}{(n-1)^2(n-2)^2(n-3)}$$

$$\begin{matrix}\times\times\times \\ \times\times\times \\ \times\times\,. \\ \times\times\,. \\ \times\,.\,\times\end{matrix} \quad \frac{n(n^2-5n-2)}{(n-1)^2(n-2)^2(n-3)}$$

$$\begin{matrix}\times\times\times \\ \times\times\times \\ \times\times\,. \\ \times\times\,. \\ \times\times\,.\end{matrix} \quad \frac{n(n^2-4n-9)}{(n-1)^2(n-2)(n-3)(n-4)}$$

$$\begin{matrix}\times\times\times \\ \times\times\times \\ \times\times\,. \\ \times\,.\,\times \\ .\,\times\times\end{matrix} \quad \frac{n(n^3-9n^2+19n+5)}{(n-1)^2(n-2)^2(n-3)^2}$$

$$\begin{matrix}\times\times\,. \\ \times\times\,. \\ \times\,.\,\times \\ \times\,.\,\times \\ .\,\times\times \\ .\,\times\times\end{matrix} \quad \frac{n(n+1)(n^2-5n+2)}{(n-1)^2(n-2)^2(n-3)^2}$$

$$\begin{matrix}\times\times\,. \\ \times\times\,. \\ \times\times\,. \\ \times\,.\,\times \\ \times\,.\,\times \\ .\,\times\times\end{matrix} \quad \frac{n^2(n+1)}{(n-1)^2(n-2)^2(n-3)}$$

$$\begin{matrix}\times\times \\ \times\times \\ \times\times \\ \times\times \\ \times\times \\ \times\times\end{matrix} \quad \frac{n(n+1)(n^2+15n-4)}{(n-1)(n-2)(n-3)(n-4)(n-5)}$$

The general formula for the two-column pattern with r rows is easily found, by enumerating the separations into 1, 2, 3, ... separates, to be

$$\sum_{p=1}^{r} \frac{(p-1)!}{p} \frac{\Delta^p(0^r)}{n(n-1)\dots(n-p+1)},$$

where $\Delta^p(0^r)$ stands for the leading p-th advancing difference of the series 0^r, 1^r, 2^r, ...

10. *Demonstration of the combinatorial method.*

To demonstrate the validity of the rules which have been stated, it is useful to consider in what manner the generating function M will

be modified by a functional transformation of the variates. In the case of a single variate x we have the function

$$M = 1+\mu_1\kappa+\mu_2\frac{t^2}{2!}+\ldots,$$

the coefficients of which give the mean value of all powers of x in the population. By what operation should the function M be transformed so as to give the corresponding function appropriate to a new variate ξ, which is a known function of x? Suppose that

$$\xi = f(x) = c_0+c_1x+c_2x^2+\ldots,$$

then the mean value of ξ is

$$\mu_1' = c_0+c_1\mu_1+c_2\mu_2+\ldots,$$

which may be written

$$c_0+c_1\frac{d}{dt}M+c_2\frac{d^2}{dt^2}M+\ldots,$$

or

$$f\left(\frac{d}{dt}\right)M,$$

where t is made to vanish after operation.

Moreover, the mean value of the r-th power of ξ will be given, at least formally, by the equation

$$\mu_r' = \left\{f\left(\frac{d}{dt}\right)\right\}^r M,$$

and the new generating function,

$$M' = 1+\mu_1'\tau+\mu_2'\frac{\tau^2}{2!}+\ldots,$$

may be written

$$e^{\tau f}M,$$

in which the operator is supposed to be expanded in powers of d/dt before attacking the operand.

The corresponding relationship for simultaneous variation is easily found. In such cases M will be a function of two or more variables t_1, t_2, ... corresponding to the variates x, y, ...; the new variates will be given functions of the old

$$\xi_1 = f_1(x, y, \ldots),$$
$$\xi_2 = f_2(x, y, \ldots),$$
$$\ldots \quad \ldots \quad \ldots$$

and the operative expression for the transformation of M is

$$M' = e^{\tau_1 f_1 + \tau_2 f_2 \cdots} M.$$

To apply this result to univariate sampling problems, consider the n observations of the sample as our n original variates, and the symmetric functions k_1, k_2, ... as the new variates the generating function of which is required. Then, considering first the operand, for the first observation x, the μ generator is $e^{K(t_1)}$, where K is the κ generator of the population sampled, *i.e.*

$$K(t) = \kappa_1 t + \kappa_2 \frac{t^2}{2!} + \ldots.$$

Moreover, since the n observations are independent, their simultaneous κ generator will be merely the sum of the individual generators, so that our operand is

$$\exp\left\{\kappa_1 s_1 + \kappa_2 \frac{s_2}{2!} + \ldots\right\},$$

in which

$$s_r = \sum_{\nu=1}^{n} (t_\nu^r).$$

We may note at once that the coefficient of $\kappa_{q_1}^{\chi_1} \kappa_{q_2}^{\chi_2} \ldots$ in the operand is

$$\frac{s_{q_1}^{\chi_1}}{(q_1!)^{\chi_1} \chi_1!} \frac{s_{q_2}^{\chi_2}}{(q_2!)^{\chi_2} \chi_2!} \ldots.$$

The μ generator of the simultaneous distribution of the k statistics will be given by the operator

$$e^{\tau_1 k_1 + \tau_2 k_2 + \ldots + \tau_n k_n},$$

in which k_ν is interpreted as the same function of d/dt_1, d/dt_2, ... as the corresponding k statistic is of x_1, x_2, ..., x_n. The property by which these statistics were defined, namely that the mean value of k_ν should be κ_ν, is now seen to imply that

$$k_\nu\left(\frac{s_\nu}{\nu!}\right) = 1;$$

but

$$k_\nu\left(\frac{s_{\nu_1}}{\nu_1!} \frac{s_{\nu_2}}{\nu_2!} \ldots\right) = 0,$$

where $(\nu_1, \nu_2, \ldots)$ is any partition of ν. If, for example, the partition is of two parts,

$$s_{\nu_1} s_{\nu_2} = \sum_{1}^{n} (t^{\nu_1 + \nu_2}) + \sum_{1}^{n(n+1)} (t^{\nu_1} t'^{\nu_2}),$$

in which t and t' are different members of the set $t_1, \ldots, t_n$, it follows that k_ν must contain, in addition to the simple term

$$\frac{1}{n} S\left(\frac{d}{dt}\right)^n,$$

terms for all two-part partitions of the form

$$\frac{-1}{n(n-1)} \frac{\nu!}{\nu_1!\,\nu_2!} S\left\{\left(\frac{d}{dt}\right)^{\nu_1}\left(\frac{d}{dt'}\right)^{\nu_2}\right\}$$

except when $\nu_1 = \nu_2$, when each operator finds two terms on which it can act, and its coefficient is therefore to be halved. Thus, if we write

$$g(p_1^{\pi_1} p_2^{\pi_2} \ldots) = \frac{(-)^{\rho-1}(\rho-1)!}{n(n-1)\ldots(n-\rho+1)} \times S\left\{\left(\frac{d}{dt_{\nu_1}}\right)^{p} \ldots \left(\frac{d}{dt_{\nu_{\pi_1}}}\right)^{p_1}\left(\frac{d}{dt_{\nu'_1}}\right)^{p_2} \ldots \left(\frac{d}{dt_{\nu'_{\pi_2}}}\right)^{p_2} \ldots\right\},$$

where $\rho = \pi_1 + \pi_2 + \ldots$ and t_{ν_1}, etc. are any selection of ρ out of the n variables t, the summation being extended over all such selections, then

$$k_p = \Sigma \frac{p!}{(p_1!)^{\pi_1}\pi_1!\,(p_2!)^{\pi_2}\pi_2!\ldots} g(p_1^{\pi_1} p_2^{\pi_2} \ldots),$$

the summation being taken over all partitions of p.

This structure of the k operator makes it possible to think of the p acts of differentiation in each operator as p separate objects, the partitions of which, represented by the g operators, occur each in as many ways as the objects can be arranged in that partition. We may thus use a two-way partition to assign how many of these operations are effective against each of a series of factors $s_{q_1} s_{q_2}$ constituting the operand.

Let now this operand product be expanded in a number of terms of the form

$$z(a, b, c) = S(t^a t'^b t''^c),$$

the summation being taken over all the $n(n-1)(n-2)$ different ways of selecting t, t', and t'' from among the set $t_1, \ldots, t_n$. This will then be a z term for every possible separation of the partition $(q_1^{\chi_1} q_2^{\chi_2} \ldots)$ into one or more separates. For each two-way partition chosen all these separations will contribute to the result, and with the same numerical coefficient, apart from that contained in the g operators, equal to the number of ways of allocating the objects in the two-way partition.

The number of terms in the z corresponding to any separation into a separates is $n(n-1)\ldots(n-a+1)$, and this, combined with the factors in the g operators, gives the functions of n corresponding to any two-way partition according to rule (5). There remains, however, in M' a number of terms corresponding to two-way partitions, in which the columns may be divided into two classes, each confined to different sets of rows. These introduce terms of a higher order in n, which are obliterated when we find $K' = \log M'$, for in these cases the additional term in M' will be of the form AB, where both A and B occur also as other terms in M'.

11. *Measures of departure from normality.*

The statistical inefficiency of moment statistics from distributions differing widely from the normal, except when they are of a special type [**10**], much reduces their practical importance for curve fitting; but since they are fully efficient for the normal distribution, they provide an ideal basis for testing if an observed sample indicates a significant departure from normality in the population sampled. Significant asymmetry should, in the first instance, be shown by an excessive value of k_3; but since the variance of the population is usually unknown, but may be estimated from the value of k_2 observed in the sample, the test of significance will usually involve not the distribution of k_3, the moments of which are given by such formulae as (4) and (20), but the distribution of the ratio $k_3 k_2^{-\frac{3}{2}}$. Since, for the normal distribution, the variance of k_3 is given by

$$\kappa(3^2) = \frac{6n}{(n-1)(n-2)}\,\kappa_2^3,$$

it will be convenient to show how the moments of such a statistic as

$$x = \sqrt{\left(\frac{(n-1)(n-2)}{6n}\right)}\, k_3 k_2^{-\frac{3}{2}}$$

may be expressed in terms of the general $\kappa(p_1^{\pi_1} p_2^{\pi_2}\ldots)$, for all distributions, and its particular value obtained for the normal distribution. This may be done by expanding the factor $k_2^{-\frac{3}{2}}$ in the form

$$\kappa_2^{-\frac{3}{2}}\left(1+\frac{k_2-\kappa_2}{\kappa_2}\right)^{-\frac{3}{2}},$$

whereupon, in virtue of the expressions connecting the moments μ with the semi-invariants κ, and the mean value of x being zero, we can at once

write down the following expansion for its variance :

$$\mu_2(x) = \kappa_2(x) = \frac{(n-1)(n-2)}{6n\kappa_2^3}\left\{\kappa(3^2) - \frac{3}{\kappa_2}\kappa(3^2 2) + \frac{6}{\kappa_2^2}\{\kappa(3^2)\kappa(2^2) + \kappa(3^2 2^2)\}\right.$$
$$\left. - \frac{10}{\kappa_2^3}\{3\kappa(3^2 2)\kappa(2^2) + \kappa(3^2)\kappa(2^3)\} + \frac{15}{\kappa^4}\{3\kappa(3^2)\kappa^2(2^2)\}\right\},$$

in which, remembering that a κ of p parts involves $n^{-(p-1)}$, terms beyond n^{-2} have been omitted, as well as the terms of odd degree which vanish for symmetrical distributions.

Similarly we have

$$\mu_4(x) = \frac{(n-1)^2(n-2)^2}{36n^2\kappa_2^6}\left\{3\kappa^2(3^2) + \kappa(3^4) - \frac{6}{\kappa_2}\{6\kappa(3^2 2)\kappa(3^2) + \kappa(3^4 2)\}\right.$$
$$+ \frac{21}{\kappa_2^2}\{3\kappa^2(3^2)\kappa(2^2) + 6\kappa^2(3^2 2)$$
$$+ \kappa(3^4)\kappa(2^2) + 6\kappa(3^2 2^2)\kappa(3^2)\}$$
$$- \frac{56}{\kappa_2^3}\{18\kappa(3^2 2)\kappa(3^2)\kappa(2^2) + 3\kappa^2(3^2)\kappa(2^3)\}$$
$$\left. + \frac{126}{\kappa_2^4}\{9\kappa^2(3^2)\kappa^2(2^2)\}\right\}$$

and

$$\mu_6(x) = \frac{(n-1)^3(n-2)^3}{216n^3\kappa_2^9}\left\{15\kappa^3(3^2) + 5\kappa(3^4)\kappa(3^2) + \kappa(3^6)\right.$$
$$- \frac{9}{\kappa_2}\{45\kappa(3^2 2)\kappa^2(3^2) + 15\kappa(3^4 2)\kappa(3^2) + 15\kappa(3^4)\kappa(3^2 2)\}$$
$$+ \frac{45}{\kappa_2^2}\{15\kappa^3(3^2)\kappa(2^2) + 90\kappa^2(3^2 2)\kappa(3^2) + 45\kappa(3^2 2^2)\kappa^2(3^2)$$
$$+ 15\kappa(3^4)\kappa(3^2)\kappa(2^2)\}$$
$$- \frac{165}{\kappa_2^3}\{15\kappa^3(3^2)\kappa(2^3) + 135\kappa(3^2 2)\kappa^2(3^2)\kappa(2^2)$$
$$\left. + \frac{495}{\kappa_2^4}\{45\kappa^3(3^2)\kappa^2(2^2)\}\right\}.$$

From these moments of the distribution of x, the semi-invariants $\kappa_4(x)$ and $\kappa_6(x)$ may be obtained by means of the relations

$$\mu_4(x) = \kappa_4(x) + 3\kappa_2^2(x),$$

$$\mu_6(x) = \kappa_6(x) + 15\kappa_4(x)\kappa_2(x) + 15\kappa_2^3(x),$$

giving

$$\kappa_4(x) = \frac{(n-1)^2(n-2)^2}{36n^2\kappa_2^6}\left\{\kappa(3^4) - \frac{18}{\kappa_2}\kappa(3^2 2)\kappa(3^2) + \frac{27}{\kappa_2^2}\kappa^2(3^2)\kappa(2^2)\right.$$
$$- \frac{6}{\kappa_2}(3^4 2) + \frac{99}{\kappa_2^2}\kappa^2(3^2 2) + \frac{21}{\kappa_2^2}\kappa(3^4)\kappa(2^2)$$
$$+ \frac{90}{\kappa_2^2}\kappa(3^2 2^2)\kappa(3^2) - \frac{720}{\kappa_2^3}\kappa(3^2 2)\kappa(3^2)\kappa(2^2)$$
$$\left. - \frac{108}{\kappa_2^3}\kappa^2(3^2)\kappa(2^3) + \frac{756}{\kappa_2^4}\kappa^2(3^2)\kappa^2(2^2)\right\}$$

and

$$\kappa_6(x) = \frac{(n-1)^3(n-2)^3}{216n^3\kappa_2^9}\left\{\kappa(3^6) - \frac{45}{\kappa_2}\kappa(3^4 2)\kappa(3^2) - \frac{90}{\kappa_2}\kappa(3^4)\kappa(3^2 2)\right.$$
$$+ \frac{1350}{\kappa_2^2}\kappa^2(3^2 2)\kappa(3^2) + \frac{405}{\kappa_2^2}\kappa(3^2 2^2)\kappa^2(3^2)$$
$$+ \frac{270}{\kappa_2^2}\kappa(3^4)\kappa(3^2)\kappa(2^2) - \frac{405}{\kappa_2^3}\kappa^3(3^2)\kappa(2^3)$$
$$\left. - \frac{5670}{\kappa_2^3}\kappa(3^2 2)\kappa^2(3^2)\kappa(2^2) + \frac{4860}{\kappa_2^4}\kappa^3(3^2)\kappa^2(2^2)\right\},$$

while no higher semi-invariants contain terms involving only n^{-2}.

The formulae tabulated give all the values required for $\kappa_2(x)$; thus for samples from the normal distribution

$$\kappa(2^2) = \frac{2}{n-1}\kappa_2^2, \qquad \kappa(3^2) = \frac{6n}{(n-1)(n-2)}\kappa_2^3, \qquad \kappa(2^3) = \frac{8}{(n-1)^2}\kappa_2^3,$$
$$\kappa(3^2 2) = \frac{6}{n-1}\kappa_2\kappa(3^2), \qquad \kappa(3^2 2^2) = \frac{48}{(n-1)^2}\kappa_2^2\kappa(3^2),$$

and substituting these values, we find

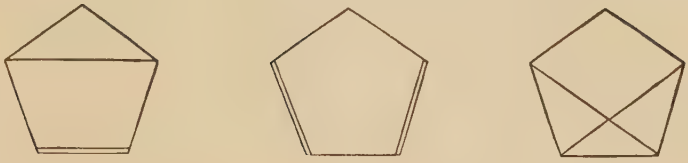
$$\kappa_2(x) = 1 - \frac{6}{n} + \frac{22}{n^2}.$$

To evaluate $\kappa_4(x)$ we need in addition

$$\kappa(3^4) = \frac{648(5n-12)n^2}{(n-1)^3(n-2)^3}\kappa_2^6,$$

and the leading term in $\kappa(3^4 2)$; this latter only requires the enumeration of the number of ways of building up two-way partitions of $(3^4 2)$ with row totals (2^7), or the number of ways of connecting up the symbolical

figures

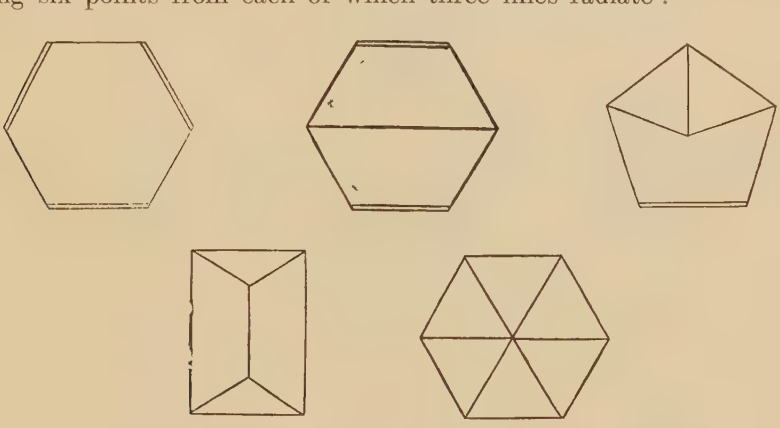


which can be done in 15548, 7774 and 15548 ways respectively, showing that $\kappa(3^42)$ from normal samples is approximately $38880n^{-4}$.

With this value, that of $\kappa_4(x)$ is evaluated as

$$\kappa_4(x) = \frac{36}{n} - \frac{1296}{n^2}.$$

Finally, for $\kappa_6(x)$ the only new κ required is $\kappa(3^6)$, involving the figures having six points from each of which three lines radiate :—



which supply a contribution of $47520n^{-2}$ to $\kappa_6(x)$, or, with the other terms, lead to the value

$$\kappa_6(x) = \frac{15120}{n^2}.$$

For the practical application of the function x in testing asymmetry we shall now require to construct a function of x which, as far as terms in n^{-2}, is distributed normally. Putting

$$x = \beta\xi + \delta(\xi^3 - 3\xi) + \eta(\xi^5 - 10\xi^3 + 15),$$

where ξ is normally distributed with unit variance, it is easy to obtain

$$\kappa_2(x) = \beta^2 + 6\delta^2,$$

$$\kappa_4(x) = 24\beta^3\delta + 216\beta^2\delta^2,$$

$$\kappa_6(x) = 720\beta^5\eta + 3240\beta^4\delta^2,$$

which are satisfied by

$$\beta = 1-\frac{3}{n}-\frac{1}{4n^2}, \qquad \delta = \frac{3}{2n}\left(1-\frac{81}{2n}\right), \qquad \eta = \frac{87}{8n^2},$$

or, inverting the relation between ξ and x, we have

$$\xi = x\left(1+\frac{3}{n}+\frac{91}{4n^2}\right)-\frac{3}{2n}\left(1-\frac{111}{2n}\right)(x^3-3x)-\frac{33}{8n^2}(x^5-10x^3+15x).$$

This translation formula makes it possible to assess the numerical effects upon tests of significance of the actual distribution; Tables 2 and 3 show the values of various possible formulae for the test deviate in the region, important for tests of significance, $x = 1\cdot8$ to $2\cdot2$, and indicate that these effects are very serious.

TABLE 2.

Comparison of deviates in five formulae for testing asymmetry.

$n = 100$.

(*a*) $\sqrt{\frac{n}{6}}\, m_3 m_2^{-\frac{3}{2}}$	(*b*) $\sqrt{\frac{n}{6}}\, k_3 k_2^{-\frac{3}{2}}$	(*c*) x	(*d*) ξ_1	(*e*) ξ_2
1·7999	1·8274	1·8	1·8475	1·8603
1·9999	2·0305	2·0	2·0300	2·0586
2·1999	2·2335	2·2	2·2053	2·2530

TABLE 3.

Comparison of deviates in five formulae for testing asymmetry.

$n = 50$.

(*a*) $\sqrt{\frac{n}{6}}\, m_3 m_2^{-\frac{3}{2}}$	(*b*) $\sqrt{\frac{n}{6}}\, k_3 k_2^{-\frac{3}{2}}$	(*c*) x	(*d*) ξ_1	(*e*) ξ_2
1·7996	1·8558	1·8	1·8950	1·9463
1·9996	2·0620	2·0	2·0600	2·1745
2·1995	2·2682	2·2	2·2106	2·4016

In this region an error of $0\cdot1$ in the deviate produces an error of about 24 per cent. in the probability deduced, and, although high accuracy in the latter is not a necessity, little reliance can be placed upon tests when the deviate may be biased by as much as $0\cdot2$. Of the formulae tested, the formula (*a*) in terms of crude moments is almost equivalent to the use of x, and these are evidently the most in error. Of the simple formulae (*b*) is least in error, and for samples of 100 this error is only about ·03. The value ξ_1 shows the effect of using terms of the first degree only in the translation formula, while ξ_2 shows the effect of using also terms in n^{-2}. There is evidently little to be gained by using ξ_1 instead of the simple formula $\sqrt{(n/6)}\, k_3 k_4^{-\frac{3}{2}}$, which latter gives

apparently the better values for deviations exceeding 2·0. For samples as small as 50 the fully corrected value ξ_2 is evidently required, and in view of the uncertainty of the effect of the omitted terms in n^{-2}, etc., no reliable test of normality for materially smaller samples can be said to be available. As in so many other cases, the adequate treatment even of moderately small samples is not well approached by series in n^{-1}.

12. *The significance of the fourth moment.*

The sampling variance of k_4 from a normal sample is

$$\frac{24n(n+1)}{(n-1)(n-2)(n-3)}\,\kappa_2^4;$$

in testing the significance of such a value, we should therefore naturally calculate

$$x = \sqrt{\left(\frac{(n-1)(n-2)(n-3)}{24n(n+1)}\right)}\,k_4 k_2^{-2}$$

as a variate which, with increasing sample number, tends to be normally distributed with unit variance. With finite samples the distribution is asymmetrical, for $\kappa(4^3)$ is not zero. The true mean value of x is zero, for with a normal distribution $\kappa(42^p)$ is zero for all values of p, whence it follows that the mean of k_4 is zero independently for all values of k_2.

The mean value of x^2 is easily expanded in the form

$$\frac{(n-1)(n-2)(n-3)}{24n(n+1)\,\kappa_2^4}\left\{\kappa(4^2)-\frac{4}{\kappa_2}\,\kappa(4^2 2)+\frac{10}{\kappa_2^2}\,\kappa(4^2)\,\kappa(2^2)+\ldots\right\}$$

or

$$1-\frac{32}{n-1}+\frac{20}{n-1}$$

as far as n^{-1}.

The mean value of x^3, as far as $n^{-\frac{3}{2}}$, is

$$\left\{\frac{(n-1)(n-2)(n-3)}{24n(n+1)\,\kappa_2^4}\right\}^{\frac{3}{2}}\left\{\kappa(4^3)-\frac{6}{\kappa_2}\,\kappa(4^3 2)+\frac{21}{\kappa_2^2}\,\kappa(4^3)\,\kappa(2^2)+\ldots\right\}.$$

Now $\kappa(4^3)$ has been evaluated by the direct combinatorial method, giving (formula 57)

$$\kappa(4^3) = \frac{1728n(n+1)(n^2-5n+2)}{(n-1)^2(n-2)\,2(n-3)^2}\,\kappa_2^6,$$

or, as near as needed,

$$\frac{1728}{n^3}(n+8);$$

while the leading term of $\kappa(4^3 2)$ is $1728n^{-3}\times 12$, giving, as the mean value of x^3,

$$\left\{\frac{(n-1)(n-2)(n-3)}{n^3(n+1)}\right\}^{\frac{3}{2}} 6\sqrt{6}\{n+8-72+42\},$$

or

$$\frac{6\sqrt{6}}{\sqrt{n}}\left(1-\frac{65}{2n}\right).$$

Next, the mean value of x^4 is, as far as n^{-1},

$$\left\{\frac{(n-1)(n-2)(n-3)}{24n(n+1)\kappa_2^4}\right\}^2\left\{3\kappa^2(4^2)+\kappa(4^4)\right.$$
$$\left.-\frac{8}{\kappa_2}\{6\kappa(4^2)\kappa(4^2 2)\}+\frac{36}{\kappa_2^2}\{3\kappa^2(4^2)\kappa(2^2)\}\right\};$$

whence, subtracting three times the square of the mean of x^2, there remains

$$\kappa_4(x)=\left\{\frac{(n-1)(n-2)(n-3)}{24n(n+1)\kappa_2^4}\right\}^2\left\{\kappa(4^4)-\frac{24}{\kappa_2}\kappa(4^2)\kappa(4^2 2)+\frac{96}{n-1}\kappa^2(4^2)\right\}.$$

The leading term in $\kappa(4^4)\div 576\kappa_2^8$ comes to $636n^{-3}$, to which the other terms add -192 and $+96$ respectively, leaving

$$\kappa_4(x)=\frac{540}{n}.$$

For the mean value of x^5 we shall need

$$\left(\frac{n}{24\kappa_2^4}\right)^{\frac{5}{2}}\left\{10\kappa(4^2)\kappa(4^3)+\kappa(4^5)-\frac{10}{\kappa_2}10\kappa(4^2 2)\kappa(4^3)\right.$$
$$\left.+10\kappa(4^2)\kappa(4^2)\kappa(4^3 2)+\frac{550}{\kappa_2^2}\kappa(4^3)\kappa(4^2)\kappa(2^2)\right\},$$

whence, deducting $10\kappa_2(x)\,.\,\kappa_3(x)$, there remains

$$\kappa_5(x)=\left(\frac{n}{24\kappa_2^4}\right)^{\frac{5}{2}}\left\{\kappa(4^5)-\frac{60}{\kappa_2}\kappa(4^3)\kappa(4^2 2)\right.$$
$$\left.-\frac{40}{\kappa_2}\kappa(4^3 2)\kappa(4^2)+\frac{240}{\kappa_2^2}\kappa(4^3)\kappa(4^2)\kappa(2^2)\right\},$$

or

$$\frac{91}{n^{\frac{5}{2}}}.144\sqrt{6}.$$

Now if ξ is normally distributed with unit variance, and x can be expressed approximately in the form

$$x=\beta\xi+\gamma(\xi^2-1)+\delta(\xi^3-3\xi)+\epsilon(\xi^4-6\xi^2+3)+\ldots,$$

we have

$$\kappa_1(x) = \mu_1(x) = 0,$$

$$\kappa_2(x) = \mu_2(x) = \beta^2+2\gamma^2+6\delta^2+24\epsilon^2+\ldots,$$

$$\kappa_3(x) = \mu_3(x) = 6\beta^2\gamma+8\gamma^3+36\beta\gamma\delta+\ldots,$$

$$\mu_4(x) = 3\beta^4+24\beta^3\delta+48\beta^2\gamma^2,$$

whence $$\kappa_4(x) = 24\beta^3\delta+48\beta^2\gamma^2,$$

and $$\mu_5(x) = 60\beta^4\gamma+120\beta^4\epsilon+1080\beta^3\gamma\delta+680\beta^2\gamma^3,$$

whence $$\kappa_5(x) = 120\beta^4\epsilon+720\beta^3\gamma\delta+560\beta^2\gamma^2;$$

and equating these to the actual values, neglecting n^{-2}, we have the translation formula

$$x = \left(1-\frac{12}{n}\right)\xi+\sqrt{\frac{6}{n}}\left(1-\frac{159}{2n}\right)(\xi^2-1) + \frac{21}{2n}(\xi^3-3\xi)+\frac{91\sqrt{6}}{5n^{\frac{3}{2}}}(\xi^4-6\xi^2+3),$$

or, inversely,

$$\xi = -\frac{21\sqrt{6}}{n^{\frac{3}{2}}}+x\left(1+\frac{36}{n}\right)-\sqrt{\frac{6}{n}}\left(1-\frac{201}{2n}\right)(x^2-1) + \frac{3}{2n}(x^3-3x)+\frac{43}{10n}\sqrt{\frac{6}{n}}(x^4-6x^2+3).$$

Summary.

The equations which connect the moment functions of the sampling distribution of moment statistics with the moment functions of the population from which the samples are drawn correspond in univariate problems to all the partitions of all the natural numbers, and in multivariate problems to all the partitions of all multipartite numbers. Very few of this system of equations have hitherto been obtained owing to the algebraical complexity of their direct evaluation. The formulae are very much simplified (i) by using the semi-invariants instead of the moments of the population, and (ii) by using the system of moment statistics, the mean sampling value of each of which is the corresponding semi-invariant. The relations which necessarily exist between the different multivariate formulae demonstrate that all of these, as well as the univariate formulae, must be derivable from a system of rules associating

different two-way partitions of multipartite and unipartite numbers with corresponding functions of the sample number n.

Rules are given and illustrated which enable any term of any of these formulae to be obtained directly from an examination of the appropriate partition. Their general validity is demonstrated by a theorem which connects the moment generating function of any distribution with the corresponding function of any functionally related set of variates. Complete univariate formulae are given up to the tenth degree, and some new results are applied to the theory of samples from a normal population.

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GENERALIZATION OF TWO THEOREMS OF STIELTJES CONCERNING LEGENDRE'S COEFFICIENTS

By E. W. HOBSON.

[Received 12 December, 1928.—Read 13 December, 1928.]

With a view to employment in the theory of convergence of series, of which the terms are multiples of Legendre's coefficients, I showed*, in 1908, that the Legendre's coefficient $P_n(\cos\theta)$ has the property that $(n\sin\theta)^{\frac{1}{2}}P_n(\cos\theta)$ is less, in absolute value, than some fixed number, independent of n and θ, where $0<\theta<\pi$. Another proof of the theorem was afterwards given† by Jolliffe. I found later that the theorem had been given‡ in 1890 by Stieltjes, in the form $|P_n(\cos\theta)|<\frac{MC_n}{(2\sin\theta)^{\frac{1}{2}}}$, where $C_n=\frac{2}{(n\pi)^{\frac{1}{2}}}(1+\epsilon)$, and ϵ converges to zero with $1/n$, and M is a number which varies between 1 and 2. In 1913 it was shown§ by Gronwall that $|P_n(\cos\theta)|<\frac{4}{(2\pi n\sin\theta)^{\frac{1}{2}}}$; this formula he deduced from the asymptotic series obtained by Stieltjes for $P_n(\cos\theta)$. More recently an elementary proof, not involving the use of asymptotic series, or of definite integrals, has been given‖ by Fejér, that $|P_n(\cos\theta)|\leqslant\frac{8}{(2\pi n\sin\theta)^{\frac{1}{2}}}$.

In the present communication it is shown that the theorem holds when n is not restricted to be integral, but may have any real value greater than or equal to 1. The theorem is further extended to the function $P_n^{\pm m}(\cos\theta)$, where n and m are real and $n\geqslant 1$, $m>0$, $n-m+1>0$. The corresponding theorems for $Q_n^m(\cos\theta)$ are also ob-

* *Proc. London Math. Soc.* (2), 7 (1908), 25.
† *Messenger of Math.*, 43 (1913), 85.
‡ *Annales de Toulouse*, 4 (1890), G. 6.
§ *Math. Annalen*, 74 (1913), 221.
‖ *Math. Zeitschrift*, 24 (1925), 290.

tained. Another theorem was given* by Stieltjes for the Legendre's coefficients, viz. $|P_n(\cos\theta)-P_{n+2}(\cos\theta)|<\frac{C}{n^{\frac{1}{2}}}$, where C is independent of n and θ and $0\leqslant\theta\leqslant\pi$. This theorem I have also shown to hold good for real values of n that are not necessarily integral; moreover, I have extended the theorem to the function $P_n^{\pm m}(\cos\theta)$, and to $Q_n^{\pm m}(\cos\theta)$. The results are deduced from a general formula for $Q_n^m(\mu)$, where μ is complex, given in my general theory† of the functions $P_n^m(\mu)$, $Q_n^m(\mu)$, for complex values of n, m, and μ. I have in this note used the methods and results of that memoir, and have quoted results there given by the letter H with the number of the formula referred to.

1. In the general expression H(40) for $Q_n^m(\mu)$, where n, m are any complex numbers, and μ is complex, on the plane of μ with a cross-cut along the real axis from $-\infty$ to 1,

$$Q_n^m(\mu)=ie^{(m-n)\pi i}\cdot 2^m\frac{\Pi(m-\frac{1}{2})\Pi(-\frac{1}{2})}{4\pi\sin(n+m)\pi}(\mu^2-1)^{\frac{1}{2}m}\times\int^{(\frac{1}{2}+,\,0+,\,\frac{1}{2}-,\,0-)}\frac{h^{n+m}}{(1-2\mu h+h^2)^{m+\frac{1}{2}}}\,dh,$$

where the point z $[=\mu+(\mu^2-1)^{\frac{1}{2}}]$ is outside the contour. If we write $h=(1-u)/z$, we have

$$Q_n^m(\mu)=-ie^{(m-n)\pi i}\cdot 2^m\frac{\Pi(m-\frac{1}{2})\Pi(-\frac{1}{2})}{4\pi\sin(n+m)\pi}\frac{z^{m-n}}{(z^2-1)^{m+\frac{1}{2}}}(\mu^2-1)^{\frac{1}{2}m}\times\int^{(0+,\,1+,\,0-,\,1-)}u^{-m-\frac{1}{2}}(1-u)^{n+m}\left(1+\frac{u}{z^2-1}\right)^{-m-\frac{1}{2}}du.$$

In this expression the initial phases of u and $1-u$, at some point on the real axis of u between the points 0 and 1, are taken to be zero.

If we assume that $R(n+m+1)>0$, $R(\frac{1}{2}-m)\geqslant 0$, the path of integration may be taken to consist of circles round the points 0 and 1, and straight portions along the real axis of u. By making the radii of the circles converge to zero, we find that the formula becomes

$$Q_n^m(\mu)=e^{m\pi i}\cdot\frac{2^m\cos m\pi}{\pi}\cdot\Pi(m-\tfrac{1}{2})\Pi(-\tfrac{1}{2})\frac{z^{m-n}}{(z^2-1)^{m+\frac{1}{2}}}(\mu^2-1)^{\frac{1}{2}m}\times\int_0^1\frac{(1-u)^{n+m}}{u^{m+\frac{1}{2}}\{1+(u/z^2-1)\}^{m+\frac{1}{2}}}\,du,$$

where $R(n+m+1)>0$, $R(\frac{1}{2}-m)\geqslant 0$.

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* *Correspondance d'Hermite et de Stieltjes* (Paris, 1905), Letters 309, 310, pp. 174–177.

† *Phil. Trans. Royal Soc.* (A), 187 (1896), 443–531.

If we take $m=0$, $R(n+1)>0$, we have

$$Q_n(\mu)=\frac{z^{-n}}{(z^2-1)^{\frac{1}{2}}}\int_0^1\frac{(1-u)^n}{u^{\frac{1}{2}}\{1+u/(z^2-1)\}^{\frac{1}{2}}}\,du.$$

Now let $\mu=\cos\theta+0\,.\,i$, in which case $z=e^{i\theta}$; we then have

$$Q_n(\cos\theta+0\,.\,i)=\frac{e^{-[(n+\frac{1}{2})\theta+\frac{1}{4}\pi]i}}{(2\sin\theta)^{\frac{1}{2}}}\int_0^1\frac{(1-u)^n}{u^{\frac{1}{2}}\{1+u/(z^2-1)\}^{\frac{1}{2}}}\,du.$$

Since, in accordance with H (31), $Q_n(\cos\theta+0\,.\,i)=Q_n(\cos\theta)-\frac{1}{2}\pi i P_n(\cos\theta)$, we see that, if n be taken to be real and greater than -1, $|Q_n(\cos\theta)|$ and $\frac{1}{2}\pi|P_n(\cos\theta)|$ are both less than

$$\frac{1}{(2\sin\theta)^{\frac{1}{2}}}\int_0^1\frac{(1-u)^n}{u^{\frac{1}{2}}}\left|\frac{1}{\{1+u/(z^2-1)\}^{\frac{1}{2}}}\right|du,$$

where $0<\theta<\pi$.

We have

$$1+\frac{u}{z^2-1}=1+\frac{ue^{-i\theta}}{2i\sin\theta}=1-\tfrac{1}{2}u-\tfrac{1}{2}iu\cot\theta,$$

and
$$|1-\tfrac{1}{2}u-\tfrac{1}{2}iu\cot\theta|=\{(1-\tfrac{1}{2}u)^2+\tfrac{1}{4}u^2\cot^2\theta\}^{\frac{1}{2}},$$

and this exceeds $1-\frac{1}{2}u$ for all values of θ. Also the least value of $1-\frac{1}{2}u$ is $\frac{1}{2}$ for all values of u in (0, 1). It thus appears that, for all values of θ within $(0, \pi)$,

$$\int_0^1\frac{(1-u)^n}{u^{\frac{1}{2}}}\left|\frac{1}{\{1+u/(z^2-1)\}^{\frac{1}{2}}}\right|du<2^{\frac{1}{2}}\int_0^1\frac{(1-u)^n}{u^{\frac{1}{2}}}\,du$$
$$<(2\pi)^{\frac{1}{2}}\frac{\Pi(n)}{\Pi(n+\frac{1}{2})}.$$

It has thus been shown that, when n is positive and real, but not necessarily integral,

$$|Q_n(\cos\theta)|<\left(\frac{\pi}{\sin\theta}\right)^{\frac{1}{2}}\frac{\Pi(n)}{\Pi(n+\frac{1}{2})},\qquad|P_n(\cos\theta)|<\frac{2}{(\pi\sin\theta)^{\frac{1}{2}}}\frac{\Pi(n)}{\Pi(n+\frac{1}{2})}.$$

It will now be shown that, for $n\geqslant 1$, $\dfrac{\Pi(n)}{\Pi(n+\frac{1}{2})}<\dfrac{1}{n^{\frac{1}{2}}}$.

We have*

$$\frac{\Pi(n)\,\Pi(-\frac{1}{2})}{\Pi(n+\frac{1}{2})}=\int_0^1 t^n(1-t)^{-\frac{1}{2}}\,dt=2\int_0^{\frac{1}{2}\pi}\cos^{2n+1}\theta\,d\theta;$$

* This proof was supplied by a referee. I had originally given a proof depending upon the use of Stirling's theorem.

now $e^{\frac{1}{2}\theta^2}\cos\theta$ is a decreasing function in $(0, \frac{1}{2}\pi)$, since it has the negative derivative $e^{\frac{1}{2}\theta^2}(\theta\cos\theta - \sin\theta)$, so that $e^{\frac{1}{2}\theta^2}\cos\theta \leqslant 1$ in the interval $(0, \frac{1}{2}\pi)$. We have therefore

$$\frac{\Pi(n)\Pi(-\frac{1}{2})}{\Pi(n+\frac{1}{2})} < 2\int_0^{\frac{1}{2}\pi} e^{-\frac{1}{2}(2n+1)\theta^2}\,d\theta < 2\int_0^{\infty} e^{-(n+\frac{1}{2})\theta^2} < \left(\frac{\pi}{n+\frac{1}{2}}\right)^{\frac{1}{2}};$$

therefore
$$\frac{\Pi(n)}{\Pi(n+\frac{1}{2})} < \frac{1}{(n+\frac{1}{2})^{\frac{1}{2}}} < \frac{1}{n^{\frac{1}{2}}}.$$

It has now been shown that,

If n be any real number (not necessarily integral) which is greater than or equal to 1, then

$$|\,Q_n(\cos\theta)\,| < \left(\frac{\pi}{n\sin\theta}\right)^{\frac{1}{2}}, \quad |\,P_n(\cos\theta)\,| < \frac{2}{(n\pi\sin\theta)^{\frac{1}{2}}}. \tag{A}$$

2. From the expression given above for $Q_n(\cos\theta + 0\,.\,i)$, we have

$$Q_n(\cos\theta+0\,.\,i) - Q_{n+1}(\cos\theta+0\,.\,i)$$
$$= \frac{e^{-[(n+\frac{1}{2})\theta+\frac{1}{4}\pi]i}}{(2\sin\theta)^{\frac{1}{2}}}\int_0^1 \frac{(1-u)^n\{1-e^{-2i\theta}(1-u)^2\}}{u^{\frac{1}{2}}\{1+u/(z^2-1)\}^{\frac{1}{2}}}\,du.$$

It follows that, when n is real and positive,

$$|\,Q_n(\cos\theta) - Q_{n+2}(\cos\theta)\,| \quad \text{and} \quad \tfrac{1}{2}\pi\,|\,P_n(\cos\theta) - P_{n+2}(\cos\theta)\,|$$

are both less than

$$\frac{1}{(2\sin\theta)^{\frac{1}{2}}}\int_0^1 \frac{(1-u)^n}{u^{\frac{1}{2}}}\left|\frac{1-e^{-2i\theta}(1-u)^2}{\{1+(u/z^2-1)\}^{\frac{1}{2}}}\right| du$$

or than

$$\frac{1}{2^{\frac{1}{2}}}\int_0^1 \frac{(1-u)^n}{u^{\frac{1}{2}}}\,\frac{\{1-2(1-u)^2\cos 2\theta+(1-u)^4\}^{\frac{1}{2}}}{\{\sin^2\theta(1-\frac{1}{2}u)^2+\frac{1}{4}u^2\cos^2\theta\}^{\frac{1}{4}}}\,du. \tag{1}$$

The expression $\sin^2\theta(1-\frac{1}{2}u)^2+\frac{1}{4}u^2\cos^2\theta$ may be written

$$(\tfrac{1}{2}u-\sin^2\theta)^2+\sin^2\theta\cos^2\theta.$$

In case $\theta = 0$ or π, the expression

$$\frac{\{1-2(1-u)^2\cos 2\theta+(1-u)^4\}^{\frac{1}{2}}}{\{(\frac{1}{2}u-\sin^2\theta)^2+\sin^2\theta\cos^2\theta\}^{\frac{1}{4}}} \quad \text{becomes} \quad \frac{1-(1-u)^2}{(\frac{1}{2}u)^{\frac{1}{2}}} \quad \text{or} \quad 2^{\frac{1}{2}}u^{\frac{1}{2}}(2-u),$$

which does not exceed $\frac{8}{3\,.\,3^{\frac{1}{2}}}$, hence the expression (1) does not exceed $\frac{4\,.\,2^{\frac{1}{2}}}{3^{\frac{3}{2}}}\int_0^1 \frac{(1-u)^n}{u^{\frac{1}{2}}}\,du$, and this is less than $\frac{4}{3}\left(\frac{2\pi}{3n}\right)^{\frac{1}{2}}$.

If θ has a fixed value such that $2\sin^2\theta > 1$, the least value of

$$(\tfrac{1}{2}u-\sin^2\theta)^2+\sin^2\theta\cos^2\theta$$

occurs when $u=1$, and this least value is $\frac{1}{4}$; hence, for all values of u in the interval (0, 1) the value of

$$\frac{\{1-2(1-u)^2\cos 2\theta+(1-u)^4\}^{\frac{1}{2}}}{\{(\frac{1}{2}u-\sin^2\theta)^2+\sin^2\theta\cos^2\theta\}^{\frac{1}{4}}}$$

is less than $2^{\frac{1}{2}}\{1-2(1-u)^2\cos 2\theta+(1-u)^4\}^{\frac{1}{2}}$, and this is less than $2^{\frac{1}{2}}[1+(1-u)^2]$ or than $2^{\frac{3}{2}}$. Hence (1) is less than $2\int_0^1 \frac{(1-u)^n}{u^{\frac{1}{2}}}\,du$, or than $2\left(\frac{\pi}{n}\right)^{\frac{1}{2}}$.

When θ has a fixed value such that $2\sin^2\theta \leqslant 1$, or $\cos 2\theta \geqslant 0$, let $u = 2\sin^2\theta+v$, and thus $1-u=\cos 2\theta-v$; the expression

$$\frac{\{1-2(1-u)^2\cos 2\theta+(1-u)^4\}^{\frac{1}{2}}}{\{(\frac{1}{2}u-\sin^2\theta)^2+\sin^2\theta\cos^2\theta\}^{\frac{1}{4}}}$$

is then less than

$$2^{\frac{1}{4}}\left\{\frac{1-2\cos 2\theta(\cos 2\theta-v)^2+(\cos 2\theta-v)^4}{\frac{1}{4}v^2+\sin^2\theta\cos^2\theta}\right\}^{\frac{1}{4}},$$

since $1-2(1-u)^2\cos 2\theta+(1-u)^4 \leqslant 2$, for $\cos 2\theta \geqslant 0$.

The expression $1-2\cos 2\theta(\cos 2\theta-v)^2+(\cos 2\theta-v)^4$ may be written

$$(1-2\cos^3 2\theta+\cos^4 2\theta)+8v\cos^2 2\theta\sin^2\theta$$
$$+v^2(6\cos^2 2\theta-2\cos 2\theta)-4v^3\cos 2\theta+v^4;$$

and the fraction obtained by dividing this expression by $\frac{1}{4}v^2+\sin^2\theta\cos^2\theta$ is less than

$$\frac{1-2\cos^3 2\theta+\cos^4 2\theta}{\sin^2\theta\cos^2\theta}+\frac{8|v|\cos^2 2\theta}{\cos^2\theta}$$
$$+4\cos 2\theta(6\cos 2\theta-2)+16|v|\cos 2\theta+4v^2.$$

or than

$$4\sin^2 2\theta+\frac{12\cos^2 2\theta}{\cos^2\theta}+8\cos 2\theta(3\cos 2\theta-2)+16+4,$$

since $|v| \leqslant 1$, and $\cos 2\theta$ is not negative. Consequently the whole expression is less than 44; thus the expression in (1) is less than $(22)^{\frac{1}{4}}(\pi/n)^{\frac{1}{2}}$.

It has now been shown that,

For all values of θ such that $0 \leqslant \theta \leqslant \pi$, and for all real values of n, not necessarily integral, but greater than or equal to 1,

$$|\, Q_n(\cos\theta) - Q_{n+2}(\cos\theta)\,| < C\left(\frac{\pi}{n}\right)^{\frac{1}{2}}$$

$$|\, P_n(\cos\theta) - P_{n+2}(\cos\theta)\,| < 2C\left(\frac{1}{n\pi}\right)^{\frac{1}{2}}, \qquad \text{(B)}$$

where C is a fixed number.

3. In the expression for $Q_n^m(\mu)$ given in § 1, if we change m into $-m$, we have then, if $R(n-m+1) > 0$, $R(m+\frac{1}{2}) > 0$,

$$Q_n^{-m}(\mu) = \frac{e^{-m\pi i}}{\pi} \cdot 2^{-m} \cos m\pi \cdot \Pi(-m-\tfrac{1}{2})\,\Pi(-\tfrac{1}{2})\,\frac{z^{-m-n}}{(z^2-1)^{\frac{1}{2}-m}}\,(\mu^2-1)^{-\frac{1}{2}m} \times \int_0^1 \frac{(1-u)^{n-m}}{u^{\frac{1}{2}-m}\{1+u/(z^2-1)\}^{\frac{1}{2}-m}}\,du.$$

On taking $\mu = \cos\theta + 0\cdot i$, we have

$$Q_n^{-m}(\cos\theta+0\cdot i) = 2^{-m} e^{-m\pi i}\,\frac{\Pi(-\frac{1}{2})}{\Pi(m-\frac{1}{2})}\,\frac{e^{-i(m+n)\theta-\frac{1}{2}(m\pi i)}}{(e^{2i\theta}-1)^{\frac{1}{2}-m}\sin^m\theta} \times \int_0^1 \frac{(1-u)^{n-m}}{u^{\frac{1}{2}-m}\{1+u/(z^2-1)\}^{\frac{1}{2}-m}}\,du$$

$$= \left(\frac{1}{2\sin\theta}\right)^{\frac{1}{2}} \frac{\Pi(-\frac{1}{2})}{\Pi(m-\frac{1}{2})}\,e^{-[(n+\frac{1}{2})\theta+\frac{1}{4}\pi]i - m\pi i}\,\frac{1}{\sin^m\theta} \times \int_0^1 \frac{(1-u)^{n-m}}{u^{\frac{1}{2}-m}}\,\frac{(\sin\theta-\frac{1}{2}iu\,e^{-i\theta})^m}{\{1-\frac{1}{2}iu(e^{-i\theta}/\sin\theta)\}^{\frac{1}{2}}}\,du.$$

Hence the moduli of the real and imaginary parts of

$$\sin^m\theta \cdot Q_n^{-m}(\cos\theta+0\cdot i)$$

are both less than

$$\frac{1}{(2\sin\theta)^{\frac{1}{2}}}\,\frac{\Pi(-\frac{1}{2})}{\Pi(m-\frac{1}{2})}\int_0^1 \frac{(1-u)^{n-m}}{u^{\frac{1}{2}-m}}\,\frac{|\sin\theta-\frac{1}{2}iu\,e^{-i\theta}|^m}{\{(1-\frac{1}{2}u)^2+\frac{1}{4}u^2\cot^2\theta\}^{\frac{1}{2}}}\,du.$$

Now
$$|\sin\theta-\tfrac{1}{2}iu\,e^{-i\theta}| = \{(1-\tfrac{1}{2}u)^2\sin^2\theta+\tfrac{1}{4}u^2\cos^2\theta\}^{\frac{1}{2}}$$
$$= \{(\tfrac{1}{2}u-\sin^2\theta)^2+\sin^2\theta\cos^2\theta\}^{\frac{1}{2}};$$

and the greatest value of this, for a given value of θ, in case $2\sin^2\theta<1$, occurs when $u=1$, and this value is then $\frac{1}{2}$. When $2\sin^2\theta\geqslant 1$, the greatest value occurs when $u=0$, and the maximum value is then $\sin\theta$. It follows that $|\sin\theta-\frac{1}{2}iu\,e^{-i\theta}|\leqslant 1$, for all values of u and θ.

It has thus been shown that, when $m\geqslant 0$, since

$$|1-\tfrac{1}{2}u-\tfrac{1}{2}iu\cot\theta|\geqslant\tfrac{1}{2},$$

the moduli of the real and imaginary parts of $\sin^m\theta\,Q_n^{-m}(\cos\theta+0\,.\,i)$ are each less than

$$\frac{1}{(\sin\theta)^{\frac{1}{2}}}\frac{\Pi(-\frac{1}{2})}{\Pi(m-\frac{1}{2})}\int_0^1\frac{(1-u)^{n-m}}{u^{\frac{1}{2}-m}}\,du,\quad\text{or than}\quad\frac{\Pi(n-m)}{\Pi(n+\frac{1}{2})}\left(\frac{\pi}{\sin\theta}\right)^{\frac{1}{2}}.$$

It is known [H(30) and (31)] that

$$Q_n^{-m}(\cos\theta+0\,.\,i)=e^{-\frac{3}{2}m\pi i}\{Q_n^{-m}(\cos\theta)-\tfrac{1}{2}\pi i\,P_n^{-m}(\cos\theta)\};$$

hence the real and imaginary parts of $Q_n^{-m}(\cos\theta+0\,.\,i)$ are

$$\cos\tfrac{3}{2}m\pi\,.\,Q_n^{-m}(\cos\theta)-\tfrac{1}{2}\pi\sin\tfrac{3}{2}m\pi\,.\,P_n^{-m}(\cos\theta)$$

and

$$-i\{\sin\tfrac{3}{2}m\pi\,.\,Q_n^{-m}(\cos\theta)+\tfrac{1}{2}\pi\cos\tfrac{3}{2}m\pi\,.\,P_n^{-m}(\cos\theta)\}.$$

Accordingly we see that

$$|\cos\tfrac{3}{2}m\pi\,.\,Q_n^{-m}(\cos\theta)-\tfrac{1}{2}\pi\sin\tfrac{3}{2}m\pi\,.\,P_n^{-m}(\cos\theta)|\sin^m\theta$$

and

$$|\sin\tfrac{3}{2}m\pi\,.\,Q_n^{-m}(\cos\theta)+\tfrac{1}{2}\pi\cos\tfrac{3}{2}m\pi\,.\,P_n^{-m}(\cos\theta)|\sin^m\theta$$

are both less than

$$\frac{\Pi(n-m)}{\Pi(n)}\left(\frac{\pi}{n\sin\theta}\right)^{\frac{1}{2}}.$$

Since $|\cos\tfrac{3}{2}m\pi|+|\sin\tfrac{3}{2}m\pi|\leqslant 2^{\frac{1}{2}}$, and, when m is integral, it is less than or equal to 1, it follows that

$$|Q_n^{-m}(\cos\theta)|\sin^m\theta<\frac{\Pi(n-m)}{\Pi(n)}\left(\frac{2\pi}{n\sin\theta}\right)^{\frac{1}{2}}\quad\text{or}\quad\frac{\Pi(n-m)}{\Pi(n)}\left(\frac{\pi}{n\sin\theta}\right)^{\frac{1}{2}},$$

and

$$|P_n^{-m}(\cos\theta|\sin^m\theta)<\frac{\Pi(n-m)}{\Pi(n)}\left(\frac{8}{n\pi\sin\theta}\right)^{\frac{1}{2}}\quad\text{or}\quad\frac{\Pi(n-m)}{\Pi(n)}\left(\frac{4}{n\pi\sin\theta}\right)^{\frac{1}{2}},$$

according as m is not restricted, or is restricted, to have an integral value. We have assumed that $n\geqslant 1$, $n-m+1>0$, $m\geqslant 0$.

Since $$\frac{e^{-m\pi i}Q_n^m(\cos\theta+0\,.\,i)}{\Pi(n+m)}=\frac{e^{m\pi i}Q_n^{-m}(\cos\theta+0\,.\,i)}{\Pi(n-m)},\quad\text{[see H (12)]},$$

we have

$$Q_n^m(\cos\theta+0\,.\,i)=\frac{1}{(2\sin\theta)^{\frac{1}{2}}}\frac{\Pi(-\frac{1}{2})}{\Pi(m-\frac{1}{2})}\frac{\Pi(n+m)}{\Pi(n-m)}\,e^{-[(n+\frac{1}{2})\theta+\frac{1}{4}\pi]+m\pi i}\frac{1}{\sin^m\theta}$$
$$\times\int_0^1\frac{(1-u)^{n-m}}{u^{\frac{1}{2}-m}}\frac{(\sin\theta-\frac{1}{2}iu\,e^{-i\theta})^m}{\{1-\frac{1}{2}iu\,(e^{-i\theta}/\sin\theta)\}^{\frac{1}{2}}}\,du.$$

Hence the moduli of the real and imaginary parts of $\sin^m\theta\,.\,Q_n^m(\cos\theta+0\,.\,i)$ are both less than

$$\frac{\Pi(n+m)}{\Pi(n+\frac{1}{2})}\left(\frac{\pi}{n\sin\theta}\right).$$

It can now be shown, as before, that

$$|\,Q_n^m(\cos\theta)\,|\sin^m\theta<\frac{\Pi(n+m)}{\Pi(n)}\left(\frac{2\pi}{n\sin\theta}\right)^{\frac{1}{2}}\quad\text{or}\quad\frac{\Pi(n+m)}{\Pi(n+\frac{1}{2})}\left(\frac{\pi}{\sin\theta}\right)^{\frac{1}{2}},$$

$$|\,P_n^m(\cos\theta)\,|\sin^m\theta<\frac{\Pi(n+m)}{\Pi(n)}\left(\frac{8}{n\pi\sin\theta}\right)^{\frac{1}{2}}\quad\text{or}\quad\frac{\Pi(n+m)}{\Pi(n+\frac{1}{2})}\left(\frac{4\pi}{\sin\theta}\right)^{\frac{1}{2}},$$

according as m is not restricted, or is restricted, to be integral. The following theorem has been established :—

If $0<\theta<\pi$, $n\geqslant 1$, $n-m+1>0$, $m\geqslant 0$, *then*

$$|\,Q_n^{\pm m}(\cos\theta)\,|\sin^m\theta<\frac{\Pi(n\pm m)}{\Pi(n)}\left(\frac{2\pi}{n\sin\theta}\right)^{\frac{1}{2}}\quad\text{or}\quad\frac{\Pi(n\pm m)}{\Pi(n+\frac{1}{2})}\left(\frac{4\pi}{\sin\theta}\right)^{\frac{1}{2}},$$

$$|\,P_n^{\pm m}(\cos\theta)\,|\sin^m\theta<\frac{\Pi(n\pm m)}{\Pi(n)}\left(\frac{8}{n\pi\sin\theta}\right)^{\frac{1}{2}}\quad\text{or}\quad\frac{\Pi(n\pm m)}{\Pi(n+\frac{1}{2})}\left(\frac{4\pi}{\sin\theta}\right)^{\frac{1}{2}},$$

according as m *is not restricted to be integral, or is so restricted; where* n *is not restricted to be integral.* (C)

4. We find, from the expression for $Q_n^{-m}(\cos\theta+0\,.\,i)$ in § 3, that

$$\{Q_n^{-m}(\cos\theta+0\,.\,i)-Q_{n+2}^{-m}(\cos\theta+0\,.\,i)\}\sin^m\theta$$

is equivalent to

$$\frac{1}{(2\sin\theta)^{\frac{1}{2}}}\frac{\Pi(-\frac{1}{2})}{\Pi(m-\frac{1}{2})}\,e^{-[(n+\frac{1}{2})\theta+\frac{1}{4}\pi]i-m\pi i}$$
$$\times\int_0^1\frac{(1-u)^{n+m}}{u^{\frac{1}{2}-m}}(\sin\theta-\tfrac{1}{2}iu\,e^{-i\theta})^m\,\frac{1-(1-u)^2\,e^{-2i\theta}}{\{1-\frac{1}{2}iu\,(e^{-i\theta}/\sin\theta)\}^{\frac{1}{2}}}\,du.$$

Referring to § 2, we see that the moduli of the real and imaginary parts of $\sin^m\theta\{Q_n^{-m}(\cos\theta+0\,.\,i)-Q_{n+2}^{-m}(\cos\theta+0\,.\,i)\}$ are both less than

$$\frac{1}{2^{\frac{1}{2}}}\int_0^1\frac{(1-u)^{n+m}}{u^{\frac{1}{2}}}\,\frac{\{1-2(1-u)^2\cos 2\theta+(1-u)^4\}^{\frac{1}{2}}}{\{\sin^2\theta(1-\frac{1}{2}u)^2+\frac{1}{4}u^2\cos^2\theta\}^{\frac{1}{2}}}\,du$$

or than $\left(\frac{\pi}{n}\right)^{\frac{1}{2}} \frac{\Pi(n-m)}{\Pi(n+\frac{1}{2})} C$, where C is a fixed number. It now follows, as in § 3, that

$$\sin^m\theta \left| Q_n^{-m}(\cos\theta) - Q_{n+2}^{-m}(\cos\theta) \right| < \frac{\Pi(n-m)}{\Pi(n)} \left(\frac{2\pi}{n}\right)^{\frac{1}{2}} C,$$

$$\sin^m\theta \left| P_n^{-m}(\cos\theta) - P_{n+2}^{-m}(\cos\theta) \right| < \frac{\Pi(n-m)}{\Pi(n)} \left(\frac{8}{n\pi}\right)^{\frac{1}{2}} C.$$

Proceeding, as in § 3, we now obtain the following theorem :—

If $n \geqslant 1$, $n-m+1 > 0$, $m \geqslant 0$, *we have, when* $0 \leqslant \theta \leqslant \pi$,

$$\left| Q_n^{\pm m}(\cos\theta) - Q_{n+2}^{\pm m}(\cos\theta) \right| \sin^m\theta < \frac{\Pi(n \pm m)}{\Pi(n)} C \left(\frac{2\pi}{n}\right)^{\frac{1}{2}},$$

$$\left| P_n^{\pm m}(\cos\theta) - P_{n+2}^{\pm m}(\cos\theta) \right| \sin^m\theta < \frac{\Pi(n \pm m)}{\Pi(n)} C \left(\frac{8}{n\pi}\right)^{\frac{1}{2}},$$

where n *and* m *are not restricted to be integral.* (D)

ON THE EXISTENCE AND UNIQUENESS OF THE SOLUTION OF CAUCHY'S PROBLEM FOR A SYSTEM OF TWO FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

By MARY TAYLOR.

[Received 28 November, 1928.—Read 13 December, 1928.]

Introduction.

The present paper gives the solution of Cauchy's problem for a system of two first order partial differential equations

$$\begin{cases} F(x, y, u, v, p, q, m, n) = 0, \\ G(x, y, u, v, p, q, m, n) = 0, \end{cases}$$

in which x, y are the independent variables, u, v the functions to be determined, and $p = u_x$, $q = u_y$, $m = v_x$, $n = v_y$. The "initial" values of u, v, p, q, m, n are given along a curve in the (x, y)-plane, which is nowhere characteristic, and are such that they satisfy the equations $F = 0$, $G = 0$, and the "strip" relations of the type

$$du = p\,dx + q\,dy,$$

$$dv = m\,dx + n\,dy,$$

under the hypothesis that, for these initial values, the system is of hyperbolic character. It is important to remark that *these initial values are not assumed to be analytic*. The definition of "hyperbolic character" and of the characteristic directions is given in § 1.

The discussion given in this paper is based on the theory of a so-called canonical hyperbolic system, *i.e.* on the following theorem* :—

If, on a segment $\alpha \leqslant \alpha_0$, $\beta \leqslant \beta_0$, $\alpha_0 + \beta_0 > 0$ *of the straight line* $\alpha + \beta = 0$ *in an* (α, β)*-plane, the functions* $\phi_1, \phi_2, \ldots, \phi_n$ *are given as sufficiently often (but not infinitely often—in general twice is sufficient)*

* H. Lewy, *Math. Annalen*, 98 (1928), 179–191.

continuously differentiable along the line, they can then be continued in a certain neighbourhood of the segment in such a way that the n partial differential equations

$$\sum_{k=1}^{n} a_{ik}\phi'_k = b_i \quad (i = 1, 2, 3, \ldots, m < n),$$

$$\sum_{k=1}^{n} a_{ik}\phi'_k = b_i \quad (i = m+1, \ldots, n)$$

(where the ' and ` indicate partial differentiations with respect to α and β respectively) are satisfied, under the assumption that the determinant $|a_{ik}|$ is not zero for the initial values and that the a_{ik}'s and b_i's are sufficiently often continuously differentiable with respect to their arguments $\phi_1, \ldots, \phi_n$. There exists only one system of such functions, which satisfies these differential equations and assumes the given initial values.

By application of a method similar to that used by K. Friedrichs and H. Lewy* we can show that the problem of the solution of our system $F = 0$, $G = 0$ can be reduced to that of the solution of a system of characteristic equations of the type referred to in the above theorem.

In our characteristic equations appear the differentials of all the quantities x, y, u, v, p, q, m, n in the directions of the characteristics. There are, however; more equations than there are unknown functions x, y, u, v, p, q, m, n when the "strip" relations, which hold identically for every solution, are included. From this system of characteristic equations it is possible to choose out a "canonical hyperbolic system" with non-vanishing determinant†.

In order to replace the given initial value problem by the problem of the canonical hyperbolic system, the initial curve and initial values (*i.e.* x, y, u, v, p, q, m, n) are considered as functions of a parameter t along the straight line $\alpha+\beta = 0$ in an (α, β)-plane, and the differentials in the directions of the two characteristics are replaced by the derivatives with respect to α or β. The quantities $x, y, \ldots, n$ are considered as unknown functions of these two parameters and have to be determined. It must then finally (§ 3) be proved that the solution of the canonical system, whose existence and uniqueness is ensured by the theorem quoted above, is really a solution of the original system of differential equations $F = 0$, $G = 0$.

* *Math. Annalen*, 99 (1928), 200–221.

† See p. 254.

Since the solution of the canonical system depends only on the given values of the functions on the segment $\alpha \leqq \alpha_0$, $\beta \leqq \beta_0$, $\alpha_0+\beta_0 > 0$ of the line $\alpha+\beta = 0$, and is independent of the values outside this segment, the solution of the original system of differential equations $F = 0$, $G = 0$ at the point (x_0, y_0) corresponding to (α_0, β_0) is completely determined by the initial values of u, v, p, q, m, n on the portion of the initial curve in the (x, y)-plane which corresponds to the above segment of $\alpha+\beta = 0$ in the (α, β)-plane, and which is cut off by the characteristic curves which pass through (x_0, y_0).

Though many of the calculations run parallel to those which must be made in the treatment of a single second order equation, it yet appears that the relations in the case of two first order equations are rather simpler in principle than those in the case of one second order equation. In practice it very frequently occurs that a problem is formulated originally by two first order equations, from which in many cases one second order equation is derived by elimination, and it seems important to show that the qualitative relations are quite independent of the possibility of this elimination, and, further, that all conceptions arising from the considerations of equations of second order occur in still simpler form when systems of first order differential equations are considered. It is the purpose of this paper to consider the simplest case of such a system.

An important application of the foregoing considerations can be made to the case of a system of elliptic character. For, as is indicated by H. Lewy*, the proof of the well known theorem concerning the analytic character of the solution of elliptic differential equations can be reduced to the proof of the continuation of the given initial solutions for complex values of x, y. This is achieved by using the corresponding canonical hyperbolic system, and the theory of the characteristic equations is just what is required; they then have, of course, exactly the same form as in the hyperbolic case, and therefore we are justified in saying also in the case of an elliptic system $F = 0$, $G = 0$ that, under the hypothesis that F and G are analytic functions of their arguments, the solutions u and v are analytic functions of x and y.

1. We therefore proceed to consider the equations

$$(1) \qquad \begin{cases} F(x, y, u, v, p, q, m, n) = 0, \\ G(x, y, u, v, p, q, m, n) = 0; \end{cases}$$

in this paragraph we deduce first the characteristic principal equation,

* *Math. Annalen*, 101 (1929), 609-619.

and then the whole system of characteristic equations which the solution $u(x, y)$, $v(x, y)$ of equations (1) must satisfy.

For this purpose we form the total differential coefficients of F and G with respect to x and y; writing for the sake of brevity

$$[F]_x = F_x + F_u.p + F_v.m,$$
$$[G]_x = G_x + G_u.p + G_v.m,$$
$$[F]_y = F_y + F_u.q + F_v.n,$$
$$[G]_y = G_y + G_u.q + G_v.n,$$

we obtain the equations

$$(2)\quad \begin{cases} (a) & [F]_x + p_x F_p + q_x F_q & + m_x F_m + n_x F_n & = 0, \\ (b) & [G]_x + p_x G_p + q_x G_q & + m_x G_m + n_x G_n & = 0, \\ (c) & [F]_y + p_y F_p + q_y F_q & + m_y F_m + n_y F_n & = 0, \\ (d) & [G]_y + p_y G_p + q_y G_q & + m_y G_m + n_y G_n & = 0. \end{cases}$$

In addition we have the "strip" relations

$$(3)\quad \begin{cases} (a) & -dp + p_x dx + p_y dy & & = 0, \\ (b) & -dq + q_x dx + q_y dy & & = 0, \\ (c) & -dm & + m_x dx + m_y dy & = 0, \\ (d) & -dn & + n_x dx + n_y dy & = 0. \end{cases}$$

To form the characteristic principal equation we choose equations (2) (a, b) and (3) (a, c) and attempt to determine special directions dy/dx by equating to zero their determinant in p_x, q_x, m_x, n_x. This gives the characteristic principal equation

$$(4)\quad \begin{vmatrix} F_p & F_q & F_m & F_n \\ G_p & G_q & G_m & G_n \\ dx & dy & 0 & 0 \\ 0 & 0 & dx & dy \end{vmatrix} = 0.$$

Dividing this determinant by dx^2 and writing $dy/dx = \rho$, we obtain for the characteristic principal equation the form

$$(5)\quad \rho^2 \Delta_{pm} - \rho(\Delta_{pn} + \Delta_{qm}) + \Delta_{qn} = 0,$$

where
$$\Delta_{pm} = F_p G_m - F_m G_p, \quad \Delta_{pn} = F_p G_n - F_n G_p,$$
$$\Delta_{qm} = F_q G_m - F_m G_q, \quad \Delta_{qn} = F_q G_n - F_n G_q.$$

Equation (5) gives for ρ two real, unequal values ρ_1 and ρ_2, not zero or

infinity, provided that

$$(\Delta_{pn}+\Delta_{qm})^2-4\Delta_{pm}\Delta_{qn}>0 \tag{6}$$

and

$$\Delta_{pm}\neq 0, \quad \Delta_{qn}\neq 0. \tag{6'}$$

The inequality (6) states that the system of differential equations is *hyperbolic*. The inequalities (6'), as well as others which arise later in the investigation, are not essential, as will be shown in § 2.

The determinant in p_x, $q_x(=p_y)$, q_y, m_x, $n_x(=m_y)$ and n_y of the six equations (2) and (3) (a, b) is

$$\begin{vmatrix} F_p & F_q & 0 & F_m & F_n & 0 \\ G_p & G_q & 0 & G_m & G_n & 0 \\ 0 & F_p & F_q & 0 & F_n & F_n \\ 0 & G_p & G_q & 0 & G_m & G_n \\ dx & dy & 0 & 0 & 0 & 0 \\ 0 & dx & dy & 0 & 0 & 0 \end{vmatrix} = \Delta_{mn}\{\rho^2\Delta_{pn}-\rho(\Delta_{pn}+\Delta_{qm})+\Delta_{qn}\}, \tag{7}$$

where $\Delta_{mn}=F_mG_n-F_nG_m$, and it is clear that this determinant (7) always vanishes for such directions dy/dx as satisfy (4). From the hypothesis that the system of equations (2) and (3) possesses a solution, we see that the vanishing of (4) implies that all determinants with six columns of the matrix

$$\begin{Vmatrix} [F]_x & F_p & F_q & 0 & F_m & F_n & 0 \\ [G]_x & G_p & G_q & 0 & G_m & G_n & 0 \\ [F]_y & 0 & F_p & F_q & 0 & F_m & F_n \\ [G]_y & 0 & G_p & G_q & 0 & G_m & G_n \\ -dp & dx & dy & 0 & 0 & 0 & 0 \\ -dq & 0 & dx & dy & 0 & 0 & 0 \end{Vmatrix} \tag{8}$$

also vanish. From these six conditions we choose

$$\begin{vmatrix} [F]_x & F_p & 0 & F_m & F_n & 0 \\ [G]_x & G_p & 0 & G_m & G_n & 0 \\ [F]_y & 0 & F_q & 0 & F_m & F_n \\ [G]_y & 0 & G_q & 0 & G_m & G_n \\ -dp & dx & 0 & 0 & 0 & 0 \\ -dq & 0 & dy & 0 & & 0 \end{vmatrix} = 0. \tag{9}$$

This equation represents two conditions, corresponding to the two values of $\rho = dy/dx$; omitting the factor Δ_{mn}, which we at present assume not to vanish (though this assumption will, in § 2, be shown to be unimportant), they are seen to be

$$(10) \qquad \begin{cases} p'\rho_1\Delta_{pm}+q'\Delta_{qn}-y'(\Delta_{mx}+\Delta_{ny}) = 0, \\ p`\rho_2\Delta_{pm}+q`\Delta_{qn}-y`(\Delta_{mx}+\Delta_{ny}) = 0, \end{cases}$$

where

$$\Delta_{mx} \equiv F_m[G]_x - [F]_x G_m,$$
$$\Delta_{ny} \equiv F_n[G]_y - [F]_y G_n;$$

$'$ and $`$ denote differentiation in the first and second characteristic directions through (x, y) respectively.

Choosing the last two strip relations (3) (c, d) instead of (3) (a, b), we obtain, for their determinant, the form

$$\Delta_{pq}\{\rho^2\Delta_{pm}-\rho(\Delta_{pn}+\Delta_{qm})+\Delta_{qn}\},$$

which again always vanishes. The equations corresponding to (9) and (10) are in this case

$$(11) \qquad \begin{vmatrix} F_p & F_q & 0 & [F]_x & F_m & 0 \\ G_p & G_q & 0 & [G]_x & G_m & 0 \\ 0 & F_p & F_q & [F]_y & 0 & F_n \\ 0 & G_p & G_q & [G]_y & 0 & G_n \\ 0 & 0 & 0 & -dm & dx & 0 \\ 0 & 0 & 0 & -dn & 0 & dy \end{vmatrix} = 0.$$

Omitting the factor Δ_{pq}, which plays a part analogous to that of Δ_{mn} above, and which we assume not to vanish, we obtain

$$(12) \qquad \begin{cases} m'\rho_1\Delta_{pm}+n'\Delta_{qn}+y'(\Delta_{px}+\Delta_{qy}) = 0, \\ m`\rho_2\Delta_{pm}+n`\Delta_{qn}+y`(\Delta_{px}+\Delta_{qy}) = 0, \end{cases}$$

where

$$\Delta_{px} \equiv F_p[G]_x - [F]_x G_p,$$
$$\Delta_{qy} \equiv F_q[G]_y - [F]_y G_q.$$

(10) and (12) are further characteristic equations, and our solution $u(x, y)$, $v(x, y)$ must satisfy all the equations which can be derived in this way from the matrix (8) and the corresponding one obtained from equations (2) and (3) (c, d). These equations, together with the equations $y' = \rho_1 x'$ and $y` = \rho_2 x`$ which define ρ_1 and ρ_2, and the "strip" relations

$$u' - p x' - q y' = 0,$$
$$v' - m x' - n y' = 0,$$

form a system of differential equations linear in the derivatives of the functions to be determined. The system consists of many more equations than there are functions to be determined, so that, in order to replace the problem of the two equations $F=0$, $G=0$ by that of the system of characteristic equations, it is necessary to choose out just as many of the equations as there are unknown functions x, y, u, v, p, q, m, n. We choose as most convenient for future work equations (10) and (12), together with the equations defining ρ_1 and ρ_2 and two "strip" relations, so that the canonical system selected has the form

$$(13)\quad \begin{cases} (\alpha) & y'-\rho_1 x' = 0, \\ (\beta) & y`-\rho_2 x` = 0, \\ (\gamma) & p'\rho_1\Delta_{pm}+q'\Delta_{qn}-y'(\Delta_{mx}+\Delta_{ny}) = 0, \\ (\delta) & p`\rho_2\Delta_{pm}+q`\Delta_{qn}-y`(\Delta_{mx}+\Delta_{ny}) = 0, \\ (\epsilon) & m'\rho_1\Delta_{pm}+n'\Delta_{qn}+y'(\Delta_{px}+\Delta_{qy}) = 0, \\ (\zeta) & m`\rho_2\Delta_{pm}+n`\Delta_{qn}+y`(\Delta_{px}+\Delta_{qy}) = 0, \\ (\eta) & u'-px'-qy' = 0, \\ (\vartheta) & v'-mx'-ny' = 0. \end{cases}$$

The initial values of the functions given along the initial curve in the (x, y)-plane are transformed in the (α, β)-plane into the initial values of the corresponding functions on the initial line $\alpha+\beta=0$, as described in the Introduction.

The solution of the initial value problem for this canonical hyperbolic system exists uniquely if the determinant of equations (13) does not vanish for the initial values; as determinant is here understood the determinant of the system found by replacing x' and $x`$ by *one* unknown quantity $x_1$, $y'$ and $y`$ by a second unknown x_2, etc., and finally n' and $n`$ by an unknown x_8. To show that the determinant of the resulting homogeneous linear system of equations does not vanish, it is only necessary to show that it follows that

$$x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = x_7 = x_8 = 0.$$

We have already $\rho_1 \neq \rho_2$ and $\rho_1 \neq 0$, $\rho_2 \neq 0$, so that, from (α) and (β), it is clear that $x_1 = x_2 = 0$. (η) and (ϑ) then give directly $x_3 = x_4 = 0$. (γ) and (δ) give $x_5 = x_6 = 0$, and (ϵ) and (ζ) give $x_7 = x_8 = 0$, provided that $\Delta_{pm} \neq 0$, $\Delta_{qn} \neq 0$. Thus it is clear that the determinant of (13) does not vanish along the initial line $\alpha+\beta=0$.

We may therefore now conclude that there exists one and only one solution x, y, u, v, p, q, m, n of (13), which assumes the given initial values on the straight line $\alpha+\beta=0$.

2. In the previous paragraph it has been assumed, and in the next paragraph also it will be assumed, that certain determinants and other expressions formed from the derivatives of F and G with respect to p, q, m, n do not vanish. We now show that, by linear transformation of the independent variables x and y, by linear transformation of the functions u and v, and by linear transformations of the expressions F and G themselves, a new equivalent system $F_1 = 0$, $G_1 = 0$ in new arguments $x_1, y_1, u_1, v_1, p_1, q_1, m_1, n_1$ can be constructed from the given system (1) of differential equations, so that the assumed inequalities

$$(14) \quad \Delta_{pm} \neq 0, \quad \Delta_{qn} \neq 0, \quad \Delta_{mn} \neq 0, \quad \Delta_{pq} \neq 0, \quad \text{and} \quad F_q F_m - F_p F_n \neq 0$$

hold for the new system, which remains hyperbolic. By these transformations one solution of $F_1 = 0$, $G_1 = 0$ is obtained from every solution of the original system $F = 0$, $G = 0$, and, conversely, one solution of $F = 0$, $G = 0$ from every solution of $F_1 = 0$, $G_1 = 0$ by means of the reverse transformations.

We now assume, for the sake of simplicity, that, at a point (x, y), say the origin of coordinates, the functions u and v have a definite value, for example, zero; this assumption imposes no restriction on the generality of the reasoning. If, by means of the transformations which we proceed to determine, we can so arrange that, at this point alone, the expressions under consideration do not vanish, it then follows from considerations of continuity that the same holds for a certain neighbourhood of this point.

We first proceed to select the determinants and other combinations of the derivatives of F and G which remain unaltered or are multiplied by a non-vanishing factor when they undergo one or more of the above-mentioned transformations.

(1) The determinants to be considered are Δ_{pm}, Δ_{qn}, Δ_{mn}, and Δ_{pq}. It is evident that these all remain unaltered on replacing the equation $F = 0$ by the equation $\Phi = 0$, where $\Phi \equiv F + \lambda G$, a linear combination of F and G, λ being a constant to be determined later by other requirements. The characteristic principal equation and the hyperbolic character of the system (cf. page 252) therefore also remain unaltered and are invariants for this transformation. All the expressions (14) are unaltered except $F_q F_m - F_p F_n$.

(2) By a linear transformation of the unknown functions u and v such that they are replaced by functions $\bar{u}$ and $\bar{v}$, where

$$(15) \qquad u = a\bar{u} + b\bar{v}, \quad v = c\bar{u} + d\bar{v}, \quad ad - bc \neq 0,$$

the determinants Δ_{pm} and Δ_{qn} for the original system take the forms

$$\frac{1}{(ad-bc)}\bar{\Delta}_{\bar{p}\bar{m}}, \quad \frac{1}{(ad-bc)}\bar{\Delta}_{\bar{q}\bar{n}},$$

where $\bar{\Delta}_{\bar{p}\bar{m}}$, $\bar{\Delta}_{\bar{q}\bar{n}}$ are the corresponding expressions in the new system. The expression $\Delta_{pn}+\Delta_{qm}$ becomes

$$\frac{1}{(ad-bc)}(\bar{\Delta}_{\bar{p}\bar{n}}+\bar{\Delta}_{\bar{q}\bar{m}}),$$

so that the characteristic principal equation and the hyperbolic character of the system are also invariant for this transformation. The expression $F_qF_m-F_pF_n$ becomes

$$\frac{1}{(ad-bc)}(\bar{F}_{\bar{q}}\bar{F}_{\bar{m}}-\bar{F}_{\bar{p}}\bar{F}_{\bar{n}})$$

and does not vanish in the new system if it does not vanish in the original one. The effect of this transformation on Δ_{mn}, Δ_{pq} is discussed below.

(3) On replacement of the independent variables x, y by the variables ξ, η by means of the relations

$$x = x(\xi, \eta), \quad y = y(\xi, \eta), \tag{16}$$

where $\partial(x, y)/\partial(\xi, \eta) \neq 0$ (this transformation includes of course as a special case the linear transformation $x = A\xi+B\eta$, $y = C\xi+D\eta$, where $AD-BC \neq 0$), we find

$$\Delta_{mn} = \Delta^*_{m^*n^*}\frac{\partial(x, y)}{\partial(\xi, \eta)},$$

$$\Delta_{pq} = \Delta^*_{p^*q^*}\frac{\partial(x, y)}{\partial(\xi, \eta)},$$

$$F_qF_m-F_pF_n = (F^*_{q^*}F^*_{m^*}-F^*_{p^*}F^*_{n^*})\frac{\partial(x, y)}{\partial(\xi, \eta)},$$

Δ^*, etc., denoting the value of Δ, etc., in the new system of coordinates, so that Δ_{mn}, Δ_{pq}, and $F_qF_m-F_pF_n$ do not vanish in the new system if they do not vanish in the original one and conversely. Since

$$\begin{vmatrix} F_p & F_q & F_m & F_n \\ G_p & G_q & G_m & G_n \\ dx & dy & 0 & 0 \\ 0 & 0 & dx & dy \end{vmatrix} \equiv \begin{vmatrix} F^*_{p^*} & F^*_{q^*} & F^*_{m^*} & F^*_{n^*} \\ G^*_{p^*} & G^*_{q^*} & G^*_{m^*} & G^*_{n^*} \\ d\xi & d\eta & 0 & 0 \\ 0 & 0 & d\xi & d\eta \end{vmatrix} \cdot \begin{vmatrix} x_\xi & x_\eta & 0 & 0 \\ y_\xi & y_\eta & 0 & 0 \\ 0 & 0 & x_\xi & x_\eta \\ 0 & 0 & y_\xi & y_\eta \end{vmatrix},$$

the characteristic principal equation assumes the form

$$\{\rho^{*2}\,\Delta^*_{p^*m^*}-\rho^*(\Delta^*_{p^*n^*}+\Delta^*_{q^*m^*})+\Delta^*_{q^*n^*}\}\left(\frac{\partial(x,\,y)}{\partial(\xi,\,\eta)}\right)^2=0,$$

and is therefore invariant for all transformations considered; these also conserve the hyperbolic character of the system.

To discuss the transformations which will enable us to assume that the inequalities (14) hold for our system of equations $F=0$, $G=0$, we first note that the terms of the left-hand side of the inequality

$$(6a)\qquad (\Delta_{pn}+\Delta_{qm})^2-4\Delta_{pm}\Delta_{qn}>0$$

can be so rearranged that (6a) may be written in either of the forms

$$(6b)\qquad (\Delta_{pn}-\Delta_{qm})^2-4\Delta_{mn}\Delta_{pq}>0$$

or

$$(6c)\quad (F_qG_m+F_mG_q-F_pG_n-F_nG_p)^2-4(F_qF_m-F_pF_n)(G_qG_m-G_pG_n)>0.$$

The inequalities (6a), (6b), (6c) are thus equivalent forms of the relation required between the first partial derivatives of F and G with regard to p, q, m, n in order that the system may be one of hyperbolic character.

Consider now the transformation (16). It is clear that new coordinates ξ, η can always be found so that

$$\Delta^*_{p^*m^*}=\xi_x^2\,\Delta_{pm}+\xi_x\,\xi_y(\Delta_{pn}+\Delta_{qm})+\xi_y^2\,\Delta_{qn}\neq 0,$$

$$\Delta^*_{q^*n^*}=\eta_x^2\,\Delta_{pm}+\eta_x\,\eta_y(\Delta_{pn}+\Delta_{qm})+\eta_y^2\,\Delta_{qn}\neq 0,$$

since inequality (6a) shows that at least one of the coefficients Δ_{pm}, $\Delta_{pn}+\Delta_{qm}$, Δ_{qn} must be different from zero. The assumptions $\Delta_{pm}\neq 0$, $\Delta_{qn}\neq 0$ can therefore always be made to hold good after transformation of the independent variables. They are left unaltered by the other transformations except for a non-vanishing factor, as is shown in (1) and (2), so that we may assume that the equations have been transformed so that $\Delta_{pm}\neq 0$, $\Delta_{qn}\neq 0$.

To justify the statement that we can also assume that Δ_{mn}, Δ_{pq} can be made non-vanishing we must subject the unknown functions u and v to a linear transformation, as in (15). This gives

$$\begin{aligned}\bar\Delta_{\bar m\bar n}&=\bar F_{\bar m}\bar G_{\bar n}-\bar F_{\bar n}\bar G_{\bar m}\\&=b^2\Delta_{pq}+bd(\Delta_{pn}-\Delta_{qm})+d^2\Delta_{mn},\\ \bar\Delta_{\bar p\bar q}&=\bar F_{\bar p}\bar G_{\bar q}-\bar F_{\bar q}\bar G_{\bar p}\\&=a^2\Delta_{pq}+ac(\Delta_{pn}-\Delta_{qm})+c^2\Delta_{mn},\end{aligned}$$

and, since at least one of Δ_{mn}, $\Delta_{pn}-\Delta_{qm}$, Δ_{pq} does not vanish by inequality

(6b), we can always transform u and v so that $\bar{\Delta}_{\bar{m}\bar{n}} \neq 0$, $\bar{\Delta}_{\bar{p}\bar{q}} \neq 0$. From (1) and (2) we see that the other transformations leave Δ_{mn}, Δ_{pq} unaltered except for a non-vanishing factor.

We finally prove that it is always possible to replace F by Φ as above, so that $\Phi_q\Phi_m - \Phi_p\Phi_n \neq 0$. For

$$\begin{aligned}\Phi_q\Phi_m - \Phi_p\Phi_n &= (F_q+\lambda G_q)(F_m+\lambda G_m) - (F_p+\lambda G_p)(F_n+\lambda G_n)\\ &= \lambda^2(G_qG_m - G_pG_n) + \lambda(F_qG_m + F_mG_q - F_pG_n - F_nG_p)\\ &\qquad + F_qF_m - F_pF_n.\end{aligned}$$

If no value of λ can be found so that $\Phi_q\Phi_m - \Phi_p\Phi_n \neq 0$, the coefficients of the powers of λ in this expression must vanish separately, that is

$$F_qF_m - F_pF_n = 0,$$

$$G_qG_m - G_pG_n = 0,$$

$$F_qG_m + F_mG_q - F_pG_n - F_nG_p = 0.$$

This is, however, impossible on account of inequality (6c). Thus it is always possible to determine λ so that

$$\Phi_q\Phi_m - \Phi_p\Phi_n \neq 0,$$

and this function Φ replaces F in the system of equations. It has already been shown that the effect of the other transformations is only to multiply $F_qF_m - F_pF_n$ by a non-vanishing factor, so that we may assume that F and G already have a form such that

$$F_qF_m - F_pF_n \neq 0.$$

From the remarks made at the beginning of this paragraph it is now clear that it may be assumed that the original form of the differential equations is such that the inequalities (14) hold good.

3. In § 1 we deduced that, if the system of differential equations $F = 0$, $G = 0$ were satisfied by functions $u(x, y)$, $v(x, y)$, the same must hold for a system of characteristic differential equations. From the various possible equations which might compose the system (which consists of more differential equations than there are functions to be determined) we chose out system (13), in which we regarded the quantities x, y, u, v, p, q, m, n as functions of the two parameters α and β, no other conditions having been imposed on x, y, ..., n than that they should fulfil these equations and assume the given initial values. We now show that functions x, y, ..., n of the variables α, β, which satisfy these equations and assume the prescribed initial values on the straight line

$\alpha+\beta=0$, represent a solution of the differential equations $F=0$, $G=0$. To do this it must be proved :

(1) that x, y may be introduced as independent variables instead of α and β;

(3) that, when the equations $F=0$, $G=0$ are satisfied for the initial values, they are also satisfied for a neighbourhood of these values. For this it is naturally necessary to show

(2) that the quantities p, q, m, n really have the meaning given to them in equation (1), that is, that they represent the derivatives of u and v with respect to x and y.

(1) x and y may be introduced as independent variables in the place of α and β if $\partial(x, y)/\partial(\alpha, \beta) \neq 0$ on the initial curve and therefore in a neighbourhood of it. This is the case, since, were

$$\frac{\partial(x, y)}{\partial(\alpha, \beta)} \equiv x'y`-x`y' = x'x`(\rho_2-\rho_1) = 0,$$

we should have $x'=0$, say, for $\rho_2-\rho_1 \neq 0$. This involves $y'=0$, since $y'-\rho_1 x'=0$. Along the initial line $\alpha+\beta=0$ we should then have*, differentiating along the line with respect to the parameter t,

$$\frac{dx}{dt} = (x'-x`)\frac{d\alpha}{dt} = -x`\frac{d\alpha}{dt},$$

$$\frac{dy}{dt} = (y'-y`)\frac{d\alpha}{dt} = -y`\frac{d\alpha}{dt}.$$

It would then follow that

$$\frac{dy}{dt} - \rho_2 \frac{dx}{dt} = 0$$

along the initial line. This is contrary to our hypothesis that the initial curve is nowhere characteristic, and therefore $\partial(x, y)/\partial(\alpha, \beta)$ must not vanish along the initial curve and in a neighbourhood of it.

(2) We have, for the given initial values,

$$\frac{du}{dt} = p\frac{dx}{dt} + q\frac{dy}{dt} = u_x\frac{dx}{dt} + u_y\frac{dy}{dt},$$

where d/dt denotes differentiation with respect to t in the direction of the initial line, and also the given strip relations on the first of the characteristic curves, which we know is not coincident with the initial curve,

$$u' = px' + qy',$$

$$v' = mx' + ny'.$$

* See p. 249.

Therefore the strip relations on the other characteristic curve

$$u` = px` + qy`,$$

$$v` = mx` + ny`,$$

are also valid for the given initial values; we wish to show, in addition, that they hold in a neighbourhood of these initial values.

Let
$$U^\alpha \equiv u' - px' - qy' = 0,$$

$$V^\alpha \equiv v' - mx' - ny' = 0,$$

(17)
$$\begin{cases} U^\beta \equiv u` - px` - qy`, \\ V^\beta \equiv v` - mx` - ny`. \end{cases}$$

Then
$$\frac{\partial U^\beta}{\partial \alpha} = \frac{\partial U^\beta}{\partial u} - \frac{\partial U^\alpha}{\partial \beta}$$

$$= u`' - p'x` - px`' - q'y` - qy`' - u`' + p`x' + px`' + q`y' + qy`'$$

$$= p`x' - p'x` + q`y' - q'y`.$$

Similarly
$$\frac{\partial V^\beta}{\partial \alpha} = m`x' - m'x` + n`y' - n'y`.$$

From equations (13) (γ, δ) we now find

$$0 = (\delta)\,\rho_1 x' - (\gamma)\,\rho_2 x` = \rho_1 x' [p`\rho_2 \Delta_{pm} + q`\Delta_{qn} - y`(\Delta_{mx} + \Delta_{ny})]$$

$$- \rho_2 x` [p'\rho_1 \Delta_{pm} + q'\Delta_{qn} - y'(\Delta_{mx} + \Delta_{ny})]$$

$$= \rho_1 \rho_2 \Delta_{pm}(p`x' - p'x`) + \Delta_{qn}(q`y' - q'y`)$$

$$= \Delta_{qn}(p`x' - p'x` + q`y' - q'y`)$$

$$= \Delta_{qn} \frac{\partial U^\beta}{\partial \alpha},$$

since $\rho_1 \rho_2 = \Delta_{qn}/\Delta_{pm}$ from the characteristic principal equation.

Since, by hypothesis, $\Delta_{qn} \neq 0$, we have at once $\partial U^\beta / \partial \alpha = 0$. Similarly from

$$0 = (13)(\xi).\rho_1 x' - (13)(\epsilon).\rho_2 x`$$

we find that
$$\frac{\partial V^\beta}{\partial \alpha} = 0.$$

Now the initial values of U^β and V^β are zero, so that the strip relations

$$U^\beta \equiv 0, \quad V^\beta \equiv 0$$

hold also in a neighbourhood of the initial values.

From
$$u`-px`-qy` = 0,$$
$$u'-px'-qy' = 0,$$
it follows at once that
$$(u_x-p)\,x`+(u_y-q)\,y` = 0,$$
$$(u_x-p)\,x'+(u_y-q)\,y' = 0,$$
and, since $x`y'-x'y` \neq 0$, we have
$$p = u_x, \quad q = u_y,$$
and similarly $m = v_x$, $n = v_y$, from the strip relations for v.

(3) It remains to show that the solution u, v, p, q, m, n of (13) is also a solution of (1). In order to prove that $F \equiv 0$, $G \equiv 0$, we proceed, in the first place, to show that F and G satisfy a canonical system of homogeneous hyperbolic equations, since, by hypothesis, $F = 0$, $G = 0$ for the initial values. We have
$$F' \equiv F_x x'+F_y y'+F_u u'+F_v v'+F_p p'+F_q q'+F_m m'+F_n n',$$
$$G' \equiv G_x x'+G_y y'+G_u u'+G_v v'+G_p p'+G_q q'+G_m m'+G_n n'.$$
The strip relations proved above give*
$$F' = [F]_x x'+[F]_y y'+F_p p'+F_q q'+F_m m'+F_n n',$$
$$G' = [G]_x x'+[G]_y y'+G_p p'+G_q q'+G_m m'+G_n n'.$$
Similarly
$$F` = [F]_x x`+[F]_y y`+F_p p`+F_q q`+F_m m`+F_n n`,$$
$$G` = [G]_x x`+[G]_y y`+G_p p`+G_q q`+G_m m`+G_n n`.$$

A canonical system of homogeneous hyperbolic equations for F and G will have the form
$$(18) \qquad \begin{cases} (a) & AF'+BG' = 0, \\ (b) & CF`+DG` = 0. \end{cases}$$
To obtain such a system we seek a linear combination of equations (13) (γ, ϵ) to form one equation of the form (18) (a), and similarly a linear combination of equations (13) (δ, ζ) to form one equation of the form (18) (b). It is thus necessary to find factors X, Y so that
$$X(13)(\gamma)+Y(13)(\epsilon) \equiv AF'+BG',$$
with corresponding factors for (13) (δ, ζ) to form (18) (b).

* Cf. the notation on p. 251.

Consider first the coefficients of $[F]_x$, $[G]_x$, $[F]_y$, $[G]_y$. These have in (18) (a) the forms Ax', Bx', Ay', By' respectively. Replacing in (13) (γ) and (13) (ϵ) y' by $\rho_1 x'$ wherever it is multiplied by $[F]_x$ or $[G]_x$, and picking out the coefficients of $[F]_x$, $[G]_x$, $[F]_y$, and $[G]_y$ in (13) (γ, ϵ), we require

$$\begin{aligned}
\rho_1(X\,G_m - \quad Y\,G_p) &= A,\\
\rho_1(X\,\{-F_m\} + Y\,F_p) &= B,\\
X\,G_n - \quad Y\,G_q &= A,\\
-X\,F_n + \quad Y\,F_q &= B.
\end{aligned}$$

This gives

$$X : Y = (G_q - \rho_1 G_p) : (G_n - \rho_1 G_m) = (F_q - \rho_1 F_p) : (F_n - \rho_1 F_m);$$

this last equation is seen to be true since, on rearrangement of the terms of the characteristic principal equation, we have

$$(F_q - \rho_1 F_p)(G_n - \rho_1 G_m) - (F_n - \rho_1 F_m)(G_q - \rho_1 G_p) = 0.$$

The coefficients of x' and y' are therefore brought into the required form $A[F]_x + B[G]_x$, $A[F]_y + B[G]_y$ by writing

$$\begin{aligned}
X &= F_q - \rho_1 F_p,\\
Y &= F_n - \rho_1 F_m,
\end{aligned}$$

which gives

$$\begin{aligned}
A &= \Delta_{qn} - \rho_1 F_p G_n + \rho_1 F_m G_q,\\
B &= -\rho_1(F_q F_m - F_p F_n).
\end{aligned}$$

It remains to be verified that the coefficients of p', q', m', n' are brought into the required forms

$$AF_p + BG_p, \quad AF_q + BG_q, \quad AF_m + BG_m, \quad AF_n + BG_n$$

respectively by the combination

$$(F_q - \rho_1 F_p)(13)(\gamma) + (F_n - \rho_1 F_m)(13)(\epsilon).$$

The coefficient of p' in this combination is

$$\begin{aligned}
(F_q - \rho_1 F_p).\rho_1 \Delta_{pm} &= F_q.\rho_1 \Delta_{pm} - F_p\{\rho_1(\Delta_{pn} + \Delta_{qm}) - \Delta_{qn}\}\\
&= F_p\{\Delta_{qn} - \rho_1(F_p G_n - F_m G_q)\} + G_p \rho_1(F_p F_n - F_q F_m),
\end{aligned}$$

since, from (5),

$$\rho_1^2 \Delta_{pm} = \rho_1(\Delta_{pn} + \Delta_{qm}) - \Delta_{qn};$$

and, by using the identity

$$F_q.\Delta_{pm}-F_p(\Delta_{pn}+\Delta_{qm})=F_p(F_mG_q-F_pG_n)-G_p(F_qF_m-F_pF_n),$$

the coefficient reduces to

$$AF_p+BG_v,$$

as required.

It may similarly be shown that the coefficients of q', m', and n' are brought into the required form by the above combination. We therefore have

$$(19)\qquad (\Delta_{qn}-\rho_1F_pG_n+\rho_1F_mG_q)F'-\rho_1(F_qF_m-F_pF_n)G'=0.$$

Similarly, from $(F_q-\rho_2F_p)(13)(\delta)+(F_n-\rho_2F_m)(13)(\xi)$,

we get

$$(20)\qquad (\Delta_{qn}-\rho_2F_pG_n+\rho_2F_mG_q)F'-\rho_2(F_qF_m-F_pF_n)G'=0$$

and (19) and (20) are the required canonical system for F and G.

Now the initial values of F and G are zero, and one solution of (19) and (20) is given by $F\equiv 0$, $G\equiv 0$. It follows from the fundamental theorem about these systems that there is only one solution of this initial value problem, provided that the determinant of (19) and (20) does not vanish, *i.e.* that

$$(\rho_2-\rho_1)\Delta_{qn}(F_qF_m-F_pF_n)\neq 0.$$

The hypotheses made in § 2 as to the form of the original system show that this condition is fulfilled, so that we have proved

$$F\equiv 0,\ G\equiv 0.$$

In conclusion, I wish to express my thanks to Dr. H. Lewy and Dr. F. Rellich for their help and advice in the preparation of this paper.

ON A PROBLEM OF FORMAL LOGIC

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[Received 28 November, 1928.—Read 13 December, 1928.]

This paper is primarily concerned with a special case of one of the leading problems of mathematical logic, the problem of finding a regular procedure to determine the truth or falsity of any given logical formula*. But in the course of this investigation it is necessary to use certain theorems on combinations which have an independent interest and are most conveniently set out by themselves beforehand.

I.

The theorems which we actually require concern finite classes only, but we shall begin with a similar theorem about infinite classes which is easier to prove and gives a simple example of the method of argument.

THEOREM A. *Let* Γ *be an infinite class, and* μ *and* r *positive integers; and let all those sub-classes of* Γ *which have exactly* r *members, or, as we may say, let all* r*-combinations of the members of* Γ *be divided in any manner into* μ *mutually exclusive classes* C_i $(i = 1, 2, \ldots, \mu)$*, so that every* r*-combination is a member of one and only one* C_i*; then, assuming the axiom of selections,* Γ *must contain an infinite sub-class* Δ *such that all the* r*-combinations of the members of* Δ *belong to the same* C_i.

Consider first the case $\mu = 2$. (If $\mu = 1$ there is nothing to prove.) The theorem is trivial when r is 1, and we prove it for all values of r by induction. Let us assume it, therefore, when $r = \rho - 1$ and deduce it for $r = \rho$, there being, since $\mu = 2$, only two classes C_i, namely C_1 and C_2.

* Called in German the *Entscheidungsproblem*; see Hilbert und Ackermann, *Grundzüge der Theoretischen Logik*, 72–81.

It may happen that Γ contains a member x_1 and an infinite sub-class Γ_1, not including x_1, such that the ρ-combinations consisting of x_1 together with any $\rho-1$ members of Γ_1, all belong to C_1. If so, Γ_1 may similarly contain a member x_2 and an infinite sub-class Γ_2, not including x_2, such that all the ρ-combinations consisting of x_2 together with $\rho-1$ members of Γ_2, belong to C_1. And, again, Γ_2 may contain an x_3 and a Γ_3 with similar properties, and so on indefinitely. We thus have two possibilities: either we can select in this way two infinite sequences of members of Γ $(x_1, x_2, \ldots, x_n, \ldots)$, and of infinite sub-classes of Γ $(\Gamma_1, \Gamma_2, \ldots, \Gamma_n, \ldots)$, in which x_n is always a member of Γ_{n-1}, and Γ_n a sub-class of Γ_{n-1} not including x_n, such that all the ρ-combinations consisting of x_n together with $\rho-1$ members of Γ_n, belong to C_1; or else the process of selection will fail at a certain stage, say the n-th, because Γ_{n-1} (or if $n=1$, Γ itself) will contain no member x_n and infinite sub-class Γ_n not including x_n such that all the ρ-combinations consisting of x_n together with $\rho-1$ members of Γ_n belong to C_1. Let us take these possibilities in turn.

If the process goes on for ever let Δ be the class $(x_1, x_2, \ldots, x_n, \ldots)$. Then all these x's are distinct, since if $r>s$, x_r is a member of Γ_{r-1} and so of Γ_{r-2}, Γ_{r-3}, ..., and ultimately of Γ_s which does not contain x_s. Hence Δ is infinite. Also all ρ-combinations of members of Δ belong to C_1; for if x_s is the term of such a combination with least suffix s, the other $\rho-1$ terms of the combination belong to Γ_s, and so form with x_s a ρ-combination belonging to C_1. Γ therefore contains an infinite sub-class Δ of the required kind.

Suppose, on the other hand, that the process of selecting the x's and Γ's fails at the n-th stage, and let y_1 be any member of Γ_{n-1}. Then the $(\rho-1)$-combinations of members of $\Gamma_{n-1}-(y_1)$ can be divided into two mutually exclusive classes C_1' and C_2' according as the ρ-combinations formed by adding to them y_1 belong to C_1 or C_2, and by our theorem (A), which we are assuming true when $r=\rho-1$ (and $\mu=2$), $\Gamma_{n-1}-(y_1)$ must contain an infinite sub-class Δ_1 such that all $(\rho-1)$-combinations of the members of Δ_1 belong to the same C_i'; *i.e.* such that the ρ-combinations formed by joining y_1 to $\rho-1$ members of Δ_1 all belong to the same C_i. Moreover, this C_i cannot be C_1, or y_1 and Δ_1 could be taken to be x_n and Γ_n and our previous process of selection would not have failed at the n-th stage. Consequently the ρ-combinations formed by joining y_1 to $\rho-1$ members of Δ_1 all belong to C_2. Consider now Δ_1 and let y_2 be any of its members. By repeating the preceding argument $\Delta_1-(y_2)$ must contain an infinite sub-class Δ_2 such that all the ρ-combinations got by joining y_2 to $\rho-1$ members of Δ_2 belong to the same C_i.

And, again, this C_i cannot be C_1, or, since y_2 is a member and Δ_2 a sub-class of Δ_1 and so of Γ_{n-1} which includes Δ_1, y_2 and Δ_2 could have been chosen as x_n and Γ_n and the process of selecting these would not have failed at the n-th stage. Now let y_3 be any member of Δ_2; then $\Delta_2-(y_3)$ must contain an infinite sub-class Δ_3 such that all ρ-combinations consisting of y_3 together with $\rho-1$ members of Δ_3, belong to the same C_i, which, as before, cannot be C_1 and must be C_2. And by continuing in this way we shall evidently find two infinite sequences $y_1, y_2, \ldots, y_n, \ldots$ and $\Delta_1, \Delta_2, \ldots, \Delta_n, \ldots$ consisting respectively of members and sub-classes of Γ, and such that y_n is always a member of Δ_{n-1}, Δ_n a sub-class of Δ_{n-1} not including y_n, and all the ρ-combinations formed by joining y_n to $\rho-1$ members of Δ_n belong to C_2; and if we denote by Δ the class $(y_1, y_2, \ldots, y_n, \ldots)$ we have, by a previous argument, that all ρ-combinations of members of Δ belong to C_2.

Hence, in either case, Γ contains an infinite sub-class Δ of the required kind, and Theorem A is proved for all values of r, provided that $\mu=2$. For higher values of μ we prove it by induction; supposing it already established for $\mu=2$ and $\mu=\nu-1$, we deduce it for $\mu=\nu$.

The r-combinations of members of Γ are then divided into ν classes C_i $(i=1, 2, \ldots, \nu)$. We define new classes C'_i for $i=1, 2, \ldots, \nu-1$ by

$$C'_i = C_i \quad (i = 1, 2, \ldots, \nu-2),$$

$$C'_{\nu-1} = C_{\nu-1}+C_\nu.$$

Then by the theorem for $\mu=\nu-1$, Γ must contain an infinite sub-class Δ such that all r-combinations of the members of Δ belong to the same C'_i. If, in this C'_i, $i\leqslant\nu-2$, they all belong to the same C_i, which is the result to be proved; otherwise they all belong to $C'_{\nu-1}$, *i.e.* either to $C_{\nu-1}$ or to C_ν. In this case, by the theorem for $\mu=2$, Δ must contain an infinite sub-class Δ' such that the r-combinations of members of Δ' either all belong to $C_{\nu-1}$ or all belong to C_ν; and our theorem is thus established.

Coming now to finite classes it will save trouble to make some conventions as to notation. Small letters other than x and y, whether Italic or Greek (*e.g.* n, r, μ, m) will always denote finite cardinals, positive unless otherwise stated. Large Greek letters (*e.g.* Γ, Δ) will denote classes, and their suffixes will indicate the number of their members (*e.g.* Γ_m is a class with m members). The letters x and y will represent members of the classes Γ, Δ, etc., and their suffixes will be used merely to distinguish them. Lastly, the letter C will stand, as before, for classes of combinations, and its suffixes will not refer to the

number of members, but serve merely to distinguish the different classes of combinations considered.

Corresponding to Theorem A we then have

THEOREM B. *Given any r, n, and μ we can find an m_0 such that, if $m \geqslant m_0$ and the r-combinations of any Γ_m are divided in any manner into μ mutually exclusive classes C_i $(i = 1, 2, \ldots, \mu)$, then Γ_m must contain a sub-class Δ_n such that all the r-combinations of members of Δ_n belong to the same C_i.*

This is the theorem which we require in our logical investigations, and we should at the same time like to have information as to how large m_0 must be taken for any given r, n, and μ. This problem I do not know how to solve, and I have little doubt that the values for m_0 obtained below are far larger than is necessary.

To prove the theorem we begin, as in Theorem A, by supposing that $\mu = 2$. We then take, not Theorem B itself, but the equivalent

THEOREM C. *Given any r, n, and k such that $n+k \geqslant r$, there is an m_0 such that, if $m \geqslant m_0$ and the r-combinations of any Γ_m are divided into two mutually exclusive classes C_1 and C_2, then Γ_m must contain two mutually exclusive sub-classes Δ_n and Λ_k such that all the combinations formed by r members of $\Delta_n + \Lambda_k$ which include at least one member from Δ_n belong to the same C_i.*

That this is equivalent to Theorem B with $\mu = 2$ is evident from the fact that, for any given r, Theorem C, for n and k, asserts more than Theorem B for n, but less than Theorem B for $n+k$.

The proof of Theorem C must be performed by mathematical induction, and can conveniently be set out as a demonstration that it is possible to define by recursion a function $f(r, n, k)$ which will serve as m_0 in the theorem.

If $r = 1$, the theorem is evidently true with m_0 equal to the greater of $2n-1$ and $n+k$, so that we may define

$$f(1, n, k) = \max(2n-1, n+k) \quad (n \geqslant 1,\ k \geqslant 0).$$

For other values of r we define $f(r, n, k)$ by recursion formulae involving an auxiliary function $g(r, n, k)$. Suppose that $f(r-1, n, k)$ has been defined for a certain $r-1$, and all n, k such that $n+k \geqslant r-1$, then we define it for r by putting

$$\begin{aligned}
f(r, 1, k) &= f(r-1, k-r+2, r-2)+1 \quad (k+1 \geqslant r),\\
g(r, 0, k) &= \max(r-1, k),\\
g(r, n, k) &= f\{r, 1, g(r, n-1, k)\} \qquad (n \geqslant 1),\\
f(r, n, k) &= f\{r, n-1, g(r, n, k)\} \qquad (n > 1).
\end{aligned}$$

These formulae can be easily seen to define $f(r, n, k)$ for all positive values of r, n and k satisfying $n+k \geqslant r$, and $g(r, n, k)$ for all values of r greater than 1, and all positive values of n and k; and we shall prove that Theorem C is true when we take m_0 to be this $f(r, n, k)$. We know that this is so when $r=1$, and we shall therefore assume it for all values up to $r-1$ and deduce it for r.

When $n=1$, and $m \geqslant m_0 = f(r-1, k-r+2, r-2)+1$, we may take any member x of Γ_m to be sole member of Δ_1 and there remain at least $f(r-1, k-r+2, r-2)$ members of $\Gamma_m-(x)$; the $(r-1)$-combinations of these members of $\Gamma_m-(x)$ can be divided into classes C_1' and C_2' according as they belong to C_1 or C_2 when x is added to them, and, by our theorem for $r-1$, $\Gamma_m-(x)$ must contain two mutually exclusive classes Δ_{k-r+2}, Λ_{r-} such that every combination of $r-1$ terms from $\Delta_{k-r+2}+\Lambda_{r-2}$ (since one of its terms must come from Δ_{k-r+2}, Λ_{r-2} having only $r-2$ members) belongs to the same C_i'. Taking Λ_k to be this $\Delta_{k-r+2}+\Lambda_{r-2}$ all combinations consisting of x, together with $r-1$ members of Λ_k, belong to the same C_i. The theorem is therefore true for r when $n=1$.

For other values of n we prove it by induction, assuming it for $n-1$ and deducing it for n. Taking

$$m \geqslant m_0 = f(r, n, k) = f\{r, n-1, g(r, n, k)\},$$

Γ_m must, by the theorem for $n-1$, contain a Δ_{n-1} and a $\Lambda_{g(r, n, k)}$ such that every combination of r members of $\Delta_{n-1}+\Lambda_{g(r, n, k)}$, at least one term of which comes from Δ_{n-1}, belongs to the same C_i, say to C_1. If, now, $\Lambda_{g(r, n, k)}$ contains a member x and a sub-class Λ_k not including x, such that every combination of x and $r-1$ members of Λ_k belongs to C_1, then, taking Δ_n to be $\Delta_{n-1}+(x)$ and Λ_k to be this Λ_k, our theorem is true. If not, there can be no member of $\Lambda_{g(r, n, k)}$ which has a sub-class of k members of $\Lambda_{g(r, n, k)}$ connected with it in this way. But since

$$g(r, n, k) = f\{r, 1, g(r, n-1, k)\},$$

$\Lambda_{g(r, n, k)}$ must contain a member x_1 and a sub-class $\Lambda_{g(r, n-1, k)}$, not including x_1, such that x_1 combined with any $r-1$ members of $\Lambda_{g(r, n-1, k)}$ gives a combination belonging to the same C_i, which cannot be C_1, or x_1 and any k members of $\Lambda_{g(r, n-1, k)}$ could have been taken as the x and Λ_k above. Hence the combinations formed by x_1 together with any $r-1$ members of $\Lambda_{g(r, n-1, k)}$ all belong to C_2. But now

$$g(r, n-1, k) = f\{r, 1, g(r, n-2, k)\},$$

and $\Lambda_{g(r, n-1, k)}$ must contain an x_2 and a $\Lambda_{g(r, n-2, k)}$, not including x_2, such that the combinations formed by x_2 and $r-1$ members of $\Lambda_{g(r, n-2, k)}$ all

belong to the same C_i, which must, as before, be C_2, since x_2 and $\Lambda_{g(r, n-2, k}$ are both contained in $\Lambda_{g(r, n, k)}$ and $g(r, n-2, k) \geqslant k$. Continuing in this way we can find n distinct terms $x_1, x_2, \ldots, x_n$ and a $\Lambda_{g(r, 0, k)}$ such that every combination of r terms from $(x_1, x_2, \ldots, x_n)+\Lambda_{g(r, 0, k)}$ belongs to C_2, provided that at least one term of the combination comes from $(x_1, x_2, \ldots, x_n)$. Since $g(r, 0, k) \geqslant k$ this proves our theorem, taking Δ_n to be $(x_1, x_2, \ldots, x_n)$ and Λ_k to be any k terms of $\Lambda_{g(r, 0, k)}$.

Theorem C is therefore established for all values of r, n, and k, with m_0 equal to $f(r, n, k)$. It follows that, if $\mu = 2$, Theorem B is true for all values of r and n with m_0 equal to $f(r, n-r+1, r-1)$, which we shall also call $h(r, n, 2)$.

For other values of μ we prove Theorem B by induction, taking m_0 to be $h(r, n, \mu)$, where

$$h(r, n, 2) = f(r, n-r+1, r-1)$$

$$h(r, n, \mu) = h\{r, h(r, n, \mu-1), 2\} \quad (\mu > 2).$$

For, assuming the theorem for $\mu-1$, we prove it for μ by defining new classes of combinations

$$C_1' = C_1,$$

$$C_2' = \sum_{i=2}^{\mu} C_i.$$

If then $m \geqslant h(r, n, \mu) = h\{r, h(r, n, \mu-1), 2\}$, by the theorem for $\mu = 2$, Γ_m must contain a $\Gamma_{h(r, n, \mu-1)}$ the r-combinations of whose members belong either all to C_1' or all to C_2'. In the first case there is no more to prove; in the second we have only to apply the theorem for $\mu-1$ to $\Gamma_{h(r, n, \mu-1)}$.

In the simplest case in which $r = \mu = 2$ the above reasoning gives m_0 equal to $h(2, n, 2)$, which is easily shown to be $2^{n(n-1)/2}$. But for this case there is a simple argument which gives the much lower value $m_0 = n!$, and shows that our value $h(r, n, \mu)$ is altogether excessive.

For, taking Theorem C first, we can prove by induction with regard to n that, for $r = 2$, we may take m_0 to be $k \,.\, (n+1)!$. (k is here supposed greater than or equal to 1.) For this is true when $n = 1$, since, if $m \geqslant 2k$, of the $m-1$ pairs obtained by combining any given member of Γ_m with the others, at least k must belong to the same C_i. Assuming it, then, for $n-1$, let us prove it for n.

If $m \geqslant k \,.\, (n+1)! = k(n+1) \,.\, n!$, Γ_m must, by the theorem for $n-1$, contain two mutually exclusive sub-classes Δ_{n-1} and $\Lambda_{k(n+1)}$ such that all pairs from $\Delta_{n-1}+\Lambda_{k(n+1)}$, at least one term of which comes from Δ_{n-1}, belong to the same C_i, say C_1. Now consider the members of $\Lambda_{k(n+1)}$; in

the first place, there may be one of these, x say, which is such that there are k other members of $\Lambda_{k(n+1)}$ which combined with x give pairs belonging to C_1. If so, the theorem is true, taking Δ_n to be $\Delta_{n-1}+(x)$; if not, let x_1 be any member of $\Lambda_{k(n+1)}$. Then there are at most $k-1$ other members of $\Lambda_{k(n+1)}$ which combined with x_1 give pairs belonging to C_1, and $\Lambda_{k(n+1)}-(x_1)$ must contain a Λ_{kn} any member of which gives when combined with x_1 a pair belonging to C_2. Let x_2 be any member of Λ_{kn}, then, since x_2 and Λ_{kn} are both contained in $\Lambda_{k(n+1)}$, there are at most $k-1$ other members of Λ_{kn} which when combined with x_2 give pairs belonging to C_1. Hence $\Lambda_{kn}-(x_2)$ contains a $\Lambda_{k(n-1)}$ any member of which combined with x_2 gives a pair belonging to C_2. Continuing in this way we obtain $x_1, x_2, \ldots, x_n$ and Λ_k, such that every pair x_i, x_j and every pair consisting of an x_i and a member of Λ_k belongs to C_2. Theorem C is therefore proved.

Theorem B for n then follows, with the m_0 of Theorem C for $n-1$ and 1, *i.e.* with m_0 equal to $n!$*; and it is an easy extension to show that, if in Theorem B $r=2$ but $\mu \neq 2$, we can take m_0 to be $n!!!, \ldots,$ where the process of taking the factorial is performed $\mu-1$ times.

II.

We shall be concerned with logical formulae containing variable propositional functions, *i.e.* predicates or relations, which we shall denote by Greek letters ϕ, χ, ψ, etc. These functions have as arguments individuals denoted by x, y, z, etc., and we shall deal with functions with any finite number of arguments, *i.e.* of any of the forms

$$\phi(x), \quad \chi(x, y), \quad \psi(x, y, z), \quad \ldots.$$

In addition to these variable functions we shall have the one constant function of identity $\quad x = y \quad$ or $\quad = (x, y).$

By operating on the values of ϕ, χ, ψ, ..., and $=$ with the logical operations

$\sim$	meaning	*not,*
$\vee$	,,	*or,*
.	,,	*and,*
(x)	,,	*for all x.*
(Ex)	,,	*there is an x for which,*

* But this value is, I think, still much too high. It can easily be lowered slightly even when following the line of argument above, by using the fact that if k is even it is impossible for every member of an odd class to have exactly $k-1$ others with which it forms a pair of C_1, for then twice the number of these pairs would be odd; we can thus start when k is even with a $\Lambda_{k(n+1)-1}$ instead of a $\Lambda_{k(n+1)}$

we can construct expressions such as

$$[(x, y)\{\phi(x, y) \vee x = y\}] \vee \{(Ez)\chi(z)\}$$

in which all the individual variables are made "apparent" by prefixes (x) or (Ex), and the only real variables left are the functions ϕ, χ, ... Such an expression we shall call a *first order formula*.

If such a formula is true for all interpretations* of the functional variables ϕ, χ, ψ, etc., we shall call it *valid*, and if it is true for no interpretations of these variables we shall call it *inconsistent*. If it is true for some interpretations (whether or not for all) we shall call it *consistent*†.

The *Entscheidungsproblem* is to find a procedure for determining whether any given formula is valid, or, alternatively, whether any given formula is consistent; for these two problems are equivalent, since the necessary and sufficient condition for a formula to be consistent is that its contradictory should not be valid. We shall find it more convenient to take the problem in this second form as an investigation of *consistency*. The consistency of a formula may, of course, depend on the number of individuals in the universe considered, and we shall have to distinguish between formulae which are consistent in every universe and those which are only consistent in universes with some particular numbers of members. Whenever the universe is infinite we shall have to assume the axiom of selections.

The problem has been solved by Behmann‡ for formulae involving only functions of one variable, and by Bernays and Schönfinkel§ for formulae involving only two individual apparent variables. It is solved below for the further case in which, when the formula is written in "normal form", there are any number of prefixes of generality (x) but none of existence (Ex)‖. By "normal form"¶ is here meant that all the prefixes stand at the beginning, with no negatives between or in front of them, and have scopes extending to the end of the formula.

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* To avoid confusion we call a constant function substituted for a variable ϕ, not a value but an *interpretation* of ϕ; the *values* of $\phi(x, y, z)$ are got by substituting constant individuals for x, y, and z.

† German *erfüllbar*.

‡ H. Behmann, "Beiträge zur Algebra der Logik und zum Entscheidungsproblem", *Math. Annalen*, 86 (1922), 163–229.

§ P. Bernays und M. Schönfinkel, "Zum Entscheidungsproblem der mathematischen Logik", *Math. Annalen*, 99 (1928), 342–372. These authors do not, however, include identity in the formulae they consider.

‖ Later we extend our solution to the case in which there are also prefixes of existence provided that these all precede all the prefixes of generality.

¶ Hilbert und Ackermann, *op. cit.*, 63–4.

The formulae to be considered are thus of the form

$$(x_1, x_2, \ldots, x_n) F(\phi, \chi, \psi, \ldots, =, x_1, x_2, \ldots, x_n),$$

where the matrix F is a truth-function of values of the functions ϕ, χ, ψ, etc., and $=$ for arguments drawn from $x_1, x_2, \ldots, x_n$.

This type of formula is interesting as being the general type of an axiom system consisting entirely of "general laws"*. The axioms for order, betweenness, and cyclic order are all of this nature, and we are thus attempting a general theory of the consistency of axiom systems of a common, if very simple, type.

If identity does not occur in F the problem is trivial, since in this case whether the formula is consistent or not can be shown to be independent of the number of individuals in the universe, and we have only the easy task of testing it for a universe with one member only†.

But when we introduce identity the question becomes much more difficult, for although it is still obvious that if the formula is consistent in a universe U it must be consistent in any universe with fewer members than U, yet it may easily be consistent in the smaller universe but not in the larger. For instance,

$$(x_1, x_2)[x_1 = x_2 \vee \{\phi(x_1) . \sim \phi(x_2)\}]$$

is consistent in a universe with only one member but not in any other.

We begin our investigation by expressing F in a special form. F is a truth-function of the values of ϕ, χ, ψ, ..., and $=$ for arguments drawn from $x_1, x_2, \ldots, x_n$. If ϕ is a function of r variables there will be n^r‡ values of ϕ which can occur in F, and F will be a truth-function of Σn^r values of ϕ, χ, ψ, ..., and $=$, which we shall call *atomic propositions*. With regard to these Σn^r atomic propositions there are $2^{\Sigma n^r}$ possibilities of truth and falsity which we shall call *alternatives*, each alternative being a conjunction of Σn^r propositions which are either atomic propositions or their contradictions. In constructing the alternatives all the Σn^r atomic propositions are to be used whether or not they occur in F. F can then be expressed as a disjunction of some of these alternatives, namely those with which it is compatible. It is well known that such an

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* C. H. Langford, "Analytic completeness of postulate sets", *Proc. London Math. Soc.* (2), 25 (1926), 115–6.

† Bernays und Schönfinkel, *op. cit.*, 359. We disregard altogether universes with no members.

‡ Here and elsewhere numbers are given not because they are relevant to the argument, but to enable the reader to check that he has in mind the same class of entities as the author.

expression is possible; indeed, it is the dual of what Hilbert and Ackermann call the "ausgezeichnete konjunktive Normalform"*, and is fundamental also in Wittgenstein's logic. The only exception is when F is a self-contradictory truth-function, in which case our formula is certainly not consistent.

F having been thus expressed as a disjunction of alternatives (in our special sense of the word), our next task is to show that some of these alternatives may be able to be removed without affecting the consistency or inconsistency of the formula. If all the alternatives can be removed in this way the formula will be inconsistent; otherwise we shall have still to consider the alternatives that remain.

In the first place an alternative may violate the laws of identity by containing parts of any of the following forms :—

$$x_i \neq x_i\dagger,$$

$$x_i = x_j \,.\, x_j \neq x_i \qquad (i \neq j),$$

$$x_i = x_j \,.\, x_j = x_k \,.\, x_i \neq x_k \quad (i \neq j,\ j \neq k,\ k \neq i),$$

or by containing $x_i = x_j$ $(i \neq j)$ and values of a function ϕ and its contradictory $\sim\phi$ for sets of arguments which become the same when x_i is substituted for x_j [*e.g.* $x_1 = x_2 \,.\, \phi(x_1, x_2, x_3) \,.\, \sim \phi(x_2, x_1, x_3)$].

Any alternative which violates these laws must always be false and can evidently be discarded without affecting the consistency of the formula. The remaining alternatives can then be classified according to the number of x's they make to be different, which may be anything from 1 up to n.

Suppose that for a given alternative this number is ν, then we can derive from it what we will call the corresponding y alternative by the following process :—

For x_1, wherever it occurs in the given alternative, write y_1; next, if in the alternative $x_2 = x_1$, for x_2 write y_1 again, if not for x_2 write y_2. In general, if x_i is in the given alternative identical with any x_j with j less than i, write for x_i the y previously written for x_j; otherwise write for x_i, y_{k+1}, where k is the number of y's already introduced. The expression which results contains ν y's all different instead of n x's, some of which are identical, and we shall call it the y alternative corresponding to the given x alternative.

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* *Op. cit.*, 16.

† We write $x \neq y$ for $\sim(x = y)$.

Thus to the alternative

$$\phi(x_1) . \sim \phi(x_2) . \phi(x_3) . \sim \phi(x_4) . x_1 = x_3 . x_2 = x_4 . x_1 \neq x_2^*$$

corresponds the y alternative

$$\phi(y_1) . \sim \phi(y_2) . y_1 \neq y_2.$$

We call two y alternatives *similar* if they contain the same number of y's and can be derived from one another by permuting those y's, and we call two x alternatives *equivalent* if they correspond to similar (or identical) y alternatives.

Thus

$$\phi(x_1) . \sim \phi(x_2) . \phi(x_3) . \sim \phi(x_4) . x_1 = x_3 . x_2 = x_4 . x_1 \neq x_2, \qquad (\alpha)$$

is equivalent to

$$\sim \phi(x_1) . \phi(x_2) . \phi(x_3) . \phi(x_4) . x_1 \neq x_2 . x_2 = x_3 = x_4, \qquad (\beta)$$

since they correspond to the similar y alternatives

$$\phi(y_1) . \sim \phi(y_2) . y_1 \neq y_2$$

$$\sim \phi(y_1) . \phi(y_2) . y_1 \neq y_2$$

derivable from one another by interchanging y_1 and y_2, although (α) and (β) are not so derivable by permuting the x's.

We now see that we can discard any alternative contained in F unless F also contains all the alternatives equivalent to it; *e.g.* if F contains (α) but not (β), (α) may be discarded from it. For omitting alternatives clearly cannot make the formula consistent if it was not so before; and we can easily prove that, if it was consistent before, omitting *these* alternatives cannot make it inconsistent.

For suppose that the formula is consistent, *i.e.* that for some particular interpretation of ϕ, χ, ψ, ..., F is true for every set of x's, and let p be an alternative contained in F, q an alternative equivalent to p but not contained in F. Then for every set of x's one and only one alternative in F will (on this interpretation of ϕ, χ, ψ, ...) be the true one, and this alternative can never be p. For if it were p, the corresponding y alternative would be true for some set of y's, and the similar y alternative corresponding to q would be true for a set of y's got by permuting this last set. Giving the x's suitable values in terms of the y's, q would then

* We take one function of one variable only for simplicity; also to save space we omit expressions which may be taken for granted, such as $x_1 = x_1$, $x_1 \neq x_4$.

be true for a certain set of x's and F would be false for these x's contrary to hypothesis. Hence p is never the true alternative and may be omitted without affecting the consistency of the formula.

When we have discarded all these alternatives from F, the remainder will fall into sets each of which is the complete set of all alternatives equivalent to a given alternative. To such a set of x alternatives will correspond a complete set of similar y alternatives, and the disjunction of such a complete set of similar y alternatives (*i.e.* of all permutations of a given y alternative) we shall call a *form**. A form containing ν y's we shall denote by an Italic capital with suffix ν, *e.g.* A_ν, B_ν.

The force of our formula can now be represented by the following conjunction, which we shall call P.

$$\left.\begin{array}{ll}
\text{For every } y_1, & A_1 \text{ or } B_1 \text{ or } \ldots \\
\text{For every distinct } y_1, y_2, & A_2 \text{ or } B_2 \text{ or } \ldots \\
\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots & \\
\text{For every distinct } y_1, y_2, \ldots, y_\nu, & A_\nu \text{ or } B_\nu \text{ or } \ldots \\
\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots & \\
\text{For every distinct } y_1, y_2, \ldots, y_n, & A_n \text{ or } B_n \text{ or } \ldots
\end{array}\right\} (P)$$

where A_ν, B_ν†, etc., are the forms corresponding to the x alternatives still remaining in F. If for any ν there are no such forms, *i.e.* if no alternatives with ν different x's remain in F, our formula implies that there are no such things as ν distinct individuals, and so cannot be consistent in a world of ν or more members.

We have now to define what is meant by saying that one form is *involved* in another. Consider a form A_ν and take one of the y alternatives contained in it. This y alternative is a conjunction of the values of ϕ, χ, ψ, ..., and their negatives for arguments drawn from $y_1, y_2, \ldots, y_\nu$. (We may leave out the values of identity and difference, since it is taken for granted that y's are always different.) If $\mu < \nu$ we can select μ of these y's in any way and leave out from the alternative all the terms in it which contain any of the $\nu - \mu$ y's not selected. We have left an alternative in μ y's which we can renumber $y_1, y_2, \ldots, y_\mu$, and the form E_μ to which this new alternative belongs we shall describe as being *involved* in the A_ν with which we started. Starting with one particular y alternative in A_ν we shall get a large number of different E_μ's by

* Cf. Langford, *op. cit.*, 116–120.

† The notation is partially misleading, since A_ν has no closer relation to A_μ than to B_μ.

choosing differently the μ y's which we select to preserve; and from whichever y alternative in A_ν we start, the E_μ's which we find to be involved in A_ν will be the same.

For example,

$$\{\phi(y_1, y_1) \cdot \phi(y_1, y_2) \cdot \phi(y_2, y_1) \cdot \sim \phi(y_2 \cdot y_2)\}$$
$$\vee \{\sim \phi(y_1, y_1) \cdot \phi(y_1, y_2) \cdot \phi(y_2, y_1) \cdot \phi(y_2, y_2)\}$$

is a form A_2 which involves the two E_1's

$$\phi(y_1, y_1)$$
$$\sim \phi(y_1, y_1).$$

It is clear that if for some distinct set of ν y's a form A_ν is true, then every form E_μ involved in A_ν will be true for some distinct set of μ y's contained in the ν.

We are now in a position to settle the consistency or inconsistency of our formula when N, the number of individuals in the universe, is less than or equal to n, the number of x's in our formula. In fact, if $N \leqslant n$, it is necessary and sufficient for the consistency of the formula that P should contain a form A_N together with all the forms E_μ involved in it for every μ less than N.

This condition is evidently necessary, since the N individuals in the universe must, taken as $y_1, y_2, \ldots, y_N$, have some form A_N in regard to any $\phi, \chi, \psi, \ldots$; and all forms involved in this A_N must be true for different selections of y's, and so contained in P if P is to be true for this $\phi, \chi, \psi, \ldots$.

Conversely, suppose that P contains a form A_N together with all forms involved in A_N; then, calling the N individuals in the universe $y_1, y_2, \ldots, y_N$, we can define functions $\phi, \chi, \psi, \ldots$ to make any assigned y alternative in A_N true; for any permutation of these N y's another alternative in A_N will be the true one, and for any subset of y's some y alternative in a form involved in A_N. Since all these y alternatives are by hypothesis contained in P, P will be true for these $\phi, \chi, \psi, \ldots$, and our formula consistent.

When, however, $N > n$ the problem is not so simple, although it clearly depends on the A_n's in P such that all forms involved in them are also contained in P. These A_n's we may call *completely contained* in P, and if there are no such A_n's a similar argument to that used when $N \leqslant n$ will show that the formula is inconsistent. But the converse argument, that if there is an A_n completely contained in P the formula must be consistent, no longer holds good; and to proceed further we have to introduce a new conception, the conception of a form being *serial*.

But before proceeding to explain this idea it is best to simplify matters by the introduction of new functions. Let ϕ be one of the variable functions in our formula, with, say, r arguments. Then, if $r < n$, ϕ will occur in P with all its arguments different [*e.g.* $\phi(y_1, y_2, \ldots, y_r)$] and also with some of them the same [*e.g.* $\phi(y_1, y_2, \ldots, y_{r-1}, y_1)$]; but we can conveniently eliminate values of the second kind by introducing new functions of fewer arguments than r, which, when all their arguments are different, take values equivalent to those of ϕ with some of its arguments identical.

E.g. we may put

$$\phi_1(y_1, y_2, \ldots, y_{r-1}) = \phi(y_1, y_2, \ldots, y_{r-1}, y_1).$$

In this way ϕ gives rise to a large number of functions with fewer arguments; each of these functions we define only for the case in which all its arguments are different, as is secured by these arguments being y's with different suffixes. If $r \geq n$, there is no difference except that ϕ can never occur with all its arguments different, and so is entirely replaced by the new functions.

If we do this for all the functions ϕ, χ, ψ,..., and replace them by new functions wherever they occur in P with some of their arguments the same, P will contain a new set of variable functions (including all the old ones which have no more than n arguments), and these will never occur in P with the same argument repeated.

It is easy to see that this transformation does not affect the consistency of the formula, for, if it were consistent before, it must be consistent afterwards, since the new functions have simply to be replaced by their definitions. And if it is consistent afterwards it must have been so before, since any function of the old set has only to be given for any set of arguments the value of the appropriate function of the new set*.

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* For instance, if $\phi(y_1, y_2, y_3)$ is a function of the old set, we have five new functions

$$\begin{aligned}
\phi_0(y_1, y_2, y_3) &= \phi(y_1, y_2, y_3),\\
\chi_0(y_1, y_2) &= \phi(y_1, y_1, y_2),\\
\psi_0(y_1, y_2) &= \phi(y_1, y_2, y_1),\\
\pi_0(y_1, y_2) &= \phi(y_1, y_1, y_2),\\
\rho_0(y_1) &= \phi(y_1, y_1, y_1),
\end{aligned}$$

and any value of ϕ is equivalent to a value of one and only one of the new functions. It must be remembered that the new functions are used only with all their arguments different; for otherwise they would not be independent, since we should have, for instance, $\chi_0(y_1, y_1)$ equivalent to $\rho_0(y_1)$. But $\chi_0(y_1, y_1)$ never occurs, and $\phi(y_1, y_1, y_1)$ is equivalent not to any value of χ_0 but only to $\rho_0(y_1)$.

In view of this fact we shall find it more convenient to take P in its new form, and denote the new set of functions by ϕ_0, χ_0, ψ_0, ...

Suppose, then, that ϕ_0 is a function of r variables; there are

$$n(n-1)\dots(n-r+1)$$

values of ϕ_0 with r different arguments drawn from $y_1, y_2, \dots, y_n$ and every y alternative must contain each of these values or its contradictory. $r!$ of these values will have as arguments permutations of $y_1, y_2, \dots, y_r$. Any other set of r y's can be arranged in the order of their suffixes as $y_{s_1}, y_{s_2}, \dots, y_{s_r}$, $s_1 < s_2 < s_3 \dots < s_r$, and it may happen that a given alternative contains the values of ϕ_0 for those and only those permutations of $y_{s_1}, y_{s_2}, \dots, y_{s_r}$ which correspond (in the obvious way) to the permutations of $y_1, y_2, \dots, y_r$ for which it (the alternative) contains the values of ϕ_0; *e.g.* if the alternative contains $\phi_0(y_1, y_2, \dots, y_r)$ and $\phi_0(y_r, y_{r-1}, \dots, y_1)$, but for every other permutation of $y_1, y_2, \dots, y_r$ contains the corresponding value of $\sim\phi_0$, then it may happen that the alternative contains $\phi_0(y_{s_1}, y_{s_2}, \dots, y_{s_r})$ and $\phi_0(y_{s_r}, y_{s_{r-1}}, \dots, y_{s_1})$, but for every other permutation of $y_{s_1}, y_{s_2}, \dots, y_{s_r}$ contains the corresponding value of $\sim\phi_0$.

If this happens, no matter how the set of r y's, $y_{s_1}, y_{s_2}, \dots, y_{s_r}$ is chosen from $y_1, y_2, \dots, y_n$, then we say that the *alternative* is *serial in* ϕ_0*, and if an alternative is serial in every function of the new set we shall call it *serial* simply.

Consider, for example, the following alternative, in which we may imagine ϕ_0 and ψ_0 to be derived from one "old" function ϕ by the definitions

$$\phi_0(y_i, y_k) = \phi(y_i, y_k),$$

$$\psi_0(y_i) = \phi(y_i, y_i),$$

$$\phi_0(y_1, y_2) \cdot \sim\phi_0(y_2, y_1) \cdot \phi_0(y_1, y_3) \cdot \sim\phi_0(y_3, y_1) \cdot \phi_0(y_2, y_3) \cdot \sim\phi_0(y_3, y_2),$$

$$\psi_0(y_1) \cdot \sim\psi_0(y_2) \cdot \psi_0(y_3).$$

This is serial in ϕ_0, since we always have $\phi_0(y_{s_1}, y_{s_2}) \cdot \sim\phi_0(y_{s_2}, y_{s_1})$; but not in ψ_0, since we sometimes have $\psi_0(y_{s_1})$, but sometimes $\sim\psi_0(y_{s_1})$. Hence it is not a serial alternative.

We call a *form serial* when it contains at least one serial alternative, and can now state our chief result as follows.

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* Thus, if ϕ_0 is a function of n variables, *all* alternatives are serial in ϕ_0.

THEOREM.—*There is a finite number m, depending on n, the number of functions ϕ, χ, ψ, ..., and the numbers of their arguments, such that the necessary and sufficient condition for our formula to be consistent in a universe with m or more members is that there should be a serial form A_n completely contained in P. For consistency in a universe of fewer than m members this condition is sufficient but not necessary.*

We shall first prove that, whatever be the number N of individuals in the universe, the condition is sufficient for the consistency of the formula. If $N \leqslant n$, this is a consequence of a previous result, since, if A_n is completely contained in P, so is any A_N involved in A_n.

If $N > n$, we suppose the universe ordered in a series by a relation R. (If N is infinite this requires the Axiom of Selections.) Let q be any serial alternative contained in A_n. If ϕ_0 is a function of r arguments, q will contain the values of either ϕ_0 or $\sim\phi_0$ (but not both) for every permutation of $y_1, y_2, \ldots, y_r$. Any such permutation can be written $y_{\rho_1}, y_{\rho_2}, \ldots, y_{\rho_r}$ where $\rho_1, \rho_2, \ldots, \rho_r$ are $1, 2, \ldots, r$ rearranged. We make a list of all those permutations $(\rho_1, \rho_2, \ldots, \rho_r)$ for which q contains the values of ϕ_0, and call this list Σ. We now give ϕ_0 the constant interpretation that $\phi_0(z_1, z_2, \ldots, z_r)$ is to be true if and only if the order of the terms $z_1, z_2, \ldots, z_r$ in the series R is given by one of the permutations $(\rho_1, \rho_2, \ldots, \rho_r)$ contained in Σ, in the sense that, for each i, z_i is the ρ_i-th of $z_1, z_2, \ldots, z_r$ as they are ordered by R.

Let us suppose now that $y_1, y_2, \ldots, y_n$ are numbered in the order in which they occur in R, *i.e.* that in the R series y_1 is the first of them, y_2 the second, and so on. Then we shall see that, if ϕ_0 is given the constant interpretation defined above, all the values of ϕ_0 and $\sim\phi_0$ in q will be true. Indeed, for values whose arguments are obtained by permuting $y_1, y_2, \ldots, y_r$ this follows at once from the way in which ϕ_0 has been defined. For $\phi_0(y_{\sigma_1}, y_{\sigma_2}, \ldots, y_{\sigma_r})$ is true if and only if the order of $y_{\sigma_1}, y_{\sigma_2}, \ldots, y_{\sigma_r}$ in the R series is given by a permutation $(\rho_1, \rho_2, \ldots, \rho_r)$ contained in Σ. But the order in the series of $y_{\sigma_1}, y_{\sigma_2}, \ldots, y_{\sigma_r}$ is in fact given (on our present hypothesis that the order of the y's is $y_1, y_2, \ldots, y_r$) by $(\sigma_1, \sigma_2, \ldots, \sigma_r)$, which is contained in Σ if and only if $\phi_0(y_{\sigma_1}, y_{\sigma_2}, \ldots, y_{\sigma_r})$ is contained in q. Hence values of ϕ_0 for arguments consisting of the first r y's are true when they are contained in q and false otherwise, *i.e.* when the corresponding values of $\sim\phi_0$ are contained in q.

For sets of arguments not confined to the first r y's our result follows from the fact that q is serial, *i.e.* that if $s_1 < s_2 < \ldots < s_r$, so that $y_{s_1}, y_{s_2}, \ldots, y_{s_r}$ are in the order given by the R series, q contains the

values of ϕ_0 for just those permutations of $y_{s_1}, y_{s_2}, \ldots, y_{s_r}$ which correspond to the permutations of $y_1, y_2, \ldots, y_r$ for which it contains the values of ϕ_0, *i.e.* by the definition of ϕ_0 and the preceding argument, for just those permutations of $y_{s_1}, y_{s_2}, \ldots, y_{s_r}$ which make ϕ_0 true.

Hence all the values of ϕ_0 and $\sim\phi_0$ in q are true when $y_1, y_2, \ldots, y_n$ are in the order given by the R series.

If, then, we define analogous constant interpretations for χ_0, ψ_0, etc., and combine these with our interpretation of ϕ_0, the whole of q will be true provided that $y_1, y_2, \ldots, y_n$ are in the order given by the R series, and if $y_1, y_2, \ldots, y_n$ are in any other order the true alternative will be obtained from q by suitably permuting the y's, *i.e.* will be an alternative similar to q and contained in the same form A_n. Hence A_n is true for any set of distinct $y_1, y_2, \ldots, y_n$. Moreover, for any set of distinct $y_1, y_2, \ldots, y_\nu$ $(\nu < n)$ the true form will be one involved in A_n, and since A_n and all forms involved in it are contained in P, P will be true for these interpretations of ϕ_0, χ_0, ψ_0, ..., and our formula must be consistent.

Having thus proved our condition for consistency sufficient in any universe, we have now to prove it necessary in any infinite or sufficiently large finite universe, and for this we have to use the Theorem B proved in the first part of the paper.

Our line of argument is as follows: we have to show that, whatever ϕ_0, χ_0, ψ_0, ... we take P will be false unless it completely contains a serial A_n. For this it is enough to show that, given any ϕ_0, χ_0, ψ_0, ..., there must be a set of n y's for which the true form is serial*, or, since a serial form is one which contains a serial alternative, that there must be a set of values of $y_1, y_2, \ldots, y_n$ for which the true alternative is serial.

Let us suppose that among our functions ϕ_0, χ_0, ψ_0, ... there are a_1 functions of one variable, a_2 of two variables, ..., and a_n of n variables, and let us order the universe by a serial relation R.

The N individuals in the universe are divided by the a_1 functions of one variable into 2^{a_1} classes according to which of these functions they make true or false, and if $N \geqslant 2^{a_1} k_1$ we can find k_1 individuals which all belong to the same class, *i.e.* agree as to which of the a_1 functions they make true and which false, where k_1 is a positive integer to be assigned later. Let us call this set of k_1 individuals Γ_{k_1}.

Now consider any two distinct members of Γ_{k_1}, z_1 and z_2 say, and let z_1 precede z_2 in the R series. Then in regard to any of the a_2 functions of two variables, ϕ_0 say, there are four possibilities. We may either

* For then P can only be true for ϕ_0, χ_0, ψ_0, ... by *completely* containing this true serial form.

have

$$(1) \quad \phi_0(z_1, z_2) \,.\, \phi_0(z_2, z_1),$$

or $$(2) \quad \phi_0(z_1, z_2) \,.\, \sim \phi_0(z_2, z_1),$$

or $$(3) \quad \sim \phi_0(z_1, z_2) \,.\, \phi_0(z_2, z_1),$$

or $$(4) \quad \sim \phi_0(z_1, z_2) \,.\, \sim \phi_0(z_2, z_1).$$

ϕ_0 thus divides the combinations two at a time of the members of Γ_{k_1} into four distinct classes according to which of these four possibilities is realised when the combination is taken as z_1, z_2 in the order in which its terms occur in the R series; and the whole set of a_2 functions of two variables divide the combinations two at a time of the members of Γ_{k_1} into 4^{a_2} classes, the combinations in each class agreeing in the possibility they realise with respect to each of the a_2 functions. Hence, by Theorem B, if $k_1 = h(2, k_2, 4^{a_2})$, Γ_{k_1} must contain a sub-class Γ_{k_2} of k_2 members such that all the pairs out of Γ_{k_2} agree in the possibilities they realise with respect to each of the a_2 functions of two variables.

We continue to reason in the same way according to the following general form :—

Consider any r distinct members of $\Gamma_{k_{r-1}}$; suppose that in the R series they have the order $z_1, z_2, \ldots, z_r$. Then with respect to any function of r variables there are $2^{r!}$ possibilities in regard to $z_1, z_2, \ldots, z_r$, and the a_r functions of r variables divide the combinations r at a time of the members of $\Gamma_{k_{r-1}}$ into $2^{r!\,a_r}$ classes. By Theorem B, if $k_{r-1} = h(r, k_r, 2^{r!\,a_r})$*, $\Gamma_{k_{r-1}}$ must contain a sub-class Γ_{k_r} of k_r members such that all the combinations r at a time of the members of Γ_{k_r} agree in the possibilities they realise with respect to each of the a_r functions of r variables.

We proceed in this way until we reach $\Gamma_{k_{n-1}}$, all combinations $n-1$ at a time of whose members agree in the possibilities they realise with respect to each of the a_{n-1} functions of $n-1$ variables. We then determine that k_{n-1} shall equal n, which fixes k_{n-2} as $h(n-1, n, 2^{(n-1)!\,a_{n-1}})$ and so on back to k_1, every k_{r-1} being determined from k_r.

If, then, $N \geqslant 2^{a_1} k_1$, the universe must contain a class $\Gamma_{k_{n-1}}$ or Γ_n (since $k_{n-1} = n$) of n members which is contained in Γ_{k_r} for every r, $r = 1, 2, \ldots, n-1$. Let its n members be, in the order given them by R, $y_1, y_2, \ldots, y_n$. Then for every r less than n, $y_1, y_2, \ldots, y_n$ are contained in Γ_{k_r} and all r combinations of them agree in the possibilities they realise

* If $a_r = 0$ we interpret $h(r, k_r, 1)$ as k_r and identify $\Gamma_{k_{r-1}}$ and Γ_{k_r}.

with respect to each function of r variables. Let $y_{s_1}, y_{s_2}, \ldots, y_{s_r}$ $(s_1 < s_2 < \ldots < s_r)$ be such a combination, and χ_0 a function of r variables. Then $y_{s_1}, y_{s_2}, \ldots, y_{s_r}$ are in the order given them by R, and so are $y_1, y_2, \ldots, y_r$; consequently the fact that these two combinations agree in the possibilities which they realise with respect to χ_0 means that χ_0 is true for the same permutations of $y_{s_1}, y_{s_2}, \ldots, y_{s_r}$ as it is of $y_1, y_2, \ldots, y_r$. The true alternative for $y_1, y_2, \ldots, y_n$ is therefore serial in χ_0, and similarly it is serial in every other function of any number r of variables*; it is therefore a serial alternative.

Our condition is, therefore, shown to be necessary in any universe of at least $2^{a_1}k_1$ members where k_1 is given by

$$\left.\begin{aligned} k_{n-1} &= n, \\ k_{r-1} &= h(r, k_r, 2^{r!a_r}) \quad \text{if} \quad a_r \neq 0 \\ &= k_r \qquad\qquad\quad \text{if} \quad a_r = 0 \end{aligned}\right\} \quad (r = n-1, n-2, \ldots, 2).$$

For universes lying between n and $2^{a_1}k_1$ we have not found a necessary and sufficient condition for the consistency of the formula, but it is evidently possible to determine by trial whether any given formula is consistent in any such universe.

III.

We will now consider what our result becomes when our formula

$$(x_1, x_2, \ldots, x_n)\, F(\phi, \chi, \psi, \ldots, =, x_1, x_2, \ldots, x_n)$$

contains in addition to identity only one function ϕ of two variables.

In this case we have two functions ϕ_0, ψ_0 given by

$$\begin{aligned} \phi_0(y_i, y_k) &= \phi(y_i, y_k) \quad (i \neq k), \\ \chi_0(y_i) \quad &= \phi(y_i, y_i), \end{aligned}$$

so that $a_1 = 1$, $a_2 = 1$, $a_r = 0$ when $r > 2$. Consequently

$$k_2 = k_3 = \ldots = k_{n-1} = n \quad \text{and} \quad k_1 = h(2, n, 4);$$

but the argument at the end of I shows that we may take instead $k_1 = n!!!$, and our necessary and sufficient condition for consistency applies to any universe with at least $2 \,.\, n!!!$ individuals.

In this simple case we can present our condition in a more striking form as follows.

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* We have shown this when $r < n$; we may also have $r = n$, but then there is nothing to prove since in a function of n variables every alternative is serial.

It is necessary and sufficient for the consistency of the formula that it should be true when ϕ is replaced by at least one of the following types of function :—

(1) The universal function $x = x \,.\, y = y.$

(2) The null function $x \neq x \,.\, y \neq y.$

(3) Identity $x = y.$

(4) Difference $x \neq y.$

(5) A serial function ordering the whole universe in a series, *i.e.* satisfying

(*a*) $(x) \sim \phi(x, x),$

(*b*) $(x, y)[x = y \vee \{\phi(x, y) \,.\, \sim \phi(y, x)\} \vee \{\phi(y, x) \,.\, \sim \phi(x, y)\}],$

(*c*) $(x, y, z)\{\sim\phi(x, y) \vee \sim \phi(y, z) \vee \phi(x, z)\}.$

(6) A function ordering the whole universe in a series, but also holding between every term and itself, *i.e.* satisfying

(*a'*) $(x)\,\phi(x, x)$

and (*b*) and (*c*) as in (5).

Types (1)–(4) include only one function each; in regard to types (5) and (6) it is immaterial what function of the type we take, since if one satisfies the formula so, we shall see, do all the others*.

We have to prove this new form of our condition by showing that P will completely contain a serial A_n if and only if it is satisfied by functions of at least one of our six types. Now an alternative in n y's is serial in χ_0 if it contains

either (i) $\chi_0(y_1) \,.\, \chi_0(y_2) \,.\, \ldots \,.\, \chi_0(y_n)$ or, for short, $\prod\limits_r \chi_0(y_r),$

or (ii) $\sim \chi_0(y_1) \,.\, \ldots \,.\, \sim \chi_0(y_n)$,, ,, $\prod\limits_r \sim \chi_0(y_r),$

but not otherwise, and it will be serial in ϕ_0 if it contains

either (*a*) $\prod\limits_{r<s} \phi_0(y_r, y_s) \,.\, \phi_0(y_s, y_r),$

or (*b*) $\prod\limits_{r<s} \phi_0(y_r, y_s) \,.\, \sim \phi_0(y_s, y_r),$

or (*c*) $\prod\limits_{r<s} \sim \phi_0(y_r, y_s) \,.\, \phi_0(y_s, y_r),$

or (*d*) $\prod\limits_{r<s} \sim \phi_0(y_r, y_s) \,.\, \sim \phi_0(y_s, y_r).$

* A result previously obtained for type (5) by Langford, *op. cit.*

There are thus altogether eight alternatives serial in both ϕ_0 and χ_0 got by combining either of (i), (ii) with any of (*a*), (*b*), (*c*), (*d*); but these eight serial alternatives only give rise to six serial forms, since the alternatives (i) (*b*) and (i) (*c*) can be obtained from one another by reversing the order of the y's and so belong to the same form, and so do the alternatives (ii) (*b*) and (ii) (*c*).

It is also easy to see that any formula completely containing one of these six serial forms will be satisfied by all functions of one of the six types according to the scheme

Form	(i) (*a*)	(i) (*b* and *c*)	(i) (*d*)	(ii) (*a*)	(ii) (*b* and *c*)	(ii) (*d*)
Type of function	1	6	3	4	5	2

and that conversely a formula satisfied by a function of one of the six types must completely contain the corresponding form. For instance, a function of type 6 will satisfy the alternative (i) (*b*) when $y_1, y_2, \ldots, y_n$ are in their order in the series determined by the function, and when $y_1, y_2, \ldots, y_n$ are in any other order the function will satisfy an alternative of the same form.

In the language of the theory of postulate systems we can interpret our universe as a class K, and conclude that a postulate system on a base (K, R) consisting only of general laws involving at most n elements will be compatible with K having as many as $2 \cdot n!!!$ members if and only if it can be satisfied by an R of one of our six types.

IV.

Let us, in conclusion, briefly indicate how to extend our method in order to determine the consistency or inconsistency of formulae of the more general type

$$(Ez_1, z_2, \ldots, z_m)(x_1, x_2, \ldots, x_n)\, F(\phi, \chi, \psi, \ldots, =, z_1, z_2, \ldots, z_m, x_1, x_2, \ldots, x_n)$$

which have in normal form both kinds of prefix, but satisfy the condition that all the prefixes of existence precede all those of generality.

As before, we can suppose F represented as a disjunction of alternatives and discard those which violate the laws of identity. Those left we can group according to the values of identity and difference for arguments drawn entirely from the z's. Such a set of values of identity and difference we can denote by $H_i\ (=, z_1, z_2, \ldots, z_m)$, and F can be put in the form

$$(H_1 . F_1) \vee (H_2 . F_2) \vee (H_3 . F_3) \vee \ldots,$$

and the whole formula is equivalent to a disjunction of formulae.

$$(Ez_1, z_2, \ldots, z_m)\{H_1(=, z_1, z_2, \ldots, z_m) \cdot (x_1, x_2, \ldots, x_n) F_1(\phi, \ldots, =, z_1, \ldots, z_m, x_1, \ldots, x_n)\}$$

$$\vee (Ez_1, z_2, \ldots, z_m)\{H_2(=, z_1, \ldots, z_m) \cdot (x_1, x_2, \ldots, x_n) F_2(\phi, \ldots, =, z_1, \ldots, z_m, x_1, \ldots, x_n)\}$$

$\vee$ etc.

Since if any one of these formulae is consistent so is their disjunction, and if their disjunction is consistent one at least of its terms must be consistent, it is enough for us to show how to determine the consistency of any one of them, say the first. In this $H_1(=, z_1, z_2, \ldots, z_m)$ is a consistent set of values of identity and difference for every pair of z's. We renumber the z's $z_1, z_2, \ldots, z_\mu$ using the same suffix for every set of z's that are identical in H_1, and our formula becomes

$$(Ez_1, z_2, \ldots, z_\mu)(x_1, x_2, \ldots, x_n) F_1(\phi, \chi, \ldots, =, z_1, z_2, \ldots, z_\mu, x_1, x_2, \ldots, x_n), \quad \text{(i)}$$

in which it is understood that two z's with different suffixes are always different.

Now supposing the universe to have at least $\mu+n$ members, we consider the different possibilities in regard to the x's being identical with the z's, and rewrite our formula

$$(Ez_1, z_2, \ldots, z_\mu)(x_1, x_2, \ldots, x_n)\left\{\prod_{\substack{i=1,\ldots,n \\ j=1,\ldots,\mu}} x_i \neq z_j \rightarrow G(\phi, \chi, \ldots, =, z_1, \ldots, z_\mu, x_1, \ldots, x_n)\right\},$$

in which $\rightarrow$ means "if, then" and

$$G(\phi, \ldots, x_n) = \Pi F_1(\phi, \chi, \ldots, =, z_1, \ldots, z_\mu, \theta_1, \theta_2, \ldots, \theta_n),$$

the product being taken for

$$\theta_1 = x_1, z_1, z_2, \ldots, z_\mu,$$

$$\theta_2 = x_2, z_1, z_2, \ldots, z_\mu,$$

$$\ldots \quad \ldots \quad \ldots$$

$$\theta_n = x_n, z_1, z_2, \ldots, z_\mu,$$

and in G any term $x_i = z_j$ is replaced by a falsehood (*e.g.* $x_i \neq x_i$) not involving any z.

Next we modify G by introducing new functions. In G occur values of, *e.g.* ϕ, with arguments some of which are z's and some x's; from

these we define functions of the x's only by simply regarding the z's as constants, and call these new functions ϕ_0, χ_0, ... Values of ϕ, χ, ψ, ..., which include no x's among their arguments, we replace by constant propositions p, q, ... The only values of identity in G are of the form $x_i = x_j$ and these we leave alone. Suppose that by this process G turns into

$$L(\phi, \chi, \psi, \ldots, \phi_0, \chi_0, \ldots, p, q, \ldots, =, x_1, x_2, \ldots, x_n).$$

Then the consistency of formula (i) in a universe of N individuals is evidently equivalent to the consistency in a universe of $N-\mu$ individuals of the formula

$$(x_1, x_2, \ldots, x_n)\, L(\phi, \chi, \psi, \ldots, \phi_0, \chi_0, \ldots, p, q, \ldots, =, x_1, x_2, \ldots, x_n).$$

But this is a formula of the type previously dealt with, except for the variable propositions p, q, ..., which are easily eliminated by considering the different cases of their truth and falsity, the formula being consistent if it is consistent in one such case.

ON THE SOLUTIONS OF MALFATTI'S PROBLEM FOR A TRIANGLE

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[Received and read 10 January, 1929.—Received in revised form 12 February, 1929.]

The literature of Malfatti's problem.

1. Having obtained certain trigonometrical results, we naturally turned for a general account of this famous problem and its history, with references, to the *Encyklopädie d. math. Wissenschaften*; we also consulted the "Treatise on the circle and the sphere", by Professor J. L. Coolidge, and other books. The impression conveyed to us was that, apart from purely geometrical solutions, the investigations were carried out almost universally by means of *algebraical* formulae involving a number of square roots, each with an ambiguity of sign. We felt this to be clumsy and confusing, particularly in the matter of the systematic study of the various solutions, and we knew it to be unnecessary. (Subsidiary angles, used in the solution given in Hobson's "Trigonometry", are superfluous; and this solution is open to criticism on other grounds.)

From the Royal Society's Catalogue of Scientific Papers, Vol. I, page 444, and the *Jahrbuch u. d. Fortschritte d. Mathematik*, we compiled a much fuller list of original writings. On reading these we found that (as was to be expected) the value of trigonometry had not been overlooked. Even before the publication of Steiner's classical memoir* in 1826 (in which the problem was solved by pure geometry, the proof being withheld), Lechmütz† (Lehmus) in 1819 used trigonometry to establish results stated by Malfatti in the original paper of 1803. Formulae free from surds and ambiguities were given by Catalan‡ in

* *Journal für Math.*, 1 (1826), 161–184; 252–288. Especially Section 4.

† Gergonne, *Annales de Math.*, 10 (1819).

‡ *Nouvelles Annales de Math.*, 5 (1845), 60.

1845; they are less convenient than those which we obtain, but are at bottom equivalent to them. But by far the most important memoir was published in 1895 by Professor J. Derousseau*, who used Catalan's formulae. In this paper the system of thirty-two solutions, hinted at by Lehmus and foreseen by Steiner, was systematically considered, typical solutions being worked out *ab initio*, each on its merits. We remark with regret that neither Catalan nor Derousseau is included in the list of authors cited in the *Encyklopädie*. A subsequent paper by Pampuch, from which typical formulae are quoted by Coolidge, is mentioned with approval. In these, as in the rest of Coolidge's very interesting account of the problem, the surds make the work needlessly difficult and unattractive. Derousseau's memoir, which extends to fifty pages, opens with the best account extant of the early development of the problem; unfortunately it was reviewed in the *Fortschritte* as a contribution to the history of mathematics rather than as a notable advance in knowledge. The memoir also contains many figures, valuable because it is far from easy to grasp the way in which the various sets of circles are disposed. That the memoir should have been overlooked by Coolidge and others is not surprising; but the reason is none the less to be deplored.

In the present paper we first obtain a solution of Malfatti's original problem in a form free from ambiguities. The result is bound to resemble other such solutions, but the method appears to us the natural one to apply to the equations which present themselves, and the result is simple. It is our main purpose to show that *the thirty-two solutions can be derived from this by means of a group of substitutions without any further work*. Incidentally it is noticed that some explanation is necessary before the statement that there are thirty-two solutions and no more can be accepted.

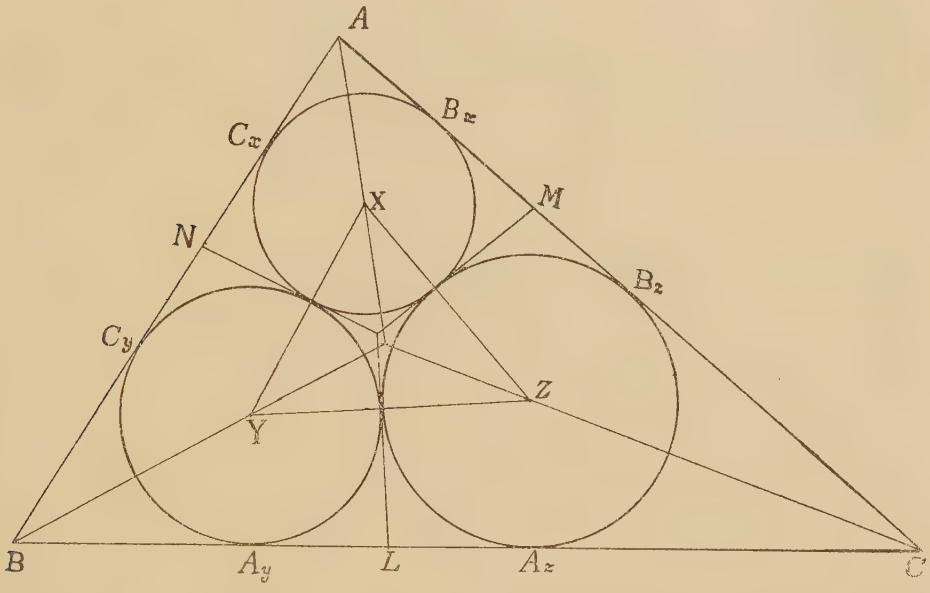
As a matter of some slight historical interest† we quote the following:—"Dato un Prisma retto triangolare di qualunque materia come di marmo, cavare da esso tre Cilindri dell' altezza del Prisma e della maggior grossezza possibile correspettivamente, e in conseguenza col minor avanzo possibile di materia avuto riguardo alla voluta grossezza". This is the original enunciation by Malfatti of the problem which bears his name. The importance of the problem, of course, lies altogether in the geometry; Malfatti himself and all the mathematicians who have written upon it recognize this. But the enunciation makes it plain that the

* *Mémoires de la Soc. R. des Sciences de Liége* (2), 18 (1895), 1–52.

† *Memorie di matematica e di fisica della Società italiana delle scienze*, 10 (1803), 235.

original problem conceived by Malfatti requires that the three circles (the bases of the cylinders) shall lie within the triangle (the base of the prism). This we proceed to solve.

2. *The original problem.* "Within a given triangle to describe three circles, each of which touches two sides of the triangle and the other two circles".



Let ABC be the given triangle. The bisectors of the angles AI, BI, CI meet in I the centre of the inscribed circle, whose radius is denoted by r. Let X, Y, Z, lying upon AI, BI, CI respectively, be the centres of the required circles, and let C_x, C_y be the points of contact of AB with the circles whose centres are X and Y. Similarly for A_y, A_z, B_z, B_x.

When two circles touch externally the length of their common tangent is the geometric mean between their diameters. In order to express the length C_xC_y without radicals, we denote the radii of the circles by x^2r, y^2r, z^2r, so that C_xC_y is $2rxy$. Thus x^2, y^2, z^2 are numbers, for which we expect to find positive values less than unity. Now

$$r(\cot\tfrac{1}{2}A+\cot\tfrac{1}{2}B) = AB = AC_x+C_xC_y+C_yB$$
$$= rx^2\cot\tfrac{1}{2}A+2rxy+ry^2\cot\tfrac{1}{2}B.$$

Hence, if l, m, n stand for $\cot\frac{1}{2}A$, $\cot\frac{1}{2}B$, $\cot\frac{1}{2}C$, the conditions to be

satisfied by x, y, z are

$$\left.\begin{aligned} my^2+2yz+nz^2 = m+n, \quad nz^2+2zx+lx^2 = n+l, \\ lx^2+2xy+my^2 = l+m \end{aligned}\right\}, \tag{i}$$

where

$$lmn = l+m+n. \tag{ii}$$

3. Equations (i) lead at once to two homogeneous relations of the second degree in x, y, z. The standard method of solution used in analytical geometry of conics for two such equations is to express them by help of the roots of the discriminating cubic as sums of the squares of three determinate linear functions. Multiply equations (i) by constants L, M, N, such that

$$L(m+n)+M(n+l)+N(l+m) = 0. \tag{iii}$$

Adding, we obtain the homogeneous relation

$$l(M+N)x^2+m(N+L)y^2+n(L+M)z^2+2Lyz+2Mzx+2Nxy = 0. \tag{iv}$$

The discriminant of this is [since, by (ii), $lmn = l+m+n$],

$$\begin{aligned} &(l+m+n)(M+N)(N+L)(L+M)+2LMN-l(M+N)L^2 \\ &\qquad -m(N+L)M^2-n(L+M)N^2 \\ &= [l(M+N)+m(N+L)+n(L+M)](MN+NL+LM)+2LMN \\ &= 2LMN, \end{aligned}$$

in virtue of (iii). If we speak of x, y, z as if they were homogeneous coordinates of a point in a plane, the left-hand side of (iv) factorizes into two straight lines if any one of the three L, M, N vanishes, the ratio of the other two being determined from (iii). The differentials of (iv) vanish at the points of intersection of the two lines, which are the vertices of the self-polar triangle of the conics (iv), viz. the points

$$\begin{aligned} -x = my = nz \quad &\text{when} \quad L = 0, \\ -y = nz = lx \quad &\text{when} \quad M = 0, \\ -z = lx = my \quad &\text{when} \quad N = 0. \end{aligned}$$

The equations of the sides of the self-polar triangle are now found, viz

$$\begin{aligned} \xi &\equiv (m+n)x+m(1+n)y+n(1+m)z = 0, \\ \eta &\equiv (n+l)y +n(1+l)z +l(1+n)x = 0, \\ \zeta &\equiv (l+m)z +l(1+m)x +m(1+l)y = 0, \end{aligned}$$

from which we deduce that

$$2lx = -l\xi+\eta+\zeta, \quad \text{etc., etc.}$$

The left-hand side of (iv), whenever (iii) is satisfied, can be expressed as a sum of multiples of ξ^2, η^2, ζ^2; and, in particular, we know that, if $L=0$, ξ^2 will not appear. But, when $L=0$, (iv) contains no term in yz and must be equivalent to

$$\eta^2 : \zeta^2 :: n(n+l) : m(l+m);$$

similar results are obtained when $M=0$ and $N=0$. Hence the values of $x:y:z$, which satisfy equations (i), must be those which make

$$\xi^2 : \eta^2 : \zeta^2 :: (m+n)\,mn : (n+l)\,nl : (l+m)\,lm$$
$$:: \sin^2\tfrac{1}{2}A : \sin^2\tfrac{1}{2}B : \sin^2\tfrac{1}{2}C, \qquad \text{(v)}$$

when their values, $\cot\frac{1}{2}A$, $\cot\frac{1}{2}B$, $\cot\frac{1}{2}C$, are restored to l, m, n. There are, therefore, four possible sets of values of $\xi:\eta:\zeta$, and of $x:y:z$. By reason of the symmetry we first consider the set in which

$$\xi : \eta : \zeta :: \sin\tfrac{1}{2}A : \sin\tfrac{1}{2}B : \sin\tfrac{1}{2}C.$$

4. Since $2lx = -l\xi+\eta+\zeta$, with similar relations for y and z, it follows that x, y, z are proportional to three analogous functions of the angles A, B, C, of which the first (omitting a factor of proportionality) is

$$2x = -\xi+(\eta+\zeta)\tan\tfrac{1}{2}A = \sin\tfrac{1}{2}A\left[-1+\frac{\sin\frac{1}{2}B+\sin\frac{1}{2}C}{\sin\frac{1}{2}(B+C)}\right]$$
$$= \sin\tfrac{1}{2}A\left[-1+\frac{\cos\frac{1}{4}(B-C)}{\cos\frac{1}{4}(B+C)}\right]$$
$$= 4\sin\tfrac{1}{4}A\cos\tfrac{1}{4}A\sin\tfrac{1}{4}B\sin\tfrac{1}{4}C\sec\tfrac{1}{4}(B+C).$$

Hence

$$x : y : z :: \cos\tfrac{1}{4}A\sec\tfrac{1}{4}(B+C) : \cos\tfrac{1}{4}B\sec\tfrac{1}{4}(C+A) : \cos\tfrac{1}{4}C\sec\tfrac{1}{4}(A+B),$$

or

$$yz : zx : xy :: \sec\tfrac{1}{4}A\cos\tfrac{1}{4}(B+C) : \sec\tfrac{1}{4}B\cos\tfrac{1}{4}(C+A) : \sec\tfrac{1}{4}C\cos\tfrac{1}{4}(A+B),$$
$$:: 1+\tan\tfrac{1}{4}A : 1+\tan\tfrac{1}{4}B : 1+\tan\tfrac{1}{4}C.$$

To find the actual values of yz, zx, xy, x^2, y^2, z^2, assume that

$$yz = k(1+\tan\tfrac{1}{4}A), \quad zx = k(1+\tan\tfrac{1}{4}B), \quad xy = k(1+\tan\tfrac{1}{4}C),$$

so that

$$x^2 = k(1+\tan\tfrac{1}{4}B)(1+\tan\tfrac{1}{4}C)/(1+\tan\tfrac{1}{4}A), \quad y^2 = \ldots,$$

and substitute in any of equations (i). The value of k is found to be $\frac{1}{2}$, so that we have finally the simple values of yz, zx, xy.

$$\left.\begin{aligned} 2yz &= 1+\tan\tfrac{1}{4}A, \quad 2zx = 1+\tan\tfrac{1}{4}B, \quad 2xy = 1+\tan\tfrac{1}{4}C, \\ \text{from which we deduce that} \\ 2x^2 &= (1+\tan\tfrac{1}{4}B)(1+\tan\tfrac{1}{4}C)/(1+\tan\tfrac{1}{4}A), \\ \text{with similar values for } y^2 &\text{ and } z^2. \end{aligned}\right\} \quad \text{(vi)}$$

These values of x^2, y^2, z^2 are positive, and we can see that they are less than unity. For if A is diminished, and, at the same time, B and C are increased so that $A+B+C$ remains equal to π, both operations increase the value of x^2; but if A is diminished to zero, and therefore $B+C=\pi$, the value of x^2 is 1. In any actual triangle x^2, y^2, z^2 are less than unity: the three circles touch the sides and one another in the prescribed manner and lie within the triangle. It follows that

Malfatti's original problem is solved by equations (vi).

5. Catalan and Derousseau preferred to use angles α, β, γ, the complements of $\frac{1}{2}A$, $\frac{1}{2}B$, $\frac{1}{2}C$. They showed that

$$C_xC_y = 2r/(1+\tan\tfrac{1}{2}\gamma),$$

a result in agreement with (vi), but less convenient on account of its fractional form. If, for example, we wish to calculate the distances of the points of contact from the angular points, AC_x, etc., equations (vi) and the three linear equations similar to

$$AC_x+C_xC_y+C_yB = AB,$$

lead immediately to the expressions

$$\begin{aligned} AC_x = AB_x &= s-a-\tfrac{1}{2}r(1-\tan\tfrac{1}{4}A+\tan\tfrac{1}{4}B+\tan\tfrac{1}{4}C) \\ &= \tfrac{1}{2}r(\cot\tfrac{1}{4}A-1-\tan\tfrac{1}{4}B-\tan\tfrac{1}{4}C). \end{aligned}$$

Also it may be observed that, although quarter angles carry little significance in connection with the ordinary geometry of a triangle, yet it is easy to modify the formulae so as to remedy the fault. For if the inscribed circle of ABC touches the sides BC, CA, AB at D, E, F,

$$r\tan\tfrac{1}{4}A = IE(\operatorname{cosec}\tfrac{1}{2}A-\cot\tfrac{1}{2}A) = AI-AE.$$

On introducing these in the last formula we arrive at the values of AC_x

and AB_x, which were originally given by Malfatti, and proved by Lehmus. But the quarter angles give the most concise and convenient formulae.

These and similar results can be made to suggest other constructions for solving the problem; yet it is doubtful whether the beautiful geometry of the method discovered by Steiner and proved by Hart will be surpassed. A purpose to which, upon consideration, our formulae appear to us to be better adapted is the careful investigation of the wider problem of the triangle, in which the conditions that the circles should lie inside the triangle are discarded, those of contact between the sides and the circles being retained. Other extensions have been suggested, but we shall confine ourselves to those which concern a plane rectilinear triangle. We note that the wording of the enunciation is sometimes loose; thus it is tacitly assumed that each of the three circles must touch a different pair of sides, a fact that should be stated explicitly. First consider the following:—"A triangle ABC being given, it is required to construct in as many ways as possible a set of three circles touching respectively AB and AC, BC and BA, CA and CB, so that each circle may touch the other two". It has been stated repeatedly that this problem has thirty-two solutions: actually it has far more. The circle inscribed in ABC touches AB and AC; we have denoted its point of contact with BC by D, and we can draw one circle touching BC at D and AB produced, and another touching BC at D and AC produced. These three circles fulfil all the conditions. Again, the circles inscribed in ABD and ACD touch one another on AD, and their points of contact with AB and AC are equidistant from A; a third circle touches AB, AC, and the two circles at these points, forming a set of three which satisfy the conditions. To rule out these unwanted solutions we shall say that the nine points of contact must be distinct. Even so, there are sixty-four solutions left; but it will be shown that if we limit ourselves to real circles we have the group of thirty-two sets which have so often been studied, since Steiner directed attention to them.

6. *The extended problem for a triangle.* "A triangle ABC being given, it is required to construct in as many ways as possible a set of three circles, which touch respectively AB and AC, BC and BA, CA and CB, so that each circle shall touch the two others; further, the nine points of contact are to be distinct, and the circles are to be real".

The circles which touch AB and AC fall into two families, according as their centres lie upon the internal or external bisector of the angle A, and a selection of one bisector of each angle of the triangle ABC can be made in eight ways. In four selections the bisectors are concurrent; in

the other four they are not. The solutions therefore fall into eight classes, four of which differ fundamentally from the other four. Each class will be found to contain eight solutions, all real when the bisectors are concurrent and all imaginary when they are not. The latter solutions are often overlooked altogether*.

When two real circles touch a straight line at two different points, and also touch one another, their centres lie upon the same side of the straight line. Let $(\alpha_1, \beta_1, \gamma_1)$ $(\alpha_2, \beta_2, \gamma_2)$ $(\alpha_3, \beta_3, \gamma_3)$ be the perpendiculars from the centres of the three circles upon the sides of ABC, with the usual convention as to sign of trilinear coordinates. For the reason just stated α_2/α_3, β_3/β_1, γ_1/γ_2 are positive; also, since the centres lie upon the bisectors of the angles, $\beta_1 = \pm\gamma_1$, $\gamma_2 = \pm\alpha_2$, $\alpha_3 = \pm\beta_3$. Hence the ambiguous signs are either all plus or else are two minus and one plus. This proves that we may dismiss the possibility of the centres lying on three non-concurrent bisectors of the angles of ABC if the circles are to be real.

We notice also that the circles must touch one another externally.

7. The bisectors of the angles of ABC upon which the centres of the three Malfatti circles lie meet either in the centre of the inscribed circle of ABC or in that of one of the escribed circles: thus the solutions of the problem we have now to consider fall into four distinct classes. The first of these classes was partly discussed in §§ 2–4; four solutions were discovered, and one was worked out further to give a solution of the original problem. We will examine this *ab initio*, discarding the notion that the circles are to lie within the triangle; the same figure and notation may be used; but the trilinear coordinates of the centres mentioned in § 6 will help to simplify the explanations.

Let the centres be $X(\alpha_1, \beta_1, \gamma_1)$, $Y(\alpha_2, \beta_2, \gamma_2)$, $Z(\alpha_3, \beta_3, \gamma_3)$. Then $\beta_1 = \gamma_1$, $\gamma_2 = \alpha_2$, $\alpha_3 = \beta_3$; also, since by § 6, X and Y lie on the same side of AB, etc., these six quantities are all of the same sign. But if all were negative each circle would lie wholly in one of the regions in which two coordinates are negative; such regions nowhere overlap, and contact between the circles is impossible. The six quantities are all positive and are the radii of the three Malfatti circles. As in § 2, we denote them by rx^2, ry^2, rz^2, r being the radius of the inscribed circle of ABC, and x, y, z being real numbers positive or negative; their signs at present are immaterial.

* Steiner himself excluded the extraneous solutions of Section 5, but did not notice the imaginary ones. They become prominent only in algebraic methods of solution.

The condition of tangency of the circles is that XY shall be equal to the sum of the radii, rx^2+ry^2, *i.e.* that

$$(rx^2+ry^2)^2 = (c-rlx^2-rmy^2)^2+(rx^2-ry^2)^2,$$

where l, m, n, as before, denote $\cot\frac{1}{2}A$, etc., and c is the length AB.

Or $$4r^2x^2y^2 = (c-rlx^2-rmy^2)^2 = r^2(l+m-lx^2-my^2)^2,$$

since $c = rl+rm$.

Or $$lx^2+my^2 \pm 2xy = l+m.$$

To solve the extended problem in the case when the centres lie upon the internal bisectors of the angles of ABC, we need do no more than modify equations (i) of § 2 by writing $\pm$ instead of $+$ before the terms $2yz$, $2zx$, $2xy$. Further, since the signs of x, y, z are not yet defined, we can, in all cases, vary them so that the ambiguities become either all $+$ or all $-$. We have only to supplement the equations (i) by the very similar set

$$\left.\begin{aligned} my^2-2yz+nz^2 = m+n, \quad nz^2-2zx+lx^2 = n+l, \\ lx^2-2xy+my^2 = l+m \end{aligned}\right\}. \qquad \text{(vii)}$$

The reduction and solution of this second set (vii) follows step by step that of (i). Equation (ii) remains. Under conditions (iii) the discriminant is found to be $-2LMN$. The vertices of the self-polar triangle are

$$x = my = nz, \text{ etc.}$$

Its sides are

$$\xi \equiv (m+n)x+m(1-n)y+n(1-m)z = 0, \text{ etc.}$$

The four common values are again determined by help of an equation identical in form with (v) in the modified values of ξ, η, ζ; and when we select once more the symmetrical case by taking the positive square roots we arrive at a result which bears an extraordinarily close resemblance* to (vi),

$$\left.\begin{aligned} 2yz = 1+\cot\tfrac{1}{4}A, \quad 2zx = 1+\cot\tfrac{1}{4}B, \quad 2xy = 1+\cot\tfrac{1}{4}C, \\ 2x^2 = (1+\cot\tfrac{1}{4}B)(1+\cot\tfrac{1}{4}C)/(1+\cot\tfrac{1}{4}A) \end{aligned}\right\} \qquad \text{(viii)}$$

with similar results for y^2 and z^2.

---

* The distance from A to the points of contact with AB or AC of the circle whose centre is X is found to be

$$\tfrac{1}{2}r(1-\tan\tfrac{1}{4}A+\cot\tfrac{1}{4}B+\cot\tfrac{1}{4}C).$$

Equations (i) and (vii) prove that there are just eight solutions of the problem when the centres of the circles lie upon the internal bisectors of the angles of ABC. We must complete the solutions and see whether they are real. This, however, can be done at once.

8. *The first eight solutions.* (The solutions in which the centres of the circles lie upon the internal bisectors of the angles of the triangle.) In the two solutions (vi) and (viii) we selected values of ξ, η, ζ, and succeeded in reducing them to very simple forms because we could use the relation $A+B+C=\pi$. If we can express the other values of ξ, η, ζ, which we passed over, in terms of new angles whose sum is π in the same way, the same reduction will be possible, and the final form can be derived from (vi) and (viii) by substituting the new angles for A, B, C.

The first eight solutions will be obtained if we substitute in (vi) and (viii) for A, B, C one of the following sets of angles :—

$$A,\ B\pm 2\pi,\ C\mp 2\pi; \quad A\mp 2\pi,\ B,\ C\pm 2\pi; \quad A\pm 2\pi,\ B\mp 2\pi,\ C;$$

for we produce the required changes of sign in the values ξ, η, ζ, and the sum of the angles in each set is π. The trigonometrical functions involved in the solutions (vi) and (viii), (viz. tangents of the quarter angles, sines, cosines, etc., of the half angles of the triangle), all have period 4π. To increase A or B or C by 4π makes no change : to increase them by 2π and to diminish them by 2π makes the same change. To complete the set of eight solutions we may add to A, B, C in all possible ways multiples of 2π, so that the sum remains equal to π, multiples of 4π being disregarded*.

But it is possible to go further and deduce all these solutions from (vi), the solution of the original problem. Consider the effect upon (vi) of substituting for A, B, C, three other angles whose sum is π,

$$-2\pi-A, \quad 2\pi-B, \quad 2\pi-C;$$

or better,

$$2\pi-A, \quad 2\pi-B, \quad 2\pi-C,$$

which because of the periodicity are equivalent to the former. The tangents of $\frac{1}{4}A$, $\frac{1}{4}B$, $\frac{1}{4}C$ are replaced by the cotangents of the same angles, and thus we arrive at solution (viii). With this substitution we may combine the three substitutions quoted earlier in this section, and so

* In the four solutions derived from (i) and (ii), we see that the four values of AC_x and AB_x given in §5 are the same as the four values of BC_y and BA_y and of CA_z and CB_z.

derive all the eight solutions from (vi) by means of substitutions of two types : either

$$A+2\alpha\pi, \qquad B+2\beta\pi, \qquad C+2\gamma\pi;$$

or

$$-A+2\alpha'\pi, \quad -B+2\beta'\pi, \quad -C+2\gamma'\pi;$$

where α, β, γ, α', β', γ' are integers positive or negative, so chosen that the sum of the three angles under consideration is either π or differs from π by a multiple of 4π. Moreover, no more than the eight are derived.

To put the matter precisely, we number the eight solutions (1), (2), (3), (4), (5), (6), (7), (8), the solution (vi) being (1), and the solution (viii) being (5); then the angles which, if substituted for A, B, C in formula (vi), lead to each solution are

(1)	A,	B,	C;	(5)	$-A+2\pi$,	$-B+2\pi$,	$-C+2\pi$;
(2)	A,	$B+2\pi$,	$C+2\pi$;	(6)	$-A+2\pi$,	$-B$,	$-C$;
(3)	$A+2\pi$,	B,	$C+2\pi$;	(7)	$-A$,	$-B+2\pi$,	$-C$;
(4)	$A+2\pi$,	$B+2\pi$,	C;	(8)	$-A$,	$-B$,	$-C+2\pi$.

The sum of each set of three angles $\equiv\pi \pmod{4\pi}$.

9. A geometrical view of this process makes its inner meaning clearer, and also suggests an extension of the principle which solves the problem completely. It will be shown first that all the thirty-two solutions, like the first eight, can be derived by a group of simple substitutions from the two solutions (vi) and (viii); and then that they can all be derived by a wider group of substitutions from the solution (vi) of the original problem, or indeed from any one solution.

Let the circle with centre I and radius r inscribed in ABC be drawn, and let D, E, F be its points of contact with the sides BC, CA, AB. The circle is to remain fixed, while the triangle is to vary.

First let E, F, and the tangents at E and F remain fixed, while D moves round the circle, carrying its tangent line BDC with it; let D pass first through E, then through F, and finally return to its starting point after making a complete circuit. The angle A has not changed, but C has steadily increased, and B has steadily diminished by 2π; yet the final triangle ABC is identical with the initial triangle. Every feature of the triangle has undergone a continuous change, which can be visualized and followed out in the formulae, and can be examined at critical positions. At the conclusion of the circuit most of the points, lines, or loci connected with the triangle, and most of the measurements, will have re-

sumed their original positions or values; the algebraic or trigonometrical formulae which define them reappear unchanged. But others (like the Malfatti circles, which depend upon fractional parts of the angles) retain their geometrical properties (which they have, in fact, never lost), but do not return to their initial position or form. In these the process deduces a new version of a theorem or a new solution of a problem.

In Malfatti's problem, by allowing each side in turn to move round the circle, the other two sides being fixed, from the solution (vi) or (1) we derive (2), (3), (4); and from the solution (viii) or (5) we derive (6), (7), (8), thus completing the system which we have described as the first eight solutions. To allow a side to make two revolutions brings back the initial configuration: to allow the three sides successively to make a revolution does the same. These are results already established in § 8.

10. *The* 32 *solutions derived from the solutions* (vi) *and* (viii).

It will now be shown that the geometrical point of view adopted in § 9 enables us to derive from one solution a set of sixteen, and from the two solutions (vi) and (viii) to derive the complete system of thirty-two foreseen by Steiner. *Suppose that the point D makes half a complete revolution.* Let DID', EIE', FIF' be the diameters of the circle through D, E, F, and suppose that D has moved past E as far as D', so that the tangent, having turned through two right angles, has taken up a position parallel to its initial position. We have a small triangle with vertex A similar to ABC; the angle A has not varied, but B has diminished by π, and C has increased by π. The two sets of three Malfatti circles (vi) and (viii) of the initial triangle have varied continuously, always touching one another and the sides of the changing triangle. The novel features are that now the final triangle is on a reduced scale from, though similar to, the initial triangle, and that the circle has ceased to be the inscribed, and has become an escribed, circle. Imagine the figure enlarged, with A as centre of similitude, until the little triangle is brought to coincide with the initial triangle ABC; the circle then is enlarged into the escribed circle opposite to A, and the three Malfatti circles have their centres on the bisectors of the angles which meet in the centre of that circle, but still satisfy the nine conditions for contact with the sides and one another.

To bring this into touch with the results (vi) and (viii) and so determine the radii of the circles, etc., we must first substitute A, $B-\pi$, $C+\pi$ for A, B, C, in order to find the value of yz, zx, xy, x^2, y^2, z^2; we must then multiply, not by r, but by r_1, the radius of the escribed circle opposite to A.

11. *The group of sixteen substitutions.* The substitutions given at the end of § 8 enable us to complete a system of eight solutions in which the centres of the Malfatti circles lie upon those bisectors of the angles of ABC which meet in the escribed centre opposite to A; symmetry supplies similar results for the other escribed centres, making up a total of 32 solutions. Each solution is associated with a substitution, *i.e.* with a set of three angles which take the places of A, B, C in the formulae of §§ 2–4 and later work, and to classify the substitutions is a simple means of classifying the solutions. The 32 substitutions (like the 8 in § 8) fall into two entirely distinct classes; in sixteen the angles A, B, C enter with positive signs, in the remaining sixteen they enter with negative signs. It is convenient to call these the "plus" group and the "minus" group. Although the minus group has almost identical properties, we shall first discuss only the plus group of sixteen substitutions.

All our formulae, and therefore all the substitutions, have a period 4π, as has been noticed previously; the words "to modulus 4π" or "(mod 4π)" must be written or understood in connection with nearly all statements. The substitutions 2, 3, 4, and that quoted at the end of § 10 belong to the plus group, and it is clear from these that the effect of a substitution of the plus group is to increase or diminish A, B, C by integer multiples of π, so chosen that the sum of the angles is not altered. Taking the periodicity into account we say that, for A, B, C, we substitute.

$$A+\alpha\pi, \quad B+\beta\pi, \quad C+\gamma\pi, \quad \text{where} \quad \alpha+\beta+\gamma \equiv 0 \pmod 4.$$

The values of α, β, γ may be restricted to 0, 1, 2, 3; and when two of the three are known the third is determined. Therefore there are just sixteen possibilities. We may, if we like, regard

$$S_1(A,\ B,\ C) = (A+2\pi,\ B+\pi,\ C+\pi);$$

$$S_2(A,\ B,\ C) = (A+\pi,\ B+2\pi,\ C+\pi);$$

$$S_3(A,\ B,\ C) = (A+\pi,\ B+\pi,\ C+2\pi);$$

$$S_1^4 = 1, \quad S_2^4 = 1, \quad S_3^4 = 1, \quad S_1 S_2 S_3 = 1,$$

as fundamental operations, and express the sixteen substitutions as

$$(1,\ S_1^2,\ S_2^2,\ S_3^2); \quad S_1(1,\ S_1^2,\ S_2^2,\ S_3^2); \quad S_2(1,\ S_1^2,\ S_2^2,\ S_3^2); \quad S_3(1,\ S_1^2,\ S_2^2,\ S_3^2);$$

a scheme which divides them into four classes in a way that harmonizes well with the geometry. But the simplest plan for identifying the various substitutions is probably to quote for each the values of α, β, γ, enclosing

them in a bracket; thus (211), (022) denote the substitutions $(A+2\pi, B+\pi, C+\pi)$, $(A, B+2\pi, C+2\pi)$ respectively. The sixteen substitutions, arranged in four classes in the same order as before, are

	$(\alpha\beta\gamma)$,	$(\alpha\beta\gamma)$,	$(\alpha\beta\gamma)$,	$(\alpha\beta\gamma)$	
Class 1:	(000),	(022),	(202),	(220)	$2yz = 1+\tan\frac{1}{4}(A+\alpha\pi)$,
Class 2:	(211),	(233),	(013),	(031)	$2zx = 1+\tan\frac{1}{4}(B+\beta\pi)$,
Class 3:	(121),	(103),	(323),	(301)	$2xy = 1+\tan\frac{1}{4}(C+\gamma\pi)$.
Class 4:	(112),	(130),	(310),	(332)	

In Class 1 the centres lie upon the internal bisectors of the angles, and the radii, etc., are found by multiplying x^2, etc., by r. In Class 2 they lie upon the bisectors which meet in the first escribed centre, and x^2, etc., must be multiplied by r_1. Classes 3 and 4 stand in the same relation to the second and third escribed centres and radii. These sixteen operations applied to the solutions (vi) and (viii) provide the 32 solutions foretold by Steiner.

12. *The group of* 32 *substitutions which derive all the solutions from the original solution.* We must now consider the bearing of the fact noted in § 8, that the solution (viii) is itself derived from (vi) by the substitution of

$$2\pi-A, \quad 2\pi-B, \quad 2\pi-C$$

for A, B, C. Clearly, from each of the sixteen substitutions tabulated above, which constitute what we have called the plus group, we may derive one substitution of the minus group by subtracting each of the angles involved from 2π, paying due regard to periodicity. The substitutions of the plus and minus groups and the solutions derived from them are thus paired off, one with one. Two substitutions, one from the plus group and one from the minus group, in which for A, B, C we write

$$A+\alpha\pi, \qquad B+\beta\pi, \qquad C+\gamma\pi;$$
$$-A+\alpha'\pi, \quad -B+\beta'\pi, \quad -C+\gamma'\pi;$$

respectively, "correspond" or "form a pair" if

$$\alpha+\alpha' \equiv \beta+\beta' \equiv \gamma+\gamma' \equiv 2 \pmod 4.$$

In order to identify the various substitutions of the minus group the simplest plan again appears to be merely to quote in brackets the values

$(\alpha', \beta', \gamma')$, as was done in the plus group. No confusion arises because

$$\alpha+\beta+\gamma \equiv 0 \pmod 4, \text{ in the plus group;}$$

$$\alpha'+\beta'+\gamma' \equiv 2 \pmod 4, \text{ in the minus group.}$$

For example, the following are "paired" substitutions, the first of each pair belonging to the plus group and the second to the minus group :—

(202), (020); (301), (321); (112), (110); (233), (033).

13. We are now in a position to draw up a table of the thirty-two solutions, with the substitutions by which each can be derived from a chosen *standard solution*. The history and the geometry of the problem alike point to the solution (vi) as the standard solution; trigonometrically (vi) and the solution (viii) which is paired with it are simpler than any other. Beyond doubt (vi) must be the standard solution. [But all the solutions can equally well be derived from any one solution.]

Having quoted the standard solution, we give concisely a list or table of the complete system of solutions. We retain the notation yz, zx, xy, x^2, y^2, z^2 for the numbers which have become familiar in §§ 2–7, but for the sake of uniformity we denote by ρ the length of the line (the radius of the inscribed or of an escribed circle) into which these numbers must be multiplied. It must be remembered that X, Y, Z, the centres of the three Malfatti circles, not only lie on three concurrent bisectors of the angles A, B, C, but either lie all on the same sides of A, B, C, as the point of concurrence, or all on the opposite sides. The latter occurs if one and only one of $\alpha\beta\gamma$ has the value 2, or if one and only one of $\alpha'\beta'\gamma'$ has the value 3, and is distinguished by the values of x^2, y^2, z^2 found from the formulae being negative.

14. *The standard solution.*

(i) X, Y, Z, the centres of the three Malfatti circles, lie upon AI, BI, CI, I being the centre of the inscribed circle of ABC.

(ii) The radii are rx^2, ry^2, rz^2, r being the radius of that circle.

(iii) The lengths intercepted on BC, CA, AB between the points of contact, or the projections of YZ, ZX, XY upon BC, CA, AB are $2ryz$, $2rzx$, $2rxy$.

(iv) $2yz = 1+\tan\frac{1}{4}A$, $\quad 2zx = 1+\tan\frac{1}{4}B$, $\quad 2xy = 1+\tan\frac{1}{4}C$,

whence $$2x^2 = (1+\tan\tfrac{1}{4}B)(1+\tan\tfrac{1}{4}C)/(1+\tan\tfrac{1}{4}A),$$

$$2y^2 = (1+\tan\tfrac{1}{4}C)(1+\tan\tfrac{1}{4}A)/(1+\tan\tfrac{1}{4}B).$$

$$2z^2 = (1+\tan\tfrac{1}{4}A)(1+\tan\tfrac{1}{4}B)/(1+\tan\tfrac{1}{4}C).$$

The figure and all its dimensions are thus defined without ambiguity.

15. *The complete system of thirty-two solutions.*

In the given triangle ABC, I, I_1, I_2, I_3 are the centres and r, r_1, r_2, r_3 are the radii of the inscribed and escribed circles.

X, Y, Z are the centres and ρx^2, ρy^2, ρz^2 are the radii of the three Malfatti circles: the values of x^2, y^2, z^2 are found immediately from those of $2yz$, $2zx$, $2xy$, which are

For the plus group.	For the minus group.
$2yz = 1+\tan\tfrac{1}{4}(A+\alpha\pi),$	$2yz = 1+\tan\tfrac{1}{4}(-A+\alpha'\pi),$
$2zx = 1+\tan\tfrac{1}{4}(B+\beta\pi),$	$2zx = 1+\tan\tfrac{1}{4}(-B+\beta'\pi),$
$2xy = 1+\tan\tfrac{1}{4}(C+\gamma\pi),$	$2xy = 1+\tan\tfrac{1}{4}(-C+\gamma'\pi),$
$\alpha+\beta+\gamma \equiv 0 \pmod 4.$	$\alpha'+\beta'+\gamma' \equiv 2 \pmod 4.$

Table of 32 sets of values of ρ, α, β, γ, α', β', γ'.

	Symbol.	α	β	γ	Symbol.	α'	β'	γ'
Class I. $\rho = r.$	(000)	0	0	0	(222)	2	2	2
	(022)	0	2	2	(200)	2	0	0
	(202)	2	0	2	(020)	0	2	0
	(220)	2	2	0	(002)	0	0	2
Class II. $\rho = r_1.$	(211)	2	1	1	(011)	0	1	1
	(233)	2	3	3	(033)	0	3	3
	(013)	0	1	3	(213)	2	1	3
	(031)	0	3	1	(231)	2	3	1
Class III. $\rho = r_2.$	(121)	1	2	1	(101)	1	0	1
	(103)	1	0	3	(123)	1	2	3
	(323)	3	2	3	(303)	3	0	3
	(301)	3	0	1	(321)	3	2	1
Class IV. $\rho = r_3.$	(112)	1	1	2	(110)	1	1	0
	(130)	1	3	0	(132)	1	3	2
	(310)	3	1	0	(312)	3	1	2
	(332)	3	3	2	(330)	3	3	0

In Class I, X, Y, Z lie upon AI, BI, CI; and $\rho = r$;

„ II, „ „ AI_1, BI_1, CI_1; „ $\rho = r_1$;

„ III, „ „ AI_2, BI_2, CI_2; „ $\rho = r_2$;

„ IV, „ „ AI_3, BI_3, CI_3; „ $\rho = r_3$.

16. This Table shows the thirty-two solutions in two groups, the "plus" and the "minus" groups, separated by the central vertical line. Two solutions which form a pair, in the way described in § 12, are in the same horizontal line. The thirty-two are also divided into four classes by the horizontal lines, so that they fall into eight sets of four solutions. In § 8, when discussing the first eight solutions, which now constitute the first class, we saw that each set of four could be derived from the four solutions of a system of three quadratics (i) or (vii). The same is true of any set of four solutions which belong to the same class and the same group. If they belong to the plus group, they satisfy equations (i), provided that l, m, n have the values

$$l = \cot \tfrac{1}{2}(A + \alpha\pi), \quad m = \cot \tfrac{1}{2}(B + \beta\pi), \quad n = \cot \tfrac{1}{2}(C + \gamma\pi),$$

α, β, γ having the values associated with any one of the four solutions, which all lead to the same values of l, m, n. If the four solutions belong to the minus group, the cotangents of the half angles are the same as before in absolute value, but with changed sign; the effect of introducing them is practically to change equations (i) into equations (vii). Equation (ii) is always valid.

The thirty-two solutions have now been classified, and each has been labelled by means of a symbol, such as (301) or (213). Each solution is associated with a substitution (also classified and labelled), which shows how the solution can be derived from the standard solution which we selected; but the same substitutions will derive the complete system of solutions from *any* one. Any geometrical fact, expressed in trigonometrical form and proved for one solution, is immediately extended to and proved for every other solution by the appropriate substitution. Further, if it is desired to carry out an investigation by means of (homogeneous) trilinear coordinates (α, β, γ), it is only necessary to change the signs of α, β, γ in conformity with those of a, b, c. Areal coordinates need no modification.

In conclusion, though we have not attempted to increase the number of elementary properties of these sets of circles, we mention two :—

"In four solutions of the same class and group the distances from A to the points of contact of the four circles whose centres lie upon the

bisector of the angle A are the same (but in a different order) as the analogous distances from B and C''.

"In eight solutions which belong to the same class, the eight centres which lie upon the bisector of one angle are placed symmetrically on either side of a certain point on that bisector. The point lies midway between the angle and the other in- or escribed centre which lies on the bisector but is not the point of concurrence".

Malfatti's statement about cutting cylinders from a block of marble with a minimum of material left over is not proved : for a value of a function which is greater than all adjacent values is not necessarily the true maximum. In an equilateral triangle the statement is untrue, for the inscribed circle, with two little circles squeezed into the angles, contain a greater area than Malfatti's three circles.

A CLASS OF QUARTIC FORMS IN FOUR DIMENSIONS

By L. Roth.

[Received 12 December, 1928.—Read 13 December, 1928.]

1.1 In the present paper we consider the locus $\mathfrak{F}$ in [4] whose equation is $|u_{rs}| = 0$, where $|u_{rs}|$ is a symmetrical determinant of order four, whose elements are linear in five homogeneous coordinates x_r. If we suppose $(\xi_1, \xi_2, \xi_3, \xi_4)$ to be coordinates in any given prime ω, it is clear that $|u_{rs}| = 0$ is the discriminant of the quadric

$$\sum_{r,s=1}^{4} u_{rs}\xi_r\xi_s = 0.$$

Since the quadric may be regarded as a member of the linear system Σ_4 obtained by taking x_r to be homogeneous parameters, we have the following representation for $\mathfrak{F}$:

The points of [4] are set in (1, 1) correspondence with the quadrics of a linear system Σ_4 in ω. Then to the cones of Σ_4 (or their vertices) correspond points of a locus $\mathfrak{F}$ which is rational, since it is birationally represented upon ω.

This correspondence between points of a space and the quadrics of a linear system affords the simplest method of discussing many well known loci, including the plane cubic, the plane quartic, as treated in Hesse's classical memoir*, the four-nodal cubic surface†, the quartic symmetroid

* *Journal für Math.*, 49 (1855), 283.

† Thus, for example, all the chief properties, as given in Baker, *Principles of geometry*, 3 (1923), 214–17, are deduced immediately from the representation. The equivalent analytical treatment is given by P. Roth, *Monatshefte für Math. und Phys.*, 22 (1911), 64–88, but he appears not to have noticed this fact, and so he obtains the required results by reciprocating Steiner's quartic surface.

surface* and its special forms, and Segre's quartic locus†, which, as will appear, is a particular case of $\mathfrak{H}$. The dual representation for quartic surfaces is considered by Reye‡ in great detail, giving among other results the chief properties of Kummer's surface.

1.2. Consider now the details of the correspondence. To the points of any line σ_1 correspond the quadrics of a pencil Σ_1; to the points of a plane σ_2 correspond the quadrics of a sheaf Σ_2; and to the points of a prime σ_3 correspond the quadrics of a net Σ_3. The order of $\mathfrak{H}$ is the number of points in which it is met by an arbitrary line σ_1, that is the number of cones in the corresponding pencil Σ_1; hence $\mathfrak{H}$ is a quartic form. Any plane section of $\mathfrak{H}$ corresponds to the Jacobian curve of the corresponding sheaf, a sextic of genus 3, and any prime section corresponds to the Jacobian of the corresponding net, a quartic surface containing ten lines, which are the axes of the plane-pairs contained in the net.

Since any two prime sections have in common a plane section, the corresponding Jacobians must have as free intersection a sextic of genus 3 and nothing more; hence all the Jacobian surfaces pass through a decimic ψ of genus 11§. And since the line common to a plane and a prime section meets $\mathfrak{H}$ in 4 points, the sextic must meet ψ in 20 points.

1.21. It is convenient to denote by (λ, μ) any surface of order λ which contains ψ as a μ-ple curve, and by $(\alpha; \beta)$ any curve of order α which meets ψ in β (not necessarily distinct) points. Thus the Jacobian surfaces have the symbol (4, 1), and the Jacobian curves have the symbol (6; 20). Evidently the order of the surface corresponding to (λ, μ) is the number of points, not on ψ, in which it meets a (6; 20), that is, $6\lambda - 20\mu$. The order of a curve $(\alpha; \beta)$ on $\mathfrak{H}$ is $4\alpha - \beta$.

1.3. *The lines on* $\mathfrak{H}$. To the points of any line on $\mathfrak{H}$ correspond the quadrics of a singular pencil, all members of which are cones. Now in Σ_4 there are ∞^1 singular pencils of two types: (i) composed of cones with a common vertex c^0; these contain three plane-pairs; (ii) composed of cones which touch along a common generator c^1; these contain only one plane-pair.

Since all the lines on $\mathfrak{H}$ meet an arbitrary prime section in one point, it follows that the points c^0 form the curve ψ and that the lines c^1 are

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* Cayley, *Collected Papers*, 7 (1894), 136, 279; Jessop, *Quartic surfaces* (1916), 166.

† Baker, *op. cit.*, 4 (1925), 125, 151 (for the dual case).

‡ *Geom. der Lage* (3rd ed.), 3 (1892), ch. 16–19; *Math. Annalen*, 48 (1897), 113.

§ This follows from the usual formulae for the partial intersection of two surfaces.

trisecants of ψ. We denote the corresponding lines by c_0 and c_1 and the scrolls which they generate by C_0 and C_1. The scroll C_0 has the symbol (0, 1); its order is the number of its intersections with an arbitrary plane section, that is, the number of points in which the corresponding (6; 20) meets ψ. Hence C_0 is of order 20.

1.4. *The nodal curve.* If p is a node of $\mathfrak{S}$, any line through p meets $\mathfrak{S}$ in two points at p, so that the corresponding cone P of Σ_4 must count twice in any pencil containing it. Hence, either its vertex lies on *every* quadric of Σ_4 (which is impossible if we suppose Σ_4 to be quite general in character), or else it is a plane-pair*. Thus the nodes of $\mathfrak{S}$ correspond to the ∞^1 axes of the plane-pairs contained in Σ_4. The order of the nodal curve δ is the number of axes contained in an arbitrary net Σ_3, that is, ten; thus every prime section of $\mathfrak{S}$ is a quartic symmetroid surface. If $\mathfrak{S}$ had a triple point, the corresponding quadric would be a double plane; and, since in general Σ_4 contains no double plane, the only singularity of $\mathfrak{S}$ is a nodal decimic δ.

1.41. Since any node of $\mathfrak{S}$ will not lie on an arbitrary prime section, the corresponding axis will have no free intersections with an arbitrary (4, 1). Hence the axes of plane-pairs are quadrisecants of ψ and generate a scroll Δ. By § 1.3, three quadrisecants pass through any point of ψ, so that ψ is a triple curve on Δ. To find the order of Δ we note that its intersection with an arbitrary (4, 1), residual to ψ, must consist of ten quadrisecants; hence it is of order 10 and is represented by (10, 3). This is otherwise obvious, for δ does not meet an arbitrary plane section of $\mathfrak{S}$ and Δ therefore has no free intersection with the corresponding (6; 20).

1.42. By taking a plane section of Δ through a generator it is seen that no two quadrisecants meet, except on ψ; this agrees with the above conclusions. The genus of δ is that of any plane section of Δ, which is $\frac{1}{2}9.8-10.3=6$.

1.5. To find the order of C_1 we note first that, if ψ were a general decimic of genus 11, the scroll of trisecants would be (80, 17). Now Δ counts quadruply in the degenerate scroll, so that the scroll of trisecants

* A similar argument shows that the plane curve of order n obtained by this type of representation cannot have a node, and is represented birationally upon the Jacobian of three quadric forms in $[n-1]$; a more precise form of this result is proved analytically by Grace, *Journal London Math. Soc.*, 2 (1927), 182.

in the present case is of order 40. By projection it is seen that 8 trisecants can be drawn through any point of ψ, since 3 quadrisecants pass through the point. Hence the scroll is (40, 8), and C_1 is therefore of order 80.

1.51. The incidence scheme of the lines may now be worked out. Since any c_1 line corresponds to a trisecant of ψ, each c_1 line meets 3 c_0 lines; and, as just remarked, each c_0 line meets 8 c_1 lines. Further, 4 c_0 lines pass through each point of δ, since a node is represented by a quadrisecant of ψ. Every c_1 line meets δ, since each trisecant of ψ meets Δ in one free point. The number of c_1 lines passing through any point of δ is the number of free points in which a quadrisecant meets (40, 8), *i.e.* $40-4.8=8$. Thus all generators of C_1 meet δ and 8 generators pass through each point of δ.

1.52. Since 3 quadrisecants pass through each point of ψ, it follows that the c_0 lines are trisecants of δ. Hence δ is not the general decimic of genus 6; it has a scroll C_0 of trisecants, on which δ is a quadruple curve.

1.53. It is interesting to note how the results obtained for the quartic polo-form Ω, discussed in a previous paper*, lead to those just established for $\mathfrak{S}$, which is a particular case of the former locus. It has been shown that Ω possesses two conjugate systems of surfaces (λ, μ) and $(\lambda, \mu)'$, whose symbols are connected by the relations $\lambda' = 11\lambda - 40\mu$, $\mu' = 3\lambda - 11\mu$. We thus have the identity

$$(\lambda, \mu)' = (\lambda, \mu) + (\lambda - 4\mu)(10, 3).$$

Hence, in the case where the surface Δ, with symbol (10, 3), exists, the conjugate system of surfaces degenerates into the original system, together with the surface Δ. The curves (2; 7) and (3; 11), which in the case of Ω represented lines on the form, now lie entirely on Δ and have no representatives on $\mathfrak{S}$.

It is shown in § 1.6 that the tangent cone to $\mathfrak{S}$ from any point meets $\mathfrak{S}$ in a surface represented by a quadric in ω. Now the net of tangent primes to $\mathfrak{S}$ from this point correspond to those members of the corresponding net of Jacobians which possess a node, so that the surface of contact corresponds to the Jacobian of the net, a surface of order 12 But since the surface of contact is represented by a quadric, it follows that the Jacobian breaks up into the surface Δ, together with a quadric.

* "A quartic form in four dimensions", *Proc. London Math. Soc.* (2), 30 (1930), 118–126.

This property is characteristic of the particular form of ψ, since it does not hold for the general ψ which occurs in the representation of Ω.

1.6. *Rank and class of* $\mathfrak{S}$. We consider now some problems of tangency. If a line σ_1 touches $\mathfrak{S}$ at p, the corresponding cone P counts twice in the pencil Σ_1; hence its vertex P_0 must lie on the base curve of Σ_1. The tangent prime σ_3 at p therefore corresponds to the net of quadrics which pass through P_0; this we denote by $\Sigma_3\ (P_0)$, a notation due to Sturm*.

1.61. A tangent to $\mathfrak{S}$ from any point p, with point of contact q, corresponds to a pencil in which the cone Q has its vertex on the quadric P. Hence the tangent cone from p meets $\mathfrak{S}$ in a surface which corresponds to the quadric P, of symbol (2, 0). Thus the surface is rational and of order 12. Since every quadrisecant of ψ meets P in two points, the surface contains δ as a nodal curve.

Similarly, the tangent planes through a line pq meet $\mathfrak{S}$ in points of a curve corresponding to the intersection (4; 0) of the quadrics P, Q; the curve of contact is therefore elliptic and of order 16.

The class of $\mathfrak{S}$ is thus the number of intersections of three arbitrary quadrics P, Q, R, that is, eight.

1.62. The present method leads at once to the distinctive property of the quartic symmetroid, already well known for its prime section†. The tangent cone to $\mathfrak{S}$ from any point p of δ has a surface of contact corresponding to the plane-pair P. Now each of these planes corresponds to a sextic surface on $\mathfrak{S}$ (§ 1.21). Hence the tangent cone from any point of the nodal curve breaks up into two cubic cones, each containing δ.

1.7. *The contact forms.* Let S_2 be an arbitrary plane in ω, and let $\Pi(S_2)$ denote the quadrics of Σ_4 which touch S_2. The corresponding form in [4], denoted by $\pi(S_2)$, is of order 3, since in an arbitrary pencil Σ_1 there are three quadrics which touch S_2. Its intersection with $\mathfrak{S}$ corresponds to the cones of Σ_4 which touch S_2. Now if a cone touches S_2 its vertex lies in S_2, so that S_2 represents the surface of intersection, a rational sextic (§ 1.21). Hence $\pi(S_2)$ touches $\mathfrak{S}$ at all common points. This may be seen otherwise. For the tangent prime to $\pi(S_2)$ at a point p corresponding to a point P_0 of S_2 is represented by those quadrics which, with the cone P, form pencils in which P satisfies doubly the relation $\Pi(S_2)$.

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* *Journal für Math.*, 70 (1869), 212.

† Cayley, *Collected papers*, 7 (1894), 258.

Hence these quadrics all pass through P_0 and form the net Σ_3 (P_0). But this (§ 1 . 6) represents the tangent prime to $\mathfrak{F}$ at p; so that $\pi(S_2)$ touches $\mathfrak{F}$ everywhere. It also contains δ, since the axis of every plane-pair meets S_2.

1 . 71. If a quadric of Σ_4 can be made to contain S_2, it counts thrice, in every pencil formed with it, in the relation $\Pi(S_2)$, and so corresponds to a triple point of $\pi(S_2)$, which is therefore a cone. This happens when S_2 is one of a plane-pair of Σ_4; $\pi(S_2)$ is then a cubic cone from a node, as mentioned in § 1 . 62.

1 . 72. In the same way consider the quadrics $\Pi(S_1)$ which touch an arbitrary line S_1. The corresponding form $\pi(S_1)$ is evidently a quadric. Now ∞^1 quadrics of Σ_4 can be made to contain S_1; whence $\pi(S_1)$ must be a line-cone. Similar reasoning to the above shows that it touches $\mathfrak{F}$ along a rational normal C^4 corresponding to S_1. Its complete surface of intersection corresponds to the cones of Σ_4 which touch S_1. Now any point of ψ is the vertex of a singular pencil, two members of which touch S_1. Hence, if the surface of intersection of $\pi(S_1)$ with $\mathfrak{F}$ is represented by (λ, μ), we have $\mu = 2^*$. And since the surface meets every c_1 line in 2 points, we have $\lambda - 3\mu = 2$, whence $\lambda = 8$. The surface is therefore represented by (8, 2).

1 . 73. Suppose that S_2, S_2' is a plane-pair of Σ_4, intersecting in S_1. Any quadric which touches S_1 forms with the plane-pair a pencil in which the latter counts thrice; hence the corresponding line has three-point intersection with $\mathfrak{F}$ at the corresponding node. Thus $\pi(S_1)$ is the quadric line-cone of tangents at the node. This agrees with a result obtained by Segre†, that the tangent cone at any point of a nodal curve must be of the second species.

1 . 74. We now show that the surfaces of contact of any two contact forms $\pi(S_2)$, $\pi(S_2')$ lie on another cubic form. For consider the quadrics of Σ_4, denoted by $\Pi(S_2, S_2')$, with regard to which S_2 and S_2' are conjugate. The corresponding form $\pi(S_2, S_2')$ is of the third order, since three such quadrics are contained in an arbitrary pencil. Now the relation $\Pi(S_2, S_2')$ is satisfied by any cone whose vertex is S_2 or S_2'; which is the required result.

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* This is otherwise evident, since every c_0 line must meet the quadric cone in two points.

† *Memori di Torino* (2), 39 (1889), 27.

2.1. *Additional singularities.* We have seen in § 1.4 that a node of $\mathfrak{S}$ must correspond to

(i) a plane-pair, or

(ii) a cone with its vertex at a base-point of Σ_4.

Now a net of quadrics cannot contain more than ten plane-pairs, unless it contains an infinity of them; hence there can be no isolated nodes of type (i). If Σ_4 has a base-point P, every Jacobian has a node at P and every prime section of $\mathfrak{S}$ has a node p; hence P corresponds to a nodal line; in fact, there is a pencil of cones with vertex P corresponding to a c_0 line of $\mathfrak{S}$. Thus $\mathfrak{S}$ cannot have an isolated node.

Nodal lines are of three kinds, corresponding to

(1) a pencil of cones with a common vertex at a base-point of Σ_4,

(2) a pencil of plane-pairs with a common axis,

(3) a pencil of plane-pairs formed by a fixed plane and a pencil of planes.

It follows from § 1.61 that the tangent cone to $\mathfrak{S}$ from any point on a nodal line of type (1) is a non-degenerate sextic, from any point on type (2) a pair of cubic cones, and in case (3), a pair of cubic cones, one fixed, the other variable. Type (2) contains two triple points, corresponding to the double planes of the pencil. By § 1.73, the nodal cone at all points on (2) is the same, whereas in the other two cases it varies from point to point.

In considering the possibility of additional singularities we need take no account of type (2), for it is clear that the presence of such a pencil in Σ_4 indicates that the nodal curve δ breaks up into a 9-ic and a line which evidently meets it in two points.

2.2. *One nodal line.* Suppose that Σ_4 contains a common point 1. Every Jacobian of the system has a node there, and 1 is the vertex of a singular pencil corresponding to a nodal line 1 on $\mathfrak{S}$. Since no arbitrary plane section meets this line, the corresponding (6; 20) does not pass through 1. Thus ψ has a node at 1: evidently this represents a coincidence of two c_0 lines, resulting in a nodal line on $\mathfrak{S}$ and on C_0. Since the pencil of cones with vertex 1 contains three plane-pairs, it follows that the line 1 is a trisecant of the decimic curve δ. Any prime section of $\mathfrak{S}$ drawn through 1 is a rational surface corresponding to a Jacobian with a triple point at 1, containing the axes of the three plane-pairs through that point.

Since every Jacobian has a node at 1, any line l through this point corresponds to a conic on $\mathfrak{F}$. Now a sheaf of Σ_4 can be made to contain l; any pencil of this sheaf contains two pairs of coincident cones with vertices on l, and thus corresponds to a bitangent of $\mathfrak{F}$. Hence the plane corresponding to the sheaf touches $\mathfrak{F}$ along a conic corresponding to l. We thus have ∞^2 conic tropes on $\mathfrak{F}$, all of them meeting 1; one trope passes through an arbitrary point of $\mathfrak{F}$.

The surface C_0 is still of order 20, but C_1 contains the scroll of lines lying in tangent planes to $\mathfrak{F}$ drawn through 1; these correspond to chords of ψ which pass through 1, forming a cone* $\begin{pmatrix} 8, & 1 \\ & \psi \end{pmatrix}$.

The class of $\mathfrak{F}$ is the number of free intersections of three arbitrary quadrics of Σ_4, that is, seven.

2.3. *Two nodal lines.* Suppose that there are two nodal lines 1, 2 corresponding to common points of Σ_4. Every Jacobian has nodes at 1 and 2 and contains the line 12; hence the fundamental curve consists of 12 and a 9-ic ψ which is found to be of genus 8; it has nodes at 1 and 2. There are two systems of tropes, meeting the lines 1 and 2 respectively; $\mathfrak{F}$ is now of class 6.

Consider the Σ_3 which contains 12; this corresponds to a prime in which all lines are bitangents of $\mathfrak{F}$. Thus the prime touches $\mathfrak{F}$ along a quadric q_{12}. All the cones of Σ_3 have their vertices on 12; they are arranged in two types of singular pencils—one with common vertex on 12, the other with common tangent plane along 12. These two systems correspond to the two reguli on q_{12}. Since 12 represents a quadric, it follows that an arbitrary (6; 20) meets 12 in two points and ψ in 18 points. In a notation to be employed later we may write it as $\begin{pmatrix} 6; & 18; & 2 \\ & \psi & 12 \end{pmatrix}$.

The two types of singular pencils just mentioned contain three and one plane-pairs respectively; their axes, which correspond to nodes, must be quadrisecants of the fundamental curve and therefore trisecants of ψ. Hence one portion of the nodal curve δ is met by the two reguli of q_{12} in three and one points respectively, and is therefore a rational quartic lying on q_{12}. Thus the scroll Δ of quadrisecants breaks up into two parts, one part Δ_{12}, corresponding to the quartic δ_{12}, generated by the trisecants of ψ, which all meet 12. Now it has been noticed that all the proper cones of Σ_3 have their vertices on 12; hence the Jacobian of Σ_3, which is

* The meaning of this symbol is evident.

known* to be a quartic scroll having 12 for triple line, must be generated by the axes of plane-pairs, and is denoted by $\Delta_{12} \equiv \begin{pmatrix} 4, & 3, & 1 \\ & 12 & \psi \end{pmatrix}$, in a notation similar to the above. It is verified that no $\begin{pmatrix} 6; & 18; & 2 \\ & \psi & 12 \end{pmatrix}$ meets this in free points, and that an arbitrary Jacobian $\begin{pmatrix} 4, & 1, & 1 \\ & \psi & 12 \end{pmatrix}$ meets it in 4 lines. The second portion of the scroll, which is generated by the quadrisecants of ψ, must be $\Delta \equiv \begin{pmatrix} 6, & 2 \\ & \psi \end{pmatrix}$. A similar verification applies to this; it is also verified that no two generators intersect, save on ψ. The genus of the corresponding nodal sextic δ is 1, being that of a plane section of Δ.

From the symbol for a plane section of $\mathfrak{F}$ it follows that C_0 consists of q_{12} and a scroll of order 18, containing 1 and 2 as nodal generators.

If S_2 is any plane through 12, the contact cubic $\pi(S_2)$ breaks up into the prime 12 and a quadric form which meets $\mathfrak{F}$ in a quartic surface corresponding to S_2. Since all quadrisecants of ψ meet S_2, the quartic surface contains the sextic δ. That S_2 represents a quartic surface follows from the fact that a surface denoted by $\begin{pmatrix} \lambda, & \mu, & \mu_1 \\ & \psi & 12 \end{pmatrix}$ is of order $6\lambda - 18\mu - 2\mu_1$. Similar considerations will apply in more complicated examples which occur later, and so these will not be stated again.

2.4. *Four nodal lines.* If Σ_4 has three common points 1, 2, 3, the fundamental curve consists of three lines 12, 23, 31, and a 7-ic ψ with nodes at 1, 2, 3. Every Jacobian has nodes at these points; there are three quadrics q_{12}, q_{23}, q_{31} of the type already discussed, and three systems of tropes. An arbitrary plane section is represented by $\begin{pmatrix} 6; & 2; & 14 \\ & rs & \psi \end{pmatrix}$, where rs is written for the three lines, each of which is a chord of (6; 20). $\mathfrak{F}$ is of class 5; ψ is found to be of genus 2.

There is a pencil of Σ_4 containing the plane 123, and thus consisting of this plane and a pencil of planes with an axis χ. This pencil of plane-pairs corresponds to a nodal line of type (3); the axes lie in 123 and pass through the point (123) where χ meets the plane. Since the axes must be quadrisecants of the fundamental curve, this point lies on ψ and is the seventh intersection of ψ with 123.

The point (123) of ψ corresponds to a nodal line on $\mathfrak{F}$, from any point of which the tangents to $\mathfrak{F}$ form a single cubic cone, corresponding to a

† *E.g.*, Reye, *op. cit.*, 3 (1892), 172.

plane of the pencil through χ. The pencil of quadrics which represents the line (123) is also the pencil containing the lines 12, 23, 31, so that the line (123) is the intersection of the primes 12, 23, 31. It is also the common transversal of the lines 1, 2, 3; the point common to 1 and (123), for instance, corresponds to the join of the points 1 and 123 in the space ω. The same reasoning as before shows that the two reguli on each quadric q_{rs} are respectively secants and trisecants of the nodal curve; thus the latter consists of three twisted cubics δ_{rs}, one on each quadric, which, with the secant (123), form degenerate rational quartics on q_{rs}, as in the previous case. The scroll Δ consists of the plane 123 and three cubic scrolls such as $\Delta_{12} \equiv \begin{pmatrix} 3, & 2, & 1 \\ & 12 & \psi \end{pmatrix}$, which is generated by trisecants of ψ which meet 12.

Any prime section through the line (123) corresponds to a Jacobian which must contain the plane 123, since this is generated by axes of plane-pairs of the system; hence the section corresponds to a cubic surface $\begin{pmatrix} 3, & 1 \\ & \psi \end{pmatrix}$ which meets each Δ_{rs} in two generators, corresponding to the points of δ_{rs}, not on (123), which lie in the prime.

It may be added that the case of three collinear points 1, 2, 3 is trivial, since every prime section is then a quadric repeated and $\mathfrak{S}$ is therefore a quadric form counted twice.

2.5. *Eight nodal lines.* If Σ_4 has four common points r, there are six quadrics q_{rs} and four nodal lines (rst) in addition to the nodal lines r; the axis χ defining (123) passes through the point 4, and three primes rs, st, tr meet in a line (rst), which is a common transversal of r, s, t. Each line such as 1 meets three lines (123), (124), (134). There are four systems of tropes; any trope of system 1, corresponding to a line l through 1, meets the lines 1, (234), since l passes through the point 1 and meets the plane 234 in a point whose join to (234) represents a point on the line (234).

The fundamental curve ψ residual to the lines r is a quartic of genus 1, passing through the points r and meeting each plane rst in a fourth point (rst). An arbitrary prime section of $\mathfrak{S}$ corresponds to a Jacobian containing the lines rs, with nodes at r, and an arbitrary plane section to $\begin{pmatrix} 6; & 2; & 8 \\ rs & & \psi \end{pmatrix}$. Evidently the scroll C_0, of order 8, contains the nodal lines as simple generators.

The nodal twisted cubics of the last case now break up into conics and secant generators. The nodal curve on q_{12}, for example, consists of (123), (124), and a conic $\delta_{12,34}$ meeting them; the corresponding

quadrisecants meet the lines 12, 34, and are therefore chords of ψ. Thus the three nodal conics correspond to quadrics such as $\Delta_{12,34} \equiv \begin{pmatrix} 2, & 1, & 1, & 1 \\ & 12 & 34 & \psi \end{pmatrix}$.

By §§ 2.2, 2.4, the prime section through the lines 1, (234) corresponds to a cubic surface with a triple point at 1, containing ψ and $1r$; this is the cubic cone with vertex 1 containing ψ.

2.6. *Segre's form.* The configuration for the case of five common points is well known, since $\mathfrak{F}$ is then Segre's quartic form. There are five nodal lines r and ten such as (123), which corresponds to the point in which the line 45 meets the plane 123. There are ten primes rs containing quadrics q_{rs}, and six systems of tropes, one system corresponding to the ∞^2 twisted cubics passing through the points r. Considerations similar to the preceding lead at once to the following results :—

Each line lies in four primes rs; thus 1 lies in 12, 13, 14, 15, and (123) lies in 12, 23, 31, 45. Each prime contains two triads of lines, forming generators of each system of q_{rs}; thus 12 contains 1, 2, (345); (123), (124), (125). Each line meets six others; thus 1 meets ($1rs$) and (123) meets 1, 2, 3, (145), (245), (345). Three lines pass through each intersection, *e.g.* 1, (123), (145). In fact, the conic $\delta_{12,34}$ of the previous case now consists of the lines (125), (345) lying in the plane common to the primes 12, 34, and meeting on 5.

Each trope of the six systems meets five nodal lines; thus a trope of system 1 meets 1, (234), (235), (245), (345), and a trope corresponding to a twisted cubic meets the five lines r, since the twisted cubic passes through the corresponding points.

3.1. *One triple point.* We consider now the effect of triple points on the form of $\mathfrak{F}$. A triple point t corresponds to a double plane τ in Σ_4. An arbitrary prime section of $\mathfrak{F}$ still corresponds to a general Jacobian containing the axes of ten plane-pairs, but any prime section through t corresponds to τ and a cubic surface s, containing six lines, which correspond to nodes. Hence the nodal decimic δ has a quadruple point at t. The fundamental curve which is the intersection, residual to a sextic of genus 3, of a Jacobian and (τ, s), consists of a sextic ψ of genus 3 and a plane quartic χ in τ; ψ and χ meet in six points. An arbitrary plane section corresponds to $\begin{pmatrix} 6; & 14; & 6 \\ & \psi & \chi \end{pmatrix}$, and a plane section through t corresponds to the free intersection of two surfaces s, so that its symbol is $\begin{pmatrix} 3; & 8 \\ & \psi \end{pmatrix}$. It meets τ in three points not on χ, as it should do, since the plane section has a triple point at t.

The curve χ is the locus of vertices of cones of Σ_4 which lie in τ; any such cone forms with τ a singular pencil containing one plane-pair. Hence the points of χ correspond to the generators of a cone of genus 3, which contains δ and has t for vertex; each generator meets δ in one point not at t. The order of the cone is the number of intersections of an arbitrary $\begin{pmatrix} 6; & 14; & 6 \\ & \psi & \chi \end{pmatrix}$ with χ, that is, six. By § 1.7, the form $\pi(\tau)$ is the cubic cone of tangents to $\mathfrak{S}$ at t. Evidently this cone meets $\mathfrak{S}$ in the sextic cone corresponding to χ.

The nodes of $\mathfrak{S}$ correspond to the trisecants of ψ, which necessarily meet χ. Since the trisecants of ψ generate a surface $\begin{pmatrix} 8, & 3 \\ & \psi \end{pmatrix}$, it follows that $\Delta \equiv \begin{pmatrix} 8, & 3, & 1 \\ & \psi & \chi \end{pmatrix}$. Any Jacobian $\begin{pmatrix} 4, & 1, & 1 \\ & \psi & \chi \end{pmatrix}$ meets this in ten lines and any $\begin{pmatrix} 3, & 1 \\ & \psi \end{pmatrix}$ meets it in six lines. The genus of δ is 3, being that of Δ. It follows that the six intersections of ψ and χ must be collinear in threes*, since the plane τ meets Δ in ψ and four generators. The scroll C_0 consists of the cone χ and a scroll of order 14.

If Σ_4 has a common point on τ, we obtain a higher type of singularity; this we shall not consider.

3.2. *Two triple points.* Suppose that there are two triple points t_1, t_2 corresponding to double planes τ_1, τ_2. Since the plane-pair (τ_1, τ_2) counts twice in any pencil, the join t_{12} of the triple points is a double line on $\mathfrak{S}$, of type (2); this corresponds to the axis τ_{12} of the plane-pair. The fundamental curve consists of the line τ_{12}, two plane cubics χ_1, χ_2 one in each of τ_1, τ_2, and a twisted cubic ψ; ψ meets χ_1 and χ_2 in three points, and an arbitrary plane section corresponds to $\begin{pmatrix} 6; & 8; & 6 \\ & \psi & \chi \end{pmatrix}$, where χ represents χ_1 and χ_2. An arbitrary prime section of $\mathfrak{S}$ corresponds to a Jacobian containing nine lines besides τ_{12}. Thus the nodal curve consists of t_{12} and a 9-ic δ. By taking arbitrary prime sections through t_1 and t_2 we see that they contain six nodes elsewhere; hence δ has triple points at t_1 and t_2. As before, we see that χ_1 and χ_2 correspond to sextic cones of genus 1, with vertices t_1, t_2 respectively, containing δ and t_{12}. The remaining part of C_0 is a scroll of order 8.

The curve δ corresponds to quadrisecants of the fundamental curve; these are chords of ψ which meet χ_1 and χ_2. Now the chords of ψ which meet χ_1 form a $\begin{pmatrix} 6, & 3, & 1 \\ & \psi & \chi_1 \end{pmatrix}$, so that $\Delta \equiv \begin{pmatrix} 6, & 3, & 1 \\ & \psi & \chi \end{pmatrix}$. It follows that δ is of genus 1.

* The type of sextic which arises here is considered by Schur, *Math. Annalen*, 18 (1881), 1.

An arbitrary prime section through t_{12} corresponds to the planes τ_1, τ_2, and a quadric $\begin{pmatrix} 2, & 1 \\ & \psi \end{pmatrix}$. Any plane section through t_1 (or t_2) corresponds to the free intersection of two surfaces $\begin{pmatrix} 3, & 1, & 1 \\ & \psi & \chi_2 \end{pmatrix}$, *i.e.* a curve $\begin{pmatrix} 3\,; & 5\,; & 3 \\ & \psi & \chi_1 \end{pmatrix}$.

3.3. *Three triple points.* If there are three triple points t, their joins t_{pq} are double lines on $\mathfrak{F}$, and the plane t_{123} lies entirely on $\mathfrak{F}$ since the double planes τ form a singular sheaf. The fundamental curve consists of the lines τ_{pq}, two in each plane τ, conics χ, one in each plane, and a line ψ meeting each conic χ. Any prime section through t_{123} corresponds to a plane $\begin{pmatrix} 1, & 1 \\ & \psi \end{pmatrix}$ and meets $\mathfrak{F}$ again in a cubic surface, whence an arbitrary (6 ; 20) meets ψ in three points ; it also passes through the point τ_{123}, since the corresponding plane section meets t_{123}. The cone with vertex t_1 containing δ consists of t_{123} and a quintic cone, so that the (6 ; 20) meets each χ in five points ; it is thus denoted by $\begin{pmatrix} 6\,; & 3\,; & 5\,; & 1 \\ & \psi & \chi & \tau_{123} \end{pmatrix}$. The quintic cone χ_1 is evidently rational and has t_{12}, t_{13} for double generators, since τ_{12} and τ_{13} meet χ_1 in two points. An arbitrary Jacobian corresponding to a prime section has a node at the intersection τ_{123} of the three lines τ which it contains. Δ is generated by the lines which meet the conics χ, and thus contains ψ also. To find it we take the scroll of secants to χ_1, χ_2, and ψ, forming a $\begin{pmatrix} 4, & 3, & 1, & 1 \\ & \psi & \chi_1 & \chi_2 \end{pmatrix}$, and we know that $\Delta \equiv \begin{pmatrix} 4, & 3, & 1 \\ & \psi & \chi \end{pmatrix}$. Any Jacobian meets Δ in seven lines, so that δ is a rational 7-ic.

3.4. *Four triple points.* By reasoning similar to the above, it is clear that the joins t_{pq} of four triple points are double lines on $\mathfrak{F}$ and that the planes t_{pqr} lie on $\mathfrak{F}$. The tetrahedra with vertices t and faces τ are evidently reciprocal, *i.e.* a vertex of one corresponds to a face of the other. An arbitrary prime section of $\mathfrak{F}$ contains four lines ; the corresponding Jacobian meets each plane τ in three lines τ_{pq} and a line χ, and has nodes at the points τ_{pqr}. One such Jacobian is the tetrahedron itself, corresponding to the prime t_{1234}. An arbitrary (6 ; 20) passes through each vertex τ_{pqr} and has each line χ for trisecant. Thus the cones on $\mathfrak{F}$ with vertices t are rational cubic cones. This also follows, as before, by taking any prime section through t_{123}, representing a cubic surface. If Σ_4 is formed by the planes τ and a quadric Q, the planes representing these cubic surfaces are the polars of τ_{123} with regard to the pencil (Q, τ_4).

The scroll Δ is generated by the quadrisecants of χ, which must lie on the same quadric regulus. Thus $\Delta \equiv \binom{2,\ 1}{\chi}$, so that δ is a rational quartic.

3.5. *Five triple points.* For five triple points the quadric Q is replaced by the double plane τ_5. The planes of the pentahedron meet in ten lines τ_{pq} forming the fundamental curve, which is contained in the Jacobian corresponding to any prime section of $\mathfrak{F}$; the ten vertices of the pentahedron are nodes on the Jacobian. The four lines τ_{pq} in each plane τ meet in six points lying on an arbitrary (6; 20), and there are ten such points in all. Evidently $\mathfrak{F}$ contains ten double lines, which are the joins of its triple points, and ten planes, each containing three triple points.

It is clear that there cannot be more than five isolated double planes in Σ_4, nor can there be an infinity, since a pencil of such planes is impossible. Hence $\mathfrak{F}$ cannot have more than five triple points.

ON THE ZEROS OF RIEMANN'S ZETA-FUNCTION

By E. C. TITCHMARSH.

[Received 14 January, 1929.—Read 14 February, 1929.]

It is well known that the complex zeros of Riemann's function $\xi(s)$, even if they do not all lie on the line $\sigma = \frac{1}{2}$, at any rate lie most densely in the neighbourhood of this line. In fact, if $N(T)$ is the number of zeros $\beta+i\gamma$ for which $0 < \gamma < T$, then $N(T) \sim AT \log T$, where A is a constant; and, on the other hand, it has been proved by Carlson* that, if $N(\sigma, T)$ is the number of zeros for which $\beta > \sigma$, $0 < \gamma < T$, then, for any fixed value of σ greater than $\frac{1}{2}$, as $T \to \infty$

$$N(\sigma, T) = O(T^{1-(2\sigma-1)^2+\epsilon})$$

for every positive ϵ.

In this note we shall improve Carlson's result slightly, proving that

$$N(\sigma, T) = O(T^{1-(2\sigma-1)/(3-2\sigma)+\epsilon}).$$

We assume that the reader is familiar with the main ideas of Carlson's work. As in the papers of Carlson and Landau we consider, instead of $\xi(s)$ itself, the function

$$f_z(s) = \xi(s) \sum_{n<z} \mu(n) n^{-s} - 1,$$

and it is sufficient to show that we can choose z as a function of T, such that

$$\int_T^{2T} |f_z(s)|^2 \, dt = O(T^{1-(2\sigma-1)/(3-2\sigma)+\epsilon}).$$

Instead of dealing with the integral, as Carlson does, by a method which applies to a general class of Dirichlet series, we use a special property of $\xi(s)$, viz. its expression by the approximate functional equation of

* F. Carlson, *Arkiv för Mat. Astr. och Fysik*, 15 (1921), No. 20. See also E. Landau, *ibid.*, 16 (1922), No. 7.

Hardy and Littlewood*. We use this well known theorem in the form

$$\zeta(s) = \sum_{n<x} n^{-s} + \chi \sum_{n<y} n^{s-1} + O(x^{-\sigma}) + O(y^{\sigma-1} t^{\frac{1}{2}-\sigma}),$$

where $2\pi xy = t > 0$, and $\chi = 2^s \pi^{s-1} \sin \frac{1}{2}s\pi \Gamma(1-s)$, the result holding uniformly for $-H \leqslant \sigma \leqslant H$, $x > K$, $y > K$, where H and K are positive constants. Since

$$\sum_{n<z} \mu(n)\, n^{-s} = O(z^{1-\sigma}),$$

we deduce that

$$f_z(s) = \phi_z(s) + \chi \psi_z(s) + O(x^{-\sigma} z^{1-\sigma}) + O(x^{1-\sigma} t^{-\frac{1}{2}} z^{1-\sigma}),$$

where

$$\phi_z(s) = \sum_{n<x} n^{-s} \sum_{n<z} \mu(n)\, n^{-s} - 1, \qquad \psi_z(s) = \sum_{n<y} n^{s-1} \sum_{n<z} \mu(n)\, n^{-s}.$$

We suppose that x is a function of T (not of t), and that $z \leqslant x$. Then

$$\phi_z(s) = \Sigma a_n n^{-s},$$

where clearly $a_1 = 0$, and

$$a_n = \sum_{r|n} \mu(r) = 0 \quad (1 < n < z),$$

$$|a_n| = \left| \sum_{r|n,\ n/x<r<z} \mu(r) \right| \leqslant \sum_{r|n} 1 = d(n) = O(n^{\epsilon})$$

for $z \leqslant n < xz$; while $a_n = 0$ for $n \geqslant xz$. Hence

$$\begin{aligned} \int_T^{2T} |\phi_z(s)|^2\, dt &= T\Sigma \frac{a_n^2}{n^{2\sigma}} + \Sigma\Sigma \frac{a_m a_n}{m^{\sigma} n^{\sigma}} \int_T^{2T} \left(\frac{n}{m}\right)^{it} dt \\ &= O\left(T \sum_{n \geqslant z} n^{\epsilon-2\sigma}\right) + O\left\{ \sum_{m<xz} \sum_{n<xz} (mn)^{\epsilon-\sigma} \Big/ \left| \log \frac{n}{m} \right| \right\} \\ &= O(Tz^{1-2\sigma+\epsilon}) + O\left\{(xz)^{2-2\sigma+\epsilon}\right\} \end{aligned}$$

by a well known lemma†.

Now consider $\psi_z(s)$. We have

$$|\psi_z(s)|^2 = \sum_{m<y} m^{\sigma+it-1} \sum_{n<y} n^{\sigma-it-1} \sum_{q<z} \mu(q)\, q^{-\sigma-it} \sum_{r<z} \mu(r)\, r^{-\sigma+it}.$$

The term (m, n) occurs if $y > \mathrm{Max}(m, n)$, *i.e.* if $t > 2\pi x\, \mathrm{Max}(m, n)$. Hence, writing $Y = T/\pi x$ and $T_1 = \mathrm{Max}(T, 2\pi xm, 2\pi xn)$,

$$\int_T^{2T} |\psi_z(s)|^2\, dt = \sum_{m<Y} m^{\sigma-1} \sum_{n<Y} n^{\sigma-1} \sum_{q<z} \mu(q)\, q^{-\sigma} \sum_{r<z} \mu(r)\, r^{-\sigma} \int_{T_1}^{T} \left(\frac{mr}{nq}\right)^{it} dt.$$

* G. H. Hardy and J. E. Littlewood, *Proc. London Math. Soc.*, (2), 21 (1922), 39–74.

† It is an obvious generalization of Lemma 5 of G. H. Hardy and J. E. Littlewood, *Math. Zeitschrift*, 10 (1921), 283–317 (287).

Let Σ_1 denote the sum of terms with $mr = nq$, and Σ_2 the remainder. Then, putting $mr = nq = \nu$ in Σ_1, we have

$$\begin{aligned}
|\Sigma_1| &\leqslant T \sum_{mr=nq} m^{\sigma-1} n^{\sigma-1} q^{-\sigma} r^{-\sigma} \\
&= T \sum_{\nu < Yz} \sum_{m < Y,\, m|\nu} \sum_{n < Y,\, n|\nu} m^{\sigma-1} n^{\sigma-1} \left(\frac{\nu}{n}\right)^{-\sigma} \left(\frac{\nu}{m}\right)^{-\sigma} \\
&= T \sum_{\nu < Yz} \nu^{-2\sigma} \Big(\sum_{m < Y,\, m|\nu} m^{2\sigma-1} \Big)^2 \\
&\leqslant T \sum_{\nu < Y} \nu^{-2\sigma} \{\nu^{2\sigma-1} d(\nu)\}^2 + T \sum_{Y \leqslant \nu < Yz} \nu^{-2\sigma} \{Y^{2\sigma-1} d(\nu)\}^2 \\
&= O\Big(T \sum_{n < Y} \nu^{2\sigma-2+\epsilon}\Big) + O\Big(T Y^{4\sigma-2} \sum_{\nu \geqslant Y} \nu^{-2\sigma+\epsilon}\Big) \\
&= O(T Y^{2\sigma-1+\epsilon}) = O(T^{2\sigma+\epsilon} x^{1-2\sigma}).
\end{aligned}$$

Again

$$\begin{aligned}
|\Sigma_2| &\leqslant 2 \sum_{mr \neq nq} m^{\sigma-1} n^{\sigma-1} q^{-\sigma} r^{-\sigma} \left| \log \frac{mr}{nq} \right|^{-1} \\
&= 2 \sum_{\mu < Yz} \sum_{\nu < Yz} \sum_{m < Y,\, m|\mu} \sum_{n < Y,\, n|\nu} m^{\sigma-1} n^{\sigma-1} \left(\frac{\nu}{n}\right)^{-\sigma} \left(\frac{\mu}{m}\right)^{-\sigma} \left| \log \frac{\mu}{\nu} \right|^{-1} \\
&< 2 Y^{4\sigma-2} \sum_{\mu < Yz} \sum_{\nu < Yz} d(\mu)\, d(\nu)\, \mu^{-\sigma} \nu^{-\sigma} \left| \log \frac{\mu}{\nu} \right|^{-1} \\
&= O\{Y^{4\sigma-2} (Yz)^{2-2\sigma+\epsilon}\} = O(T^{2\sigma+\epsilon} x^{-2\sigma} z^{2-2\sigma+\epsilon}).
\end{aligned}$$

Since $\chi = O(T^{\frac{1}{2}-\sigma})$, it follows that

$$\int_T^{2T} |\chi \psi_z(s)|^2 \, dt = O(T^{1+\epsilon} x^{1-2\sigma}) + O(T^{1+\epsilon} x^{-2\sigma} z^{2-2\sigma+\epsilon}).$$

Including obvious contributions from the error-terms, we now obtain

$$\begin{aligned}
\int_T^{2T} |f_z(s)|^2 \, dt = {} & O(T z^{1-2\sigma+\epsilon}) + O(x^{2-2\sigma+\epsilon} z^{2-2\sigma+\epsilon}) \\
& + O(T^{1+\epsilon} x^{1-2\sigma}) + O(T^{1+\epsilon} x^{-2\sigma} z^{2-2\sigma+\epsilon}) + O(T x^{-2\sigma} z^{2-2\sigma}) \\
& + O(T^{\epsilon} x^{2-2\sigma} z^{2-2\sigma}).
\end{aligned}$$

The first and third terms are of the same order if $x = z$, and then the first two terms are of the same order if $T = z^{3-2\sigma}$. The remaining terms are then of the same or smaller order, and the result follows.

ON QUADRICS SATISFYING NINE CONDITIONS

By H. D. URSELL.

[Received 20 October, 1928.—Read 8 November, 1928.]

Introduction.

The following investigations began with the consideration of the problem of drawing quadrics through five points to touch four planes. In the general case twenty-one such quadrics can be drawn : after proving this I shall discuss completely the various special cases, as they lead to at least one result in plane geometry of some interest. I then consider the effect of replacing one or more of these nine conditions by the condition of touching a given straight line. The method employed appears to be essentially equivalent to the symbolical method of Schubert*, but it brings the geometrical ideas more to the fore. I should like to thank Professor Baker for his very valuable encouragement.

In § 6 I incidentally obtain a result relating to plane conics which, at any rate, is not well known. The result is that, given a pencil of conics $S = S_1 + \lambda S_2$ and a range of conics $\Sigma = \Sigma_1 + \mu\Sigma_2$, it is possible in five ways to find a pair S, Σ which have double contact.

I have constructed a direct proof of this result. In point coordinates the range Σ is represented by $U + 2\mu V + \mu^2 W$. Consider the representation of conics in the plane by points in five dimensions : this is *relative* to a Veronese surface V_2^4 in the $[5]$ which corresponds to the double lines in the plane.

The range Σ is represented by a plane conic in $[5]$, the pencil S by a straight line. A pair S, Σ having double contact are represented by two points whose join meets V_2^4. Our problem is solved by taking the intersections of V_2^4 with the three-fold cone projecting the conic (Σ) from the line (S) in the five dimensions. These loci have eight intersections, three of which are on the conic (Σ) and correspond to the double lines or point pairs in the range Σ.

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* *Abzählende Geometrie* (Leipzig, 1879), 103.

1. Quadrics through five given points form a linear system of freedom four,

$$\lambda_0 S_0+\lambda_1 S_1+\lambda_2 S_2+\lambda_3 S_3+\lambda_4 S_4 = 0,$$

and we imagine them to be represented by points in four dimensions. In the [4] there are ten lines (ij) $(i \neq j : 1 \leqslant i, j \leqslant 5)$: the line (12) corresponds to the quadrics consisting of the plane of the points 3, 4, 5, together with a plane through the points 1, 2 of the five points which form the base of the system.

The condition that the quadric (λ_i) touches a given plane is the vanishing of a homogeneous cubic polynomial in (λ_i) : such quadrics are represented by the points of a cubic primal in [4]. This cubic primal contains the ten lines (ij).

The three cubics corresponding to three of the four given planes intersect in the ten lines and in a curve of degree $27-10 = 17$ beside. Hence, in particular, there are seventeen quadrics through six points touching three planes.

To find the number of quadrics touching four planes we must first determine the number of times the curve of order 17 meets the ten lines. Take a particular line (12), and through it draw a prime : this prime will meet the cubics in cubic surfaces through the line. Two of them meet in this line and in a curve of order 8, and this curve meets the line four times. Hence it meets the third cubic surface in $3 . 8-4 = 20$ points outside the line (12). Of these twenty points, six lie on the lines (13), (14), (15), (23), (24), (25), and the remaining fourteen lie on the curve of order 17. Thus the latter curve meets the line (12) in $17-14 = 3$ points.

These points are not the points of intersection of (12) with (45), (53), (34). For take a section by a prime through the common point of (12), (45). We obtain three cubic surfaces with a common tangent at this point : they have two intersections here, eight on the other eight lines, and seventeen others.

The curve of order 17, then, meets a fourth cubic primal through the ten lines in $17 . 3$ points in all, of which $10 . 3$ lie on the ten lines : thus there are $17 . 3-10 . 3 = 21$ points (outside the ten lines) common to the four primals, and so twenty-one quadrics touching four planes.

2. There is in the [4] a quartic primal V_4, which is the locus of points representing cones of the system of quadrics : clearly V_4 is rational. It touches each cubic primal wherever it meets it, for the tangential equation of a cone with vertex P is $P^2 = 0$. Hence, if P_0 be a point of [3] on a plane ϖ, and if (P) be any curve through P_0, the curve on V_4

arising from cones with their vertices on (P) touches the cubic primal corresponding to ϖ.

When the four given planes are concurrent, the four cubic primals meet in a point of V_4; since they all touch V_4 there, they have $2^3 = 8$ coincident intersections there. In such a case we have $21-8 = 13$ solutions other than the cone.

When three of the planes go through a line l, there are only three non-degenerate solutions—quadrics through the five points and the line l, which also touch the remaining plane. The first three cubic primals now intersect in the ten lines, the line which represents the pencil of quadrics through the five points and the line l, and a curve σ representing cones with their vertices on l. σ touches the fourth cubic primal and meets it further in ten points, one on each of the ten lines. It is therefore of the fourth order—as is otherwise clear. The three cubics touch along σ: hence σ counts four times in their intersection, which is of order $4 \cdot 4+1+10 = 27$ in all, as it should be. In particular, we verify again that σ is of order 4 and that it does not *touch* the last cubic primal on any of the ten lines.

As we have remarked, there are seventeen quadrics touching three planes and containing six given points. If the six points lie on a cone with its vertex at the common point of the three planes, four of the seventeen solutions degenerate into this cone, and there are thirteen others. If the three planes have a common line, there are four cones through the six points whose vertex lies on the line (the curve σ above was a quartic); each counts four times, and there is one non-degenerate solution.

3. Returning to the case of five points and four planes, we have next to consider the possibility of four of the points lying in a plane. In this case the cubic primals have a plane and six lines in common. The lines form three intersecting pairs and meet the plane in the vertices of a complete quadrilateral. An arbitrary prime meets three cubic primals in three cubic surfaces with a common generator; as above, these have further twenty points in common, of which six lie on the six lines. Thus three cubic primals meet in a curve of order 14 besides the lines and plane; there are fourteen quadrics touching three planes through six points, of which four are coplanar.

Taking a prime through one of the six lines we find three cubic surfaces with two intersecting generators in common. The residual intersection of two of these is a curve of order 7 which meets each of the lines three times. Hence it meets a third cubic surface in

$7 . 3-3-3-4 = 11$ points outside the six lines and the plane, and so our curve of order 14 in [4] meets each of the six lines three times. Taking a prime through the plane, we next find eight points of the curve of order 14 outside the plane : thus the curve meets the plane in six points. Finally we have $14 . 3-6 . 3-6 = 18$ points common to four such cubic primals beside the six lines and the plane, and eighteen quadrics through the five points touch the four planes.

If the four planes have a common point but no three go through a line, eight of the eighteen solutions degenerate into a cone with the point of concurrence as vertex : there are ten others. If three of the planes go through a line l, the corresponding three cubic primals intersect in another line (representing a pencil of quadrics with l as a generator), and a curve σ, quadruple, representing cones with their vertices on l. σ (previously a quartic) now touches the last cubic primal, and meets it in one point on each of the six lines and on the plane : it is a twisted cubic. The quartic of the general case now breaks up into a cubic and a straight line *which lies in the plane.*

These results may be interpreted geometrically. Given the point 5 and the conic in which the plane (1234) meets the cone, there are two possible positions of the vertex of the cone on the line l; given the vertex of the cone, the conic is determined uniquely. Hence, given two points 5, 6, we have a (2, 2) relation of points on the line l, whose coincidences give the possible positions of the vertex of a cone through 5 and 6. In this correspondence the point a, in which l meets the plane (1234), is double, and there are three other coincidences. Hence σ meets a prime (6) in three points : σ is a cubic. The line which, with σ, makes up the quartic corresponds to the plane (234), together with planes through ($a5$).

The rest of the intersection of the three cubic primals corresponds to quadrics with l as a generator. These contain also the conic (1234a) and a generator through 5. Thus the line in [4] corresponding to this pencil of quadrics meets the plane in [4], and meets the last cubic primal in two other points; and there are two quadrics satisfying the conditions.

4. We may also discuss the case in which three points are collinear. In this case, the cubic primals in [4] have in common a quadric twofold, corresponding to pairs of planes through the lines (123) and (45), and two planes which meet the quadric in two generators of the same system and each other in a single point. Two primals meet further in a quintic surface which meets the quadric in two conics and contains their common chord. It also meets the planes, say ϖ_1, ϖ_2, in cubic curves. For let

O be the intersection of ϖ_1, ϖ_2, and let l_1, l_2 be the generators of the quadric Q which lie on ϖ_1, ϖ_2. Then the cubic primals all have a double point at O. A prime through ϖ_1 meets them further in quadrics (through O) which have in common a generator of Q and a line of ϖ_2 meeting the generator. Two such quadrics meet further in a conic. Thus the section of the quintic surface by such a prime consists of this conic and a cubic curve C_1 in ϖ_1. So also the surface meets ϖ_2 in a cubic curve C_2.

A prime through O meets the first two cubic primals in cubic surfaces having in common two generators through their double point at O and a conic (of Q) meeting these. They meet further in a C^5 with a double point at O. C^5 contains a single point in the plane of the conic not lying on the conic : hence it meets the conic in four points. It meets each of the two lines through O in one point other than O, since in a plane through one of these lines the two cubic surfaces have further two conics through O meeting also in a point of Q. Hence C^5 has two points in this plane outside the line : it has two at O, and one other on the line.

It follows that C_1, C_2 have double points at O.

Consider now the section of the figure by the third cubic primal. The quintic surface meets this in C_1, C_2, and two conics of Q, and hence in a quintic curve F beside. F contains a single point in the [3]-space of Q, other than on Q : hence F meets Q in four points. If we cut the figure by a prime through ϖ_1, each of the cubic primals is met further in a quadric. The three quadrics have two intersecting generators in common; the first two quadrics meet further in a conic, which meets the third quadric in two points on the two common generators and in two other points. Thus three intersections of F with any prime through ϖ_1 lie on ϖ_1. If we cut the figure by a prime through O, the C^5 common to two cubic surfaces meets the third cubic surface four times at O, four times on Q, and once each beside on ϖ_1, ϖ_2. Hence it has five further intersections with the third cubic surface : F does not contain O.

Finally, F meets a fourth cubic primal four times on Q, three times each on ϖ_1, ϖ_2, and in five other points; there are five non-degenerate quadrics through three collinear points and two other points, touching four planes.

5. Consider a singly infinite system of quadrics, and suppose that, as a particular case, a quadric of the system may become a pair of planes. We shall find that it is possible to formulate a method of discovering whether such a degenerate case counts as a quadric of the system satisfying some further condition, and, if so, how many times.

Let us describe the quadrics of the system in the neighbourhood of the degenerate case by means of a complex parameter δ, the degenerate case being given by $\delta = 0$. Through any point P_δ of the quadric (δ) can be drawn two generators (λ, δ), (μ, δ) of (δ). We shall assume that, if P_δ tends to a definite point P_0 of the pair of planes as $\delta \to 0$, then, except for special positions of P_0, the two generators (λ, δ), (μ, δ) tend to a definite pair of straight lines $(\lambda, 0)$, $(\mu, 0)$, passing through P_0. Thus there will be a line $(\lambda, 0)$ and a line $(\mu, 0)$ through each point A or B. Let the generators $(\mu_1, 0)$ $(\mu_2, 0)$, lying in A, B respectively, meet the line AB in the unique points a, b. Then every line $(\lambda, 0)$ in A must meet $(\mu_2, 0)$, and hence passes through b; every line $(\lambda, 0)$ in B must meet $(\mu_1, 0)$, and hence passes through a. So also every $(\mu, 0)$ in A must meet every $(\lambda, 0)$ in B, and hence must pass through a; while every $(\mu, 0)$ in B must pass through b.

Analytically, the quadrics for small δ may be written

$$xy + \delta(xt + yz) + \delta^2 P_2(z, t) = 0$$

or

$$(x + \delta z)(y + \delta t) + \delta^2 z' t' = 0.$$

The forms z', t' have definite limits as $\delta \to 0$. The generator

$$x + \delta z = \lambda \delta z', \quad \lambda(y + \delta t) + \delta t' = 0,$$

where λ is fixed and not zero or infinite, has as its limit, as $\delta \to 0$, the line

$$x = 0, \quad y = 0.$$

If, however, we put $\lambda\delta$ for λ, we get as the limit the line

$$x = 0, \quad \lambda y + t' = 0,$$

while, if we put λ for $\lambda\delta$, we get as the limit the line

$$y = 0, \quad x = \lambda z'.$$

Similarly for the μ generators. The points a, b on AB are given by the the planes $z' = 0$, $t' = 0$ respectively.

As an example consider the case of quadrics through four points 1, 2, 3, and 4, touching four planes A, B, C, and D. Let (5) be a variable point on a line l which meets the plane (234) in a point 5. As (5) $\to$ 5, certain of the twenty-one quadrics through (5) may degenerate into a pair of planes of which (234) is one. Suppose in such a case that Q is the remaining plane, and a, b the two points discussed above. Since A meets the non-degenerate quadric in a pair of generators, it must meet the pair of planes in a pair of lines belonging to the systems of lines $(\lambda, 0)$, $(\mu, 0)$. In other words, it must pass through one or other of the points a, b,

The same is true of the planes B, C, D. Since no three of the planes A, B, C, D will, in general, meet in a point of (234), each of the points a, b must lie on two of the four planes. The possible pairs of points a, b are hence the intersections of (234) with the pairs of opposite edges of the tetrad $ABCD$; the plane Q in each case is the plane $(1ab)$. Thus three of the twenty-one quadrics degenerate into pairs of planes when four of the points are coplanar, and there are eighteen others.

The same argument applies when the four planes are concurrent: since thirteen quadrics through five general points touch four concurrent planes, there are ten quadrics when four of the points are coplanar.

We may also discuss the case in which three planes go through a line l. If the five points are general, we lose precisely ten of the thirteen quadrics for the general case of four concurrent planes. One degenerate solution consists of the plane of the points 1, 2, 3 with that of the points 4, 5, and the intersection of l with [123]. The last point is one of the points a, b; the other lies on the remaining plane D.

Through six points of general position, seventeen quadrics can be drawn to touch three planes. Consider how many of these are degenerate when the points 1, 2, 3, 4 lie on a plane X. For this we must find a plane Y through the points 5 and 6 such that the line XY meets the three planes A, B, and C in two points a, b only. Thus ab is a line in X which meets 56 and one of the three lines BC, CA, AB. As before, there are three degenerate quadrics, and hence fourteen others.

More complex examples are provided by the case of three points lying on a line. If of six general points the point (3) moves up to a position 3 on (12), twelve of the seventeen quadrics degenerate into pairs of planes. Of the twelve, three go into the plane [456], together with a plane through (12) and an intersection of [456] with one of the lines BC, CA, AB. The other nine* go into pairs of planes such as [1234] taken with the plane through (56) and the intersection of BC with [1234]. Hence there are $17-12=5$ quadrics through three points and a generator and touching three planes.

This corresponds to the fact that three cubic primals above met in a quintic curve F. We found that F had three intersections each with ϖ_1, ϖ_2, and 4 with Q: these results may now be interpreted. When a quadric of the system F (passing through the points 1, 2, 3, 4, 5, and touching the planes A, B, C) degenerates into the plane $X=[1234]$ and a plane Y through 5, the point a must lie on the line

* If we suppose that (3) moves up to (12) on the plane [124], the reduction on this count would be only six instead of nine.

(123) and on, say, the plane A; and then b lies on X and BC, Y joins 5 to ab. Thus there are three points of F on ϖ_1. Again, if a quadric of the system F degenerates into a plane X through (123) and a plane Y through (45), the point a must lie on (123). If b is the point (ABC), the planes X, Y, and hence a, are determined uniquely; if not, a must lie on, say, A, and then ab is the transversal from a to (45), BC. This determines the four points of F on Q.

There are thus $3.5-3-3-4=5$ quadrics through a line, and two points 4, 5, which touch four general planes A, B, C, D. When the four planes are concurrent in a point b, one of the five solutions degenerates into the pair of planes joining b to the lines (123) and (45): b is one of the special points and the other lies on (123). Thus there are four quadrics through a line and two points touching four concurrent planes.

These numbers 5 and 4 may be obtained more simply in three dimensions, but the above figure in four is not without interest. If we consider the dual problems, quadrics through a line and four general points form a net. Accordingly, nine can be drawn to touch a pair of planes; but of these four degenerate into a complex of a pair of planes and a pair of points on each plane. When the four points are coplanar, the section of the quadric by this plane is at once determined; if, then, the first of the given planes meet the conic in q_1, q_2, and the given line in p, either pq_1 or pq_2 is the second generator through p. So also we find two straight lines $p'q_1'$, $p'q_2$ in the second plane. But a quadric is determined uniquely by a conic, two skew lines pq, $p'q'$ meeting it (and a line pp' meeting all three).

The results for five points and four planes may be tabulated thus :—

Points	General	4 coplanar	3 collinear
General planes	21	18	5
4 concurrent ...	13	10	4
3 through a line	3	2	1

6. Just as we discussed the degeneration of a quadric into a pair of planes, so we may discuss the degeneration of a quadric into a pair of coincident planes. The generators of either system give in the limit a single infinity of lines in the double plane; the λ "generators" describe related ranges on two fixed μ generators, and therefore envelop a conic which touches the two μ generators. We may speak of this conic as the generator conic of the double plane.

A plane-section of a quadric corresponds exactly to a (1, 1) correspondence between the generators of opposite systems of the quadric; hence the same is true of the limit of a plane-section. This will therefore be given by the intersections of corresponding tangents in a homography of the tangents of the generating cone, *i.e.* it will be a second conic having double contact with the first. Of course, unless the plane of the section varies, the limit of the section will be simply a double line.

If a particular plane is a tangent plane of the whole system of quadrics, of which the pair of coincident planes is a particular case, it meets every quadric of the system in a pair of generators: likewise it meets the double plane in a tangent of the generator conic. If a line touches every quadric of the system, only one λ generator can be drawn through its intersections with any quadric: likewise only one tangent of the generator conic meets it—that is, the line meets the generator conic.

Analytically we may write the system of quadrics in the form

$$t^2+2\delta tK+\delta^2 T=0$$

or

$$(t+\delta K)^2+\delta^2(T-K^2)=0,$$

where K is linear and T quadratic in x, y, z. Clearly a line which touches every quadric of the system must meet $t=0$ in a point on

$$T-K^2=0,$$

T being given its limiting form. This is the equation of the generator conic; it has double contact with $T=0$, the section by the (fixed) plane $t=0$.

This apparatus leads to an interesting result when we compare the number of quadrics through five points and touching four planes when (*a*) the four planes are concurrent, or (*b*) four of the points are coplanar. These numbers are respectively 13 and 18. Consider the problem of drawing quadrics through four points to touch a line and four concurrent planes. Quadrics through the four points and touching the planes determine on the line pairs of points which are in a (13, 13) correspondence; there are twenty-six coincidences of the correspondence, and hence twenty-six quadrics through four points touching a line and four concurrent planes.

There are therefore twenty-six quadrics through four coplanar points which touch a line and four general planes. But to determine these we may consider in the same way an (18, 18) correspondence on the line.

This correspondence has thirty-six coincidences: ten are therefore irrelevant. Clearly they are found by letting the quadric through the points and touching the planes degenerate into the plane of the four points taken twice. The generating conic is a conic touching the four planes, and it has double contact with a conic through the four points. There are, as a matter of fact, five such conics, and hence five series of quadrics of the system which degenerate into the double plane: each such series gives two coincidences of the (18, 18) correspondence.

But our argument works in the reverse direction. We have proved that it is possible in five ways to find two conics, one belonging to a given plane pencil, the other to a given range in the same plane, so that they shall have double contact.

7. I shall now apply the methods sketched above to obtain the numbers of quadrics which can be drawn through m points to touch n planes and p lines, where $m+n+p=9$. These numbers it seems natural to determine successively, starting from the case $p=0$, by considering correspondences on a line. Thus one quadric can be drawn through nine general points; quadrics through eight points determine pairs of points on a line in (1, 1) correspondence; there are therefore two which touch the line, namely at the coincidences of the correspondence. Again, quadrics which pass through seven points and touch a line meet a second line in pairs of points which are in a (2, 2) correspondence: there are therefore four such quadrics which touch the second line. In the same way we may show successively that eight quadrics pass through six points and touch three lines, sixteen through five points touch four lines, and thirty-two through four points touch five lines; and when the points and lines are quite general in their mutual relations, none of these quadrics will be degenerate. If we now consider quadrics through three points touching five lines, they determine on a sixth line pairs of points in a (32, 32) correspondence; but of the sixty-four coincidences of this correspondence some are irrelevant to the problem of drawing quadrics through the three points to touch all six lines. In fact, the plane of the three points taken doubly constitutes a quadric locus which satisfies formally the conditions of the problem and gives rise to coincidences of the correspondence. We must find out *how many* coincidences arise in this manner, and this we can do with the help of the considerations of § 6.

In the first place the generator conic connected with any coincidence of the correspondence which lies on the plane is clearly determinate: it is the conic in the plane which meets the five lines. Again, the conic

$T = 0$ must contain the three given points and have double contact with the generator conic; there are, in general, four such conics. Analytically, we write $S = K^2 - T$; the conic $S = 0$ is determinate, and we have three equations

$$K_1 = \pm\sqrt{S_1}, \quad K_2 = \pm\sqrt{S_2}, \quad K_3 = \pm\sqrt{S_3}$$

to determine the coefficients in K. There are eight solutions for K and four for K^2 and T. Geometrically, T is determined by a homography between the tangents to S; if then (s_1, t_1), (s_2, t_2), (s_3, t_3) be the pairs of tangents to S from the three given points, the four conics T are determined by making the triplets (s_1, s_2, s_3), (s_1, s_2, t_3), (s_1, t_2, s_3), (s_1, t_2, t_3) correspond in order to the complementary triplets.

There are therefore four series of quadrics of the system considered which have the double plane as a limit. Two quadrics of each series pass through a point P on the sixth line lying near to the plane, so that eight of the points corresponding to P tend to coincidence with it as P approaches the double plane. There remain $64 - 8 = 56$ coincidences of the correspondence on the sixth line, each of which gives a quadric satisfying the conditions of passing through the three points and touching the six lines.

In just the same way we find that there are forty-eight quadrics through four points touching four lines and a plane. Quadrics then through three points touching four lines and a plane determine a (48, 48) correspondence on a fifth line; this has in all ninety-six coincidences, of which sixteen are irrelevant. In fact, there are now two possible generator conics of the double plane, and hence eight branches of the system of quadrics at the plane; as before, each branch gives two coincidences. Hence eighty quadrics can be drawn through three points to touch five lines and a plane.

Nine quadrics through seven points touch two planes in general; from this fact we find that seventy-two quadrics through four points touch three lines and two planes. If we proceed as above to calculate the number of quadrics through three points which touch four lines and two planes by means of a correspondence on one of the lines, the reduction to be made is now thirty-two, since four conics (in our case the generator conics) can be drawn in a plane to meet four lines and touch two planes. Thus there are $144 - 32 = 112$ quadrics of the kind required. In the same way there are sixty-eight quadrics through four points which touch two lines and three planes; there being again four possible generator conics, there remain $2 \cdot 68 - 32 = 104$ non-degenerate quadrics which pass through three points and touch three lines and three planes. Lastly, forty-two quadrics through four points touch four planes

and a line : hence $2 \cdot 42 - 16 = 68$ quadrics through three points touch four planes and two lines, in agreement with the principle of duality.

8. Quadrics through two points touching six lines meet a seventh line in pairs of points which, as we have seen, are in a (56, 56) correspondence. This correspondence has 112 coincidences; some of these give quadrics through the two points touching all seven lines, but some give degenerate quadrics of the system, each consisting of a double plane. Let A, B be the two given points, and P_0 an irrelevant coincidence on the seventh line; then, by what has been said, the plane ABP_0 meets the other six lines in six points of a conic. Of the fifty-six points Q on the seventh line corresponding to P_0, a certain number—in fact, four—will coincide with it, the remainder in general will not; of the fifty-six quadrics through A, B, P_0 touching the six lines, a certain number—in fact, four—will degenerate into the double plane, the remainder in general will not. To show that the number is four we reconsider the problem of drawing quadrics through A, B, P_0 to touch the six lines. Quadrics through A, B, P_0 touching five of the lines meet the sixth in pairs of points R, R' which are in a (32, 32) symmetrical correspondence : a certain number of coincidences, on the plane ABP_0, are irrelevant. We have seen how to determine the number in the general case; we found that there were four branches of the system of quadrics at the double plane, and each branch gave two coincidences of the correspondence. In fact, each branch of the system of quadrics gave two branches (symmetrical with respect to interchange of R and R') of the correspondence (or of the curve representing the correspondence). In our case, however, in which the sixth line meets the generator conic, the two branches coalesce, and a double point on the curve is replaced by a cusp. Hence there are three coincidences of the correspondence for each branch of the system of quadrics, instead of two as in the general case. There are therefore only fifty-two quadrics through A, B, P_0 which touch the six lines : four of the points Q corresponding to P_0 in our first correspondence coincide with it. Now there are eight planes through AB which meet the six lines in points of a conic : hence there remain $2 \cdot 56 - 8 \cdot 4 = 80$ quadrics through two points which touch seven lines.

Quadrics through two points which touch five lines and a plane meet a sixth line in pairs of points which are in an (80, 80) correspondence. Of the 160 coincidences of this correspondence, some give quadrics through the two points touching the sixth line; the others give degenerate quadrics, each consisting of a double plane, which are to be rejected.

Such a plane must pass through the two points and meet the five lines and the plane in five points and a tangent of a conic. There are fourteen such planes, and, as before, it appears that each provides four coincidences of the correspondence on the sixth line. Hence there are $160-4.14=104$ quadrics through two points which touch six lines and a plane.

Again, twenty-four planes through a line meet four lines and two planes in four points and two tangents of a conic; as before, we infer that the number of quadrics through two points touching five lines and two planes is $2.112-4.24=128$.

Since by the principle of duality the number of quadrics through two points which touch three planes and four lines is 112, we may deduce that the number of planes through a line which meet three planes and three lines in tangents and points of a conic is $\frac{1}{4}(2.104-112)=24$. Cf. § 10 below.

Quadrics through two points touching six lines meet a seventh line in pairs of points P, Q which are in a (56, 56) correspondence; when the plane ABP meets the six lines in points of a conic, four of the points Q corresponding to P coincide with it in the general case, as we have seen. We infer that there are, in general, two branches of the system of quadrics near the double plane ABP. If the point P happens to lie on the conic, each of these branches will provide three coincidences of the correspondence between P and Q instead of two: there is therefore an extra pair of coincidences of the correspondence at P.

Bearing this in mind we draw quadrics through a point to touch seven lines; these meet an eighth line in pairs of points in an (80, 80) correspondence. Of the 160 coincidences of the correspondence, some are given by quadrics through the point which touch all eight lines; the others are given by double planes through the point which meet the seven lines in points of a conic. There are thirty-four such planes, and by the same argument as in previous cases we may show that each of them gives two coincidences of the correspondence. Hence the number of quadrics through a point which touch eight lines is $160-2.34=92$.

There are fifty-two planes through a point which meet six lines and a plane in six points and a tangent of a conic; hence the number of quadrics through a point which touch seven lines and a plane is $2.104-2.52=104$.

Finally, we have seen that quadrics through a point touching seven lines meet an eighth in pairs of points in an (80, 80) correspondence, and that, in general, if P is a point on the eighth line which lies on a plane through the given point meeting the seven lines in points of a conic,

then two of the points corresponding to P coincide with it. We infer that, in general, there is one series of quadrics of the system which has as its limit the double-plane through P. When, however, P happens to lie on the conic, the correspondence on the eighth line has three coincidences at P instead of two (although, of course, only two of the points corresponding to P coincide with it).

Consider then quadrics touching eight given lines; they meet a ninth line in pairs of points which are in a (92, 92) correspondence. Of the coincidences, some are given by quadrics which touch all nine lines, the others by the planes, ninety-two in number, which meet the eight lines in points of a conic, and by the last paragraph we see that each such plane gives one coincidence of the correspondence. There are therefore ninety-two quadrics which touch nine lines.

Points	0	1	2	3	4	5	6	7	8	9
0	92	92	80	56	32	16	8	4	2	1
1		104	104	80	48	24	12	6	3	
2			128	112	72	36	18	9		
3 or				104	68	34	17			
4 planes					42	21				

9. The fact that there is only one coincidence to each double plane in the last case presents, at first sight, a difficulty; we cannot get less than two coincidences at a time out of quadrics which behave like

$$t^2+2\delta tK+\delta^2 T=0.$$

But this is not the only way in which a quadric can degenerate into a double plane; clearly the quadrics $t^2+\delta T+\delta tK=0$ also have this property. In this case the generator conic coincides with the limit of the section by the plane $t=0$; in our applications this could only occur in the last case of all, since in all other cases we are given one point at least on the section $T=0$ which will, in general, lie outside the generator conic. If we develop equations to determine the coefficients in K when we have more than five lines meeting the generator conic, they do, in fact, lead us to $\lim K=0$ in the final case, and in that case only. But even then the analysis does not show us, as the geometrical reasoning does, that there is, in fact, in this case only one coincidence per double plane (in other words, that the equation assumed is of appropriate form in terms of the parameter).

It seems necessary to go into the determination of $\lim K$ in those cases in which it is not immediately determinate. Put $K^2-T=S$, as before, so that S is in all cases to be supposed known (except for a constant factor) from the elementary theory. For simplicity we consider only cases in which we are dealing with points and lines only. On a particular plane $t=0$ we have 2, 1, or 0 points given, and 6, 7, or 8 lines respectively meeting the plane in points of a conic $S=0$. Let P be one of the latter points and P' any other point on the corresponding line; choose the coordinates of P' so that $t=1$, and express the points of the line in the form $P+\lambda P'$. Then the points in which the line meets the quadric

$$(t+\delta K)^2=\delta^2 S$$

are given by $$(\lambda+\delta K+\delta\lambda K')^2=\delta^2(S+2\lambda S'+\lambda^2 S'');$$

where K, S refer to P; K', S'' to P'; and $S'=0$ is the condition of conjugacy for P, P'. For tangency we have

$$[(1+\delta K')^2-\delta^2 S''][K^2-S]=[K+\delta KK'-\delta S']^2$$

or $$0=S+2\delta(SK'-S'K)+\delta^2(S'^2-SS''-2KK'S'+K'^2S+K^2S'').$$

We are therefore confronted with a particular case of the following general problem: we are given n equations $f(\theta_j;\ k)=0$ to determine n quantities θ_j—determine the limits of the solutions (θ_j) as $k\to 0$, when the equations $f_i=0$ are soluble for $k=0$, but not all independent. In our case $n=8$; we have written k in place of δ, and θ_j for the coefficients in K, S, one of which we may suppose given a constant value. Let (ϕ_i) be functions of the (θ_j) such that

$$\Sigma\phi_i f_i(\theta_j;\ 0)\equiv 0.$$

Then for any value of k we have, for the solutions (θ_j),

$$\Sigma\phi_i\frac{f_i(\theta_j;\ k)-f_i(\theta_j;\ 0)}{k}=0.$$

Hence, for the limits of the (θ_j) as $k\to 0$,

$$\Sigma\phi_i\left(\frac{\partial f_i}{\partial k}\right)_{k=0}=0.$$

For solutions of $f_i(\theta_j;\ 0)=0$, we have

$$\Sigma\phi_i\frac{\partial f_i}{\partial\theta_j}=0\quad(j=1,2,\ldots,n),$$

and these conditions are, in general, sufficient for the solutions ϕ_i to be

the particular values of a solution (ϕ_i) of the identity. Hence

$$\frac{\partial f_i}{\partial k} = \sum_{(j)} \psi_j \frac{\partial f_i}{\partial \theta_j} \quad (i = 1, 2, \ldots, n).$$

These conditions provide us with exactly the number of equations we lack. Applying them in our case, we get

$$SK' - S'K = \Sigma \psi_{jk} x_j x_k = (abcfgh\mathfrak{X}xyz)^2,$$

an equation which must hold at each of the points P. Using the relation $S = 0$, which holds at each of the points P, we may eliminate one of the new unknowns, say h : we then have six, seven, or eight equations

$$-S'K = (abcfg0\mathfrak{X}xyz)^2,$$

and two, one, or no equations

$$K^2 = -S$$

to determine the eight unknowns $a, b, c, f, g; u, v, w$ (the latter being the coefficients in K). Now in the last of the three cases all the eight equations are of the first type and homogeneous : hence they can be satisfied only by $K \equiv 0$.

10. We have reduced the problem of the number of quadrics satisfying nine conditions of the type considered to certain cases of the number of conics satisfying eight conditions. Lüroth* has solved the problems of the numbers of planes through a line, a point, or arbitrary, which meet six, seven, or eight lines respectively in points of a conic. A simple proof of his first result may be obtained analytically as follows. We are given seven lines a, b, c, d, e, f, x. Take any point V and any plane ϖ through x. A variable plane λ through x meets a in a point A ; the projection of A from V on ϖ describes on a line $a' = (Va, \varpi)$ a range related to the pencil of planes λ. Similarly for the other five lines. Moreover, the point (a', x) corresponds to the plane Vx : hence each range of points in ϖ may be written in the form $(\lambda x', y+\lambda y', z+\lambda z')$, equal values of λ giving corresponding points. The values of λ for which the six points lie on a conic are therefore given by a determinantal equation

$$|\xi_1^2 \ \eta_2^2 \ \xi_3^2 \ \eta_4\xi_4 \ \zeta_5\xi_5 \ \xi_6\eta_6| = 0,$$

where $\xi = \lambda x', \quad \eta = y+\lambda y', \quad \zeta = z+\lambda z'$;

which, on removal of the extraneous factor λ^4, is of the eighth degree.

It is now a simple matter to deduce the number of planes through a line which meet five lines and a plane in points and a tangent of a

* *Journal für Math.*, 68 (1867), 185–190.

conic. Take any point V on the plane; any plane through the line x meets the five lines in points of a conic which, in turn, meets the plane in two points P, Q. By the last result, there is an (8, 8) correspondence between the lines VP, VQ: of the sixteen coincidences, fourteen are given by the planes which we are seeking, and two by the plane Vx.

In the same way, given a line x, four other lines, and two planes, we take a point V on the second plane, and draw planes through x; in each of these there are two conics meeting the lines and touching the first plane, and either of these meets the second plane in a pair of points P, Q. The lines VP, VQ are in a (14, 14) correspondence, and now four of the coincidences are irrelevant, lying on the plane Vx; hence there are twenty-four planes through x which meet the four lines and two planes in points and tangents of a conic.

Likewise there are $2 \cdot 24 - 8 = 40$ planes through a line which meet three lines and three planes in points and tangents of a conic; but of these sixteen are formed of the two planes through the line meeting the three lines in three collinear points, each taken eight times. In fact, the conics in either of these planes must reduce to a double line; as there should be four conics, *i.e.* four pairs of corresponding points P, Q in each plane, we see that there are eight irrelevant coincidences of the correspondence for each plane. Thus there remain in reality twenty-four planes through a line satisfying the conditions. Cf. the deduction in § 8 above.

To show that there are fifty-two planes through a point which meet six lines and a plane in points and a tangent of a conic, we employ the same method. Let V be any point of the plane; there are thirty-four planes through the given point which meet the six lines and any line VP of the plane in points of a conic: such conics meet the plane again on a line VQ. Of the coincidences of the correspondence between VP, VQ, sixteen are irrelevant. For there are eight planes through V and the given point which meet the six lines in points of a conic, and each contributes two coincidences to our correspondence. Thus there remain $2 \cdot 34 - 16 = 52$ planes through a point meeting six lines and a plane in points and a tangent of a conic.

The number of planes through a point which meet five lines and two planes in points and tangents of a conic is $\frac{1}{2}(2 \cdot 128 - 104) = 76$ from the numbers for quadrics. This is the same as $2 \cdot 52 - 2 \cdot 14$. There are fifty-two planes through a point meeting six lines and a plane in elements of a conic, and there are fourteen planes through two points meeting five lines and a plane in elements of a conic.

COMBINATORY TOPOLOGY AND EUCLIDEAN n-SPACE

By M. H. A. NEWMAN.

[Received 20 March, 1929.—Read 25 April, 1929.]

In this paper the combinatory structure of rectilinear subdivisions of Euclidean n-space (R_n) is investigated. It is shown that, if R_n be simplicially subdivided in any way into rectilinear simplexes†, this set of simplexes forms an unbounded manifold, as that expression is understood in combinatory topology‡. Since the simplexes at a vertex of the simplicially divided R_n are isomorphic with a set filling a convex region, and the boundary of a combinatory n-element is a combinatory n-sphere, this result is a special case of the theorem, which it is our main object to prove, that *if a convex region in R_n can be simplicially divided into rectilinear simplexes, these simplexes form a combinatory n-element.*

In § 2 certain other facts concerning arrays in R_n are noticed, and the relation of the combinatory definitions of "manifold" here used to the definitions of Brouwer and Alexandroff is made clear.

§ 1.

The "moves" employed are the *addition* and *removal* of n-simplexes from an n-array§, under the following conditions:—an n-simplex, S_n, and a bounded n-array, Γ_n, not containing S_n, have *regular contact* if S_n has

† A rectilinear n-simplex is a finite closed n-dimensional region in R_n bounded by $n+1$ flat $(n-1)$-folds.

‡ *I.e.* an *n-element* is a set of n-simplexes obtainable from a single simplex by the repetition of certain "moves", an *n-sphere* is a set isomorphic with the boundary of an $(n+1)$-element, a complete *n-star* is the join of an $(n-1)$-sphere to a vertex not contained in it, and an *unbounded n-manifold* is a connected set of simplexes such that the simplexes at any vertex form a complete star. Two sets of simplexes are *isomorphic* if their vertices can be set in (1, 1) correspondence so that vertices of a simplex in either correspond to vertices of a simplex in the other.

§ *I.e.* set of n-simplexes. For a fuller statement of the definitions in this paragraph, with diagrams, see *Proc. Roy. Acad. Amsterdam*, 29 (1926), 610-641 (here cited as F I and F II).

two opposite components†, of which one, U_k, is a component of the boundary of Γ_n and is interior to Γ_n+S_n, while the other, V_{n-k-1}, is not a component of Γ_n‡. A *move of type* 1 is the addition of an n-simplex S to a bounded n-array Γ with which it has regular contact: we write $\Gamma \xrightarrow[1]{} \Gamma+S$. A *move of type* 2 is the removal of an n-simplex S from a bounded array with the rest of which it has regular contact: $\Gamma \xrightarrow[2]{} \Gamma-S$. In general, if a succession of moves of both types turns Γ into Δ, we write $\Gamma \xrightarrow[12]{} \Delta$. An E_n is any array such that $E_n \xrightarrow[12]{} S_n$, where S_n is an n-simplex. A Σ_n is any array isomorphic with the boundary of an E_{n+1}§.

THEOREM 1. *Let A_n be a finite collection of rectilinear n-simplexes in R_n, satisfying the conditions*:

(R 1) *Any two of the simplexes have either nothing or a k-component of each in common* $(0 \leqslant k \leqslant n-1)$;

(R 2) *The points of A_n form a convex region in R_n*‖.

Then the array A_n is an E_n.

Let flat $(n-1)$-folds

$$x_i = r\epsilon \quad (i = 1, \ldots, n;\ r = 0, \pm 1, \pm 2, \ldots)$$

be drawn in R_n, dividing it into "cubical" blocks (closed regions), in such a way that no two vertices of A_n lie in the same block. Let certain of these blocks be further "cubically" subdivided until no two *edges* (1-components) of A_n have points in the same block unless they have a common vertex there. Proceed in this way until R_n is subdivided into cubical blocks, B_n, such that for no value of k have two k-components of A_n points in the same block unless they have a common point there.

By an elementary property of convex regions, the simplexes of A_n are divided by this process into a number of convex n-dimensional regions whose boundary k-cells are themselves convex k-dimensional regions. This collection of "n-cells" with their bounding k-cells we call A_n': no two boundary $(k-1)$-cells of a k-cell of A_n' lie in the same flat $(k-1)$-fold. The 0-cells are already 0-simplexes. Suppose that every j-cell, for

† A k-component of an n-simplex is the k-simplex determined by any $k+1$ of its vertices. A k-component of an array Γ is a k-component of a simplex of Γ.

‡ The limiting cases where U_k or V_{n-k-1} is absent are *not included*.

§ The names *n-element* for E_n, etc., used in the first paragraph and in other contexts might lead to confusion with the point-theory definitions and so are avoided.

‖ *I.e.* any line meeting A_n meets it in a point or closed segment. The reader will not confuse the points of the region occupied by A_n with the vertices of the array A_n.

$i = 1, 2, \ldots, k-1$, has been divided into a set of rectilinear j-simplexes with a common vertex, the "centre" of the cell. In simplicially subdividing a k-cell, C_k, of A'_n, we proceed differently according to whether or not there is a component of A_n containing one or more points, but not the whole, of the closed set C_k. If there is not, choose a point in the interior of C_k and join it to the $(k-1)$-simplexes into which the boundary of C_k has by this time been divided. If on the other hand there are such components, there is one with lower dimension number than all others. The boundary cell of C_k formed by the points of this component in C_k will already have a "centre", a, which will also be the centre of all $(k-1)$-cells containing it, and so the convex cell C_k may be divided non-singularly by joining a to the boundary $(k-1)$-simplexes which do not contain it.

In this way A'_n is divided into a set of n-simplexes, A^*_n, satisfying the conditions R 1 and R 2, imposed on A_n, and such that the simplexes of A_n that lie in a cubical block B_n have a common vertex. We shall show that A^*_n *is an* E_n.

To do so we assume Theorem 1 to be true of spaces of less than n dimensions (the proof for $n = 1$ being trivial).

(a) Let B^1_n be a block all of whose points belong to A_n. Let the origin be shifted so that B^1_n becomes the set of points $0 \leqslant x_i \leqslant \epsilon$ $(i = 1, \ldots, n)$. The k-components of A^*_n lying in any k-dimensional side of B^1_n $(0 \leqslant k \leqslant n-1)$ are a set of k-simplexes satisfying R 1 and R 2, with k in place of n, and therefore form an E_k. We will call this and similar E_k's "the e_k's of A^*_n". The two e_{n-1}'s of B^1_n lying in $x_1 = 0$ and $x_2 = 0$ respectively have in common the E_{n-2} lying in their boundaries, and nothing else. From a known theorem† it follows that their sum is an E_{n-1}. More generally *any number of* e_{n-1}*'s belonging to* B^1_n *and containing the origin together form an* E_{n-1}. An inductive proof is simplest. The analogous theorem for those e_1's in the 2-fold $x_3 = x_4 = \ldots = x_n = 0$ which contain the origin is evidently true, for there are only two of them, and they have a boundary vertex (the origin) in common. Assume that any number of the e_{k-1}'s lying in $x_{k+1} = \ldots = x_n = 0$, all containing the origin, together form an E_{k-1}; and also that any $q-1$ of the e_k's in R_{k+1} $(x_{k+2} = \ldots = x_n = 0)$ containing the origin form an E_k. Consider the sum of the q e_k's whose points form the intersections of R_{k+1} with the $(n-1)$-folds

$$x_{k+1} = 0, \quad x_k = 0, \quad \ldots, \quad x_{k-q+2} = 0$$

† F II, Corollary to Theorem 8 a.

respectively. The sum of the last $q-1$ is an E_k, and has in common with the first the sum of the $q-1$ e_{k-1}'s whose points form, for $i = k-q+2, \ldots, k$, the intersections of

$$R_{k+1}, \quad x_{k+1} = 0, \quad \text{and} \quad x_i = 0.$$

By hypothesis this array is an E_{k-1}, and therefore, by the theorem quoted above, the addition of the first e_k gives an E_k.

Thus the sum of all the e_{n-1}'s of B_n^1 containing $(0, 0, \ldots, 0)$ is an E_{n-1}, and similarly the sum of all containing $(\epsilon, \epsilon, \ldots, \epsilon)$ is an E_{n-1}. These two E_{n-1}'s have in common their complete boundaries, and nothing more, and therefore† their sum, which is the complete boundary of B_n^1, is a Σ_{n-1}—say Σ_{n-1}^1. Now B_n^1 is the join of a to the simplexes of Σ_{n-1}^1 which do not contain a. Hence‡ B_n^1 is itself an E_n.

(*b*) Let B_n^2 be a block not entirely contained in A_n^*, but containing at least one of its n-simplexes. Let C_n^2 be the set of n-simplexes of A_n^* in B_n^2. The common vertex, a, of these simplexes lies on the boundary of C_n^2, and is the centre of a k-cell of A_n' lying in a k-component of A_n $(k < n)$. Although C_n^2 is a convex region it is not necessarily "strictly" convex at a, *i.e.* there may be two boundary j-components of A_n^* at a which lie in the same flat j-fold, π_j, of R_n. But C_n^2 can be made strictly convex at a by choosing any flat $(n-1)$-fold, π_{n-1}, through π_j, containing an interior point of A_n, and replacing a by a neighbouring point a' of π_{n-1} not in A_n^*, and all components aX of C_n^2 by $a'X$. The resulting set of simplexes C_n' is isomorphic with C_n^2, and its points fill a convex region, strictly convex at a'. There is, therefore, an $(n-1)$-fold ρ_{n-1} through a' which meets C_n' in no other point; and a neighbouring parallel $(n-1)$-fold, ρ_{n-1}', meets each edge (1-component) of C_n' through a' in just one point. The part of ρ_{n-1}' in C_n' is divided by the boundaries of the C_n-simplexes at a' into a set of $(n-1)$-simplexes satisfying R 1 and R 2, with $n-1$ written for n. They, therefore, form an E_{n-1}, say E_{n-1}^2. But E_{n-1}^2 is evidently isomorphic with the set of $(n-1)$-simplexes of C_n' lying in the $(n-1)$-sides of B_n^2, for this is simply the projection of E_{n-1}^2 from a'. Hence this part of the boundary of C_n' is an E_{n-1}, and so§ C_n^2, which is its join to a, is an E_n.

(*c*) We have now shown that the part of A_n^* in each block is an E_n; it remains to assemble the pieces. First, suppose that the first subdivision

† F II, Theorem 6, Corollary 1.

‡ F I, 12 with F II, Theorem 6, Corollary 2.

§ F I, 12.

of R_n, by $x_i = r\epsilon$, achieved the complete separation of the components of A_n, so that the "further subdivisions" (p. 340) were unnecessary. The blocks B_n are all equal in this case. Consider any row of them, *i.e.* a set of all those whose centres are on a line $x_r = a_r$ $(r = 2, 3, \ldots, n)$. Their centres have x_1-coordinates

$$c,\ c+\epsilon,\ c+2\epsilon,\ \ldots,\ c+k\epsilon,$$

for if there were any gap the line of centres would meet A_n in two or more segments. If the pieces of A_n in the k blocks are assembled in order of increasing x_1, adding one at a time, the piece of A_n in the $(p+1)$-th has the part of A_n in a certain $(n-1)$-dimensional block in common with the sum of the preceding p pieces—*i.e.* the common points form an E_{n-1} (by our inductive hypothesis). Hence the result is at each stage an E_n. In this way all the blocks B_n may be collected into "bars" with their axes parallel to the x_1-axis. All the bars with axes in the same $(n-2)$-fold, $x_r = \text{const.}$ $(r = 3, \ldots, n)$, together form an n-element, for the common points of two adjacent bars themselves form an $(n-1)$-dimensional bar, and therefore an $(n-1)$-element. It will now be clear how the assembling is carried out: the $(k-1)$-th step gives a number of slabs each containing the points of A_n in all blocks whose centres lie on a certain $(k-1)$-fold,

$$x_r = \text{const.} \quad (r = k, \ldots, n),$$

and the k-th assembles sets of slabs with fixed x_r $(r = k+1, \ldots, n)$ in order of increasing x_k.

This completes step (*c*) if there were no "further subdivisions". If there were, we can distinguish between the "first order" blocks, bounded by $x_i = r\epsilon$; the "second order" blocks formed by subdividing certain first order blocks, and so on. If the highest order is h, we first show that the part of A_n^* in any subdivided block of order $h-1$ is an E_n, by applying the above process to its blocks of order h; then deal with the blocks of order $h-2$; and, finally, having proved the parts of A_n^* in the blocks of order 1 to be E_n's, assemble them by the process just described.

(*d*) It has now been shown that A_n^* is an E_n. To complete the proof of Theorem 1 we have still to prove that A_n is itself an E_n.

We first notice that the argument of (*a*), (*b*), (*c*), being applicable to any convex array, shows in particular that the arrays which appear in A_n^* in place of single k-components of A_n are E_k's. The arrays A_n and A_n^* have therefore the property that a set of E_k's $(k = 0, 1, \ldots, n)$ can be found in A_n^* such that (i) they fit together like the *components* of A_n,

(ii) every n-simplex of A_n^* belongs to one of the E_n's†. Also A_n^* is an E_n. I have shown elsewhere‡ that in these circumstances it follows that A_n is also an E_n. The proof of Theorem 1 is therefore complete.

It will be noticed that the only properties of E_n's and Σ_n's used in proving A_n^* an n-element are these: (1) two E_n's with a boundary E_{n-1} and nothing else in common together form an E_n; (2) two E_n's with their complete boundaries and nothing else in common form a Σ_n; (3) (to start the inductions) a 1-simplex is an E_1; (4) if a is a vertex, aE_{n-1} and $a\Sigma_{n-1}$ are E_n's (cf. Weyl's axioms A, B, C in his axiomatic development of the subject)§.

§ 2.

Some questions connected with the converse problem—the representation of an assigned E_n in R_n—may suitably be settled here.

First it may be noted that a set of continuous n-simplexes‖ satisfying R 1 and isomorphic to an E_n is an *n-element* in the point-set sense¶. To prove this it is only necessary to follow the construction of the E_n from a simplex by moves of types 1 and 2, using the known theorem that the sum of two (continuous) n-elements with a boundary $(n-1)$-element in common is an n-element††.

Not every E_n can be represented as a non-overlapping set of rectilinear simplexes in R_n. But in topological investigations it is usually permissible to replace an n-array by any that can be obtained from it by subdivision, and so a more important question is:

Given an E_n, can it be subdivided so that the resulting array can be represented without singularities in R_n?

By a *general subdivision* of an n-array is meant the replacement of every k-component T_k ($k = 1, 2, \ldots, n$) by an E_k, $\mathfrak{T}_k$, in such a way that the conditions of the foot-note †, below are satisfied. (Thus A_n^* of § 1 is obtained from A_n by a general subdivision.) Notice that if Γ_n, a finite set of rectilinear

† More precisely: a set of E_k's of all dimension numbers from 0 to n—call them the $\mathfrak{T}_k$'s—exists in A_n^* such that (i) the boundary of each $\mathfrak{T}_k$ is made up of complete $\mathfrak{T}_{k-1}$'s, (ii) no internal component of $\mathfrak{T}_k$ belongs to $\mathfrak{T}_j$ unless $\mathfrak{T}_k$ lies in the boundary of $\mathfrak{T}_j$; (iii) A^* is the sum of the $\mathfrak{T}_n$'s; (iv) the $\mathfrak{T}_k$'s can be correlated with the components, T_k, of A_n in such a way that $\mathfrak{T}_j$ contains $\mathfrak{T}_k$ if, and only if, T_j contains T_k.

‡ *Math. Annalen*, 99 (1928), 399, Theorem 1.

§ *Rev. Mat. Hispano-Amer.* (1923).

‖ Topological images of (closed) rectilinear n-simplexes.

¶ *I.e.* homœomorphic to a continuous n-simplex.

†† Provided that the remaining distinct parts of the boundaries of the n-elements are also (with their boundaries) $(n-1)$-elements. This is satisfied in our case.

n-simplexes in R_n, satisfying conditions R 1 (p. 340) is rectilinearly *simplicially subdivided*, *i.e.* cut up into smaller rectilinear simplexes so that the resulting set Γ'_n still satisfies R 1, then, by Theorem 1 of § 1, the array Γ'_n is obtained from Γ_n by a general subdivision. This is the connexion between the "point-set" and combinatory definitions of subdivision. In a *regular* subdivision of an n-array all the k-components are replaced by *complete stars* (*e.g.* the figure resulting from the regular subdivision of a 3-simplex is that obtained by joining each edge by a plane to the mid-point of the opposite edge). The following is a known property of n-manifolds†: If $M^1 \underset{12}{\dashrightarrow} M^2$, there is a third manifold M^3 such that M^3 can be obtained from M^1 by a general subdivision, and from M^2 by a finite number of *regular* subdivisions. In particular, taking M^1 to be an E_n and M^2 to be an n-simplex, it follows that by suitable subdivision any E_n can be turned into a "regular E_n", *i.e.* an array obtainable from an n-simplex by repeated regular subdivision. From this the answer to our question at once follows, since any rectilinear simplex can be *regularly* subdivided by flat $(n-1)$-folds:

THEOREM 2*a*. *Any E_n can be transformed by a general subdivision into an E_n which can be represented in R_n by a non-overlapping set of rectilinear n-simplexes, occupying a convex region.*

Hence immediately

THEOREM 2*b*. *Any Σ_n can, after a suitable general subdivision, be represented in R_{n+1} by a non-overlapping set of rectilinear n-simplexes, forming the boundary of a convex region.*

This settles a question raised by the definitions of n-manifolds introduced by Brouwer and Alexandroff‡. Brouwer stipulates that the n-simplexes at any point of the manifold shall fit together like a set of rectilinear n-simplexes in R_n having a common vertex a and filling the space near a. It is not obvious *a priori* that this definition is as wide as one which would allow the simplexes to fit together so as to form any complete n-star, in the combinatory sense. Theorem 2*b* shows that a general subdivision reduces any arrangement admitted by the combinatory definition to one admitted by Brouwer. The simplexes of the new array that make up a single simplex of the old form an E_n, and therefore, by a previous remark, a continuous n-element. Hence the general sub-

† See *Journal London Math. Soc.*, 2 (1927), 56–64.

‡ Brouwer, *Math. Ann.*, 71 (1912), 97; Alexandroff, *Math. Ann.*, 94 (1925), 296.

division is also a *simplicial* subdivision. On the other hand, if Γ_n is a rectilinear simplex-star in R_n, as defined by Brouwer, a its centre, and Γ'_n the set of simplexes formed by replacing each n-simplex $a\beta \dots \mu$ of Γ_n by $a\beta' \dots \mu'$, where β' is the point in which $a\beta$ cuts the $(n-1)$-sphere with centre a and radius 1, then Γ_n and Γ'_n are isomorphic and Γ'_n is convex. Hence, by Theorem 1, Γ_n is an E_n. Thus Brouwer's definition is also not *more* general than the combinatory one.

Alexandroff avoids comparison with R_n by using a quasi-combinatory process of subdivision, but since only regular subdivisions are employed by him it again needs Theorem 2*b* to show that there is no essential gain in generality by referring to Σ_{n-1}. Alexandroff has in fact shown (*loc. cit.*) that his definition is equivalent to Brouwer's. Hence we may enunciate

THEOREM 3. *The definitions of "unbounded n-manifold" of Brouwer and Alexandroff are not more general, and, if a preliminary simplicial subdivision be allowed, not less general than the following one:—*

"An unbounded n-manifold is a connected set of continuous n-simplexes such that (1) *the common part of any two is either nothing or a k-component* $(0 \leqslant k \leqslant n-1)$ *of both*; (2) *the simplexes at any vertex form a complete n-star in the combinatory sense, i.e. the join of a vertex to a* Σ_{n-1} *not containing it."*

ON THE 5-TANGENT CONICS OF A PLANE QUINTIC CURVE

By F. P. White.

[Received 14 March, 1929.—Read 8 November, 1928.]

1. *Introduction.* The fact that the planes through a line lying on a cubic three-fold in four dimensions which meet the variety again in line-pairs are the planes of a quintic line-cone and meet an arbitrary plane in the points of a quintic curve must have been noticed by many geometers; it is obviously analogous to Geiser's well known generation of a quartic curve from a cubic surface. I cannot, however, find that it has been stated in print. Mr. H. W. Richmond, whom I consulted after remarking the fact for myself, sent me the following simple algebraical treatment. The equation of a cubic three-fold containing the line $x = y = z = 0$ may be written

$$(a_1, b_1, c_3, f_2, g_2, h_1 \mathfrak{X} t, u, 1)^2 = 0,$$

where a, b, c, f, g, h are quantics in x, y, z of the orders shown by the suffixes. A plane through the line cuts the cubic in a conic, which breaks up into two lines if

$$\Phi \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$$

Φ is homogeneous in x, y, z of degree 5; thus $\Phi = 0$ may be regarded as a plane quintic curve. Also $ab - h^2 = 0$ is a conic which touches the quintic at the five points where

$$\frac{a}{h} = \frac{h}{b} = \frac{g}{f}.$$

Any cubic through these five points is given by

$$\begin{vmatrix} a & h & g \\ h & b & f \\ \lambda & \mu & \rho \end{vmatrix} = 0,$$

where λ, μ are constants and ρ is linear in x, y, z; it meets Φ again in ten further points at which

$$\begin{vmatrix} a & h & g & \lambda \\ h & b & f & \mu \\ g & f & c & \rho \end{vmatrix} = 0.$$

Consider now the quartic curve

$$\begin{vmatrix} a & h & g & \lambda \\ h & b & f & \mu \\ g & f & c & \rho \\ \lambda & \mu & \rho & 0 \end{vmatrix} = 0.$$

We have the identity*

$$\begin{vmatrix} a & h & g & \lambda \\ h & b & f & \mu \\ g & f & c & \rho \\ \lambda & \mu & \rho & 0 \end{vmatrix} \times \begin{vmatrix} a & h \\ h & b \end{vmatrix} \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \times \begin{vmatrix} a & h & \lambda \\ h & b & \mu \\ \lambda & \mu & 0 \end{vmatrix} - \begin{vmatrix} a & h & g \\ h & b & f \\ \lambda & \mu & \rho \end{vmatrix}^2.$$

This proves immediately that the quartic curve touches Φ at the ten points in question and further that the line

$$\begin{vmatrix} a & h & \lambda \\ h & b & \mu \\ \lambda & \mu & 0 \end{vmatrix} = 0,$$

the tangent to the conic at the sixth point on which the cubic meets it, is a double tangent of the contact quartic.

It did not appear to have been remarked, however, that the contact quartic curves obtained as above arise very simply from the four dimen-

* This theorem in determinants is well known in the discussion of Lagrange's determinant in the theory of oscillations; see Routh, *Advanced rigid dynamics* (1905), 50. It is a particular case of Jacobi's ratio theorem, for which see Turnbull, *Determinants, matrices, and invariants* (1928), 77–79, or Salmon, *Modern higher algebra*, 3rd ed. (1876), 29.

sions; each comes, in fact, from the quartic enveloping cone drawn to the section of the cubic three-fold by a solid from the point in which the solid meets the line on the cubic three-fold. It is then clear that if the section is a four-nodal cubic surface the quartic breaks up into two conics. In this way there arise 495 pairs of 5-tangent conics, which, with the conic $(ab-h^2=0$ in the above algebraical treatment) which arises from the tangent solids at the points of the line, account for 991 of the 2015 contact conics which are known (on function-theory grounds) to exist. I have not succeeded in discovering how the remaining 1024 conics arise from the four-dimensional figure.

In paragraphs 2–4 it is shown how, beginning with the quintic curve and one contact conic, we are naturally led, by what is essentially the consideration of radical functions*, to the cubic three-fold in four dimensions†.

Paragraphs 5–8 deal with the four-dimensional figure, principally in a geometrical way, and lead up to the proof of the existence of 495 pairs of contact conics associated with any contact-conic S, each pair having a common tangent which also touches S. This number I believe to be new, although Mr. Richmond tells me that he had heard many years ago from Prof. Frank Morley that there were triads of contact conics with a common tangent; this result, so far as I am aware, has never been published, but is an immediate consequence of the algebra (due to Mr. Richmond) which is outlined above.

The final paragraph verifies that the number 495 is what would have been expected from the function-theory point of view.

2. Suppose that $S=0$ is the equation of a 5-tangent conic of a general plane quintic curve $\Phi=0$, touching it at the points $A_1, A_2, \ldots, A_5$. Through the five points A draw a cubic curve $C=0$, meeting the quintic again in ten points $B_1, B_2, \ldots, B_{10}$. The cubic curve will meet the conic in a sixth point R; let the tangent to S at R be $L=0$. Then consider the sextic curve

$$L\Phi-\lambda C^2=0,$$

where λ is an arbitrary parameter. It meets the conic S in the five points A, each counted twice, and also in R, counted twice, *i.e.* in twelve fixed points. We can therefore choose λ so as to make the sextic curve have

* See Baker, *Abel's theorem* (1897), ch. 13.

† W. P. Milne's derivation of the 5-tangent conics from a net of quadrics in four dimensions [*Journal London Math. Soc.*, 2 (1927), 79–84] is related to the present investigation in the same way as Hesse's treatment of the bitangents of a quartic curve is related to Geiser's. See the paper by Klein referred to in § 4 below.

thirteen points of intersection with the conic; it will then break up into the conic S and a quartic Ψ. Thus, absorbing the parameter λ, we have the identity

$$L\Phi \equiv C^2 + S\Psi. \tag{2.1}$$

The quartic curve $\Psi = 0$ meets the composite sextic $L\Phi$ where $C = 0$, each of the twelve points of intersection counting twice. Hence Ψ is a contact quartic of the quintic Φ, touching it in the ten points B^*, and L is a bitangent of Ψ touching it in the two further intersections of L and C, other than R.

From the last remark it follows that we may write

$$\Psi \equiv LK + P^2, \tag{2.2}$$

where $P = 0$ is a conic passing through the points of contact of L with Ψ and meeting Ψ again in six points which are the points of contact of the cubic $K = 0$ with Ψ.

Now C passes through the points of intersection of P and L; hence we may write

$$C \equiv LQ + MP, \tag{2.3}$$

where $Q = 0$ is a conic through the four further intersections of P and C, and $M = 0$ is a straight line passing through the third intersection of L and C, *i.e.* the point R at which L touches S. Hence we may write

$$S \equiv LN - M^2, \tag{2.4}$$

where $N = 0$ is the tangent to S at the second intersection of M and S. With this notation, the five points A are given by

$$\frac{L}{M} = \frac{M}{N} = -\frac{P}{Q}. \tag{2.5}$$

Using these expressions for Ψ, C, S in (2.1), we easily find that

$$\Phi \equiv LQ^2 + 2MPQ + NP^2 + K(LN - M^2). \tag{2.6}$$

Again
$$C' \equiv MQ + NP = 0 \tag{2.3}'$$

is clearly another cubic curve through the points A, and we easily have

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* That the points B are the points of contact of a quartic curve follows from a general result on contact curves which may be proved by a simple application of Abel's theorem. We shall return to this side of the matter later.

the results

$$N\Phi \equiv C'^2 + S\Psi', \tag{2.1}'$$

$$\Psi' \equiv NK + Q^2, \tag{2.2}'$$

$\Psi' = 0$ being another contact quartic of the quintic curve.

More generally, any cubic curve through the points A may be taken in the form

$$\Gamma \equiv C + \lambda C' + TS, \tag{2.7}$$

where $T = 0$ is a straight line, and may be written

$$T \equiv lL + mM + nN = 0. \tag{2.8}$$

We obtain a corresponding identity

$$(L + 2\lambda M + \lambda^2 N)\Phi \equiv \Gamma^2 + S\Psi'', \tag{2.9}$$

where

$$\Psi'' \equiv (L + 2\lambda M + \lambda^2 N)\{K - 2TQ - NT^2\} + \{P - \lambda Q - T(M + \lambda N)\}^2. \tag{2.91}$$

The quartic Ψ'', whose equation can be written in several forms of the same type as (2.91), is the most general contact quartic associated with the given contact conic S.

3. It is perhaps worth while to verify our conclusions by elementary algebra.

Take the conic S in the form $xz - y^2 = 0$, or $x : y : z = \theta^2 : \theta : 1$, where θ is a parameter.

Let the parameters of the five points A be the roots of the quintic equation

$$\theta^5 + a\theta^4 + \beta\theta^3 + \gamma\theta^2 + \delta\theta + \epsilon = 0. \tag{3.1}$$

Then, writing down the general polynomial of order 5 in the variables x, y, z, involving 21 coefficients, and expressing that the polynomial of the tenth degree in θ, obtained by replacing x, y, z by θ^2, θ, 1, is the square of (3.1), we find, after some arithmetic, that we can write the left-hand side of the equation of the quintic in the form

$$\Phi \equiv K(xz - y^2) + x(x^2 + \beta xz + \delta z^2)^2 + 2y(x^2 + \beta xz + \delta z^2)(ax^2 + \gamma xz + \epsilon z^2) + 2(ax^2 + \gamma xz + \epsilon z^2)^2, \tag{3.2}$$

wherein K is a general polynomial of order 3 in x, y, z.

This is precisely our expression (2.6), in which x, y, z replace L, M, N and for the cubics C, C' we have taken the special curves

$$x(x^2+\beta xz+\delta z^2)+y(\alpha x^2+\gamma xz+\epsilon z^2) = 0,$$

$$y(x^2+\beta xz+\delta z^2)+z(\alpha x^2+\gamma xz+\epsilon z^2) = 0.$$

These cubics may be characterized geometrically as the cubics through the the five points A, having a double-point at Y, and passing respectively through Z, X. The conics P, Q are here line-pairs passing through Y.

4. We shall in future write x, y, z in place of L, M, N, but, unless it is particularly mentioned, we shall not use the special forms of C, C' (*i.e.* of P, Q) obtained in § 3.

Now put

$$t = \frac{C}{S} = \frac{xQ+yP}{xz-y^2},$$

$$u = \frac{C'}{S} = \frac{yQ+zP}{xz-y^2},$$

and interpret x, y, z, t, u as homogeneous coordinates in space of four dimensions*.

We obtain at once

$$P = ux-ty, \quad Q = tz-uy, \tag{4.1}$$

and, from (2.6),

$$\Phi \equiv (xz-y^2)(K+zt^2-2ytu+xu^2).$$

Hence the quintic curve may be obtained as the projection from the line $x = y = z = 0$ of the curve of intersection of the cubic three-dimensional variety

$$K+zt^2-2ytu+xu^2 = 0, \tag{4.2}$$

and the quadrics (4.1).

Instead of (4.2) we may equally use the cubic variety

$$V \equiv zt^2-2ytu+xu^2-2tQ-2uP-K = 0, \tag{4.3}$$

and then the quadrics (4.1) are the polar quadrics of the points U, T respectively, with regard to V. Thus the curve on the cubic variety is the

* This is analogous to what Klein does for the plane quartic curve. See Klein. *Math. Annalen*, 36 (1890), 1–83; *Gesammelte Abhandlungen*, 3, 388–474 (441), and compare Baker, *Abel's theorem* (1897), 381–3.

locus of points, the tangent solids at which contain the line $x = y = z = 0$ (or TU).

V is a perfectly general cubic variety containing the line $x = y = z = 0$.

5. Let us now consider the geometry of the four-dimensional figure which we have obtained.

The tangent solid at any point P of a cubic variety V, a V_3^3, meets V in a cubic surface with a node at P, through which pass six lines of the surface. Thus six lines lying on V pass through any point of it.

Consider then a line l of V. The tangent solids to V at points of l contain l and are such that two pass through an arbitrary point P, namely, the tangent solids at the points in which l meets the conic, which is the further intersection of the plane lP with V. Hence the tangent solids at points of l are the tangent solids of a quadric line cone, of vertex l, and meet an arbitrary plane ϖ in tangents of a conic S. The planes of the line-cone, which give rise to the points of S, are easily seen to be the planes through l which meet V again in conics touching l.

Next, consider the points of V such that the tangent solids thereat contain l. Clearly such points lie on the polar quadric of any point of l with respect to V and their locus Φ' is the curve in which V is met by the quartic surface of intersection of the polar quadrics of any two points of l the *polar quartic surface* of l. This quartic surface contains l and its tangent plane at any point of l lies in the tangent solid to V at this point. Hence l counts twice in the intersection, and Φ' is of order 10. Moreover, if P be a point of Φ', then the tangent solid to V at P meets U in a cubic surface with a node at P and containing l. Then the plane Pl meets the cubic surface, and hence V, in l and a line-pair whose intersection is P. Thus Φ' is projected from l by planes which meet V again in line-pairs.

Suppose now that Φ is the projection of Φ' from l on to the plane ϖ. A line of ϖ is joined to l by a solid which meets V in a cubic surface containing l. There are five planes through l in this solid which meet the cubic surface again in line-pairs. We have thus verified that Φ is a quintic curve, and we also see that Φ' meets l in five points.

Now let P be a point common to the quintic Φ and the conic S. The plane Pl must meet V in l and in two lines which meet on l, say in R. The point of Φ' of which P is the projection is R, *i.e.* is on l; hence there are five such points P arising from the five points R in which Φ' meets l. We can verify that S touches Φ at these five points as follows: the tangent solid at R meets V in a cubic surface with a node at R and having three of the six lines through R in the plane Pl. This cubic surface thus has

a biplanar node at R. Of the five planes through l which meet this surface again in line-pairs, two coincide with Pl. Hence the tangent to S at P also touches Φ. The five points arise from the joins of l to the vertices of the five quadric cones through the polar quartic surface.

Summing up:

The quintic curve Φ arises by projection from a curve Φ' which meets l in five points R; the conic S touches Φ at five points arising from the points R.

6. Now consider the section of V by a general solid Π. Let Π meet l in P, and let the cubic surface of section be called Σ.

The solid Π meets the polar quartic surface of l in a quartic curve Γ' through P which is projected from l into a cubic curve Γ of ϖ. A plane of projection meets Π in a line which is conjugate to l with regard to the conic of intersection of the plane and V. If this conic consists of two lines meeting on l, every line of the plane is conjugate to l with regard to the conic; hence we see that Γ passes through the five points of contact of Φ and S. Moreover, Φ' lies on the polar quartic surface of l, and thus the ten further intersections of Γ with Φ arise from the ten intersections of Π with Φ'.

Further, the lines in Π through P which touch Σ elsewhere generate a quartic cone, which meets ϖ in a quartic Ψ, the projection of a sextic curve Ψ' in Π on Σ with a double point at P. A plane joining a point of Ψ, arising from a point Q of Ψ', to l, meets V in a conic which touches PQ at Q. A common point T of Ψ and Φ thus arises from a plane through l, which meets V again in a line-pair intersecting in a point Q on the line in which the plane meets Π. Hence Ψ passes through the ten points common to Γ and Φ which arise from the ten intersections of Π with Φ'. We can verify further that Ψ touches Φ at these ten points, so that all the 20 intersections are accounted for. For the tangent to Ψ at T arises from the join to l of the tangent plane to Σ at Q, and, since the tangent solid at Q to V contains l, it is this tangent solid which meets ϖ in the tangent to Ψ at T. This solid meets V in a cubic surface with a node at Q and two of the planes through l, which meet this cubic surface again in line-pairs, coincide with Ql. Hence *the quartic Ψ is a contact quartic of the quintic curve Φ, touching it in ten points which arise from the ten intersections of Φ' with Π; the cubic curve Γ passes through the points of contact of S and of Ψ with Φ.*

Moreover, the tangent solid at P to V gives rise to a tangent of S, and also, since it meets Π in the tangent plane to Σ at P, to a bitangent of Ψ.

And it easily follows that the points of contact of the line with S and Ψ are the three points in which it meets Γ.

7. The algebra corresponding to the work of the last paragraph is very simple.

Suppose that the equation of the solid Π is

$$t+\lambda u+lx+my+nz=0. \qquad (7.1)$$

This meets the polar quartic surface (4.1) in a curve which projects into the cubic curve

$$\Gamma \equiv C+\lambda C'+(lx+my+nz)S=0,$$

which is the most general cubic curve through the five points of contact of S and Φ.

To find the corresponding contact quartic we substitute for t from (7.1) in the equation to the cubic variety, and get

$$u^2(x+2\lambda y+\lambda^2 z)-2u\{P-\lambda Q-(y+\lambda z)(lx+my+nz)\}$$
$$-\{K-2(lx+my+nz)Q-z(lx+my+nz)^2\}=0,$$

and the enveloping cone from the point $(0, 0, 0, -\lambda, 1)$ is

$$\Psi'' \equiv (x+2\lambda y+\lambda^2 z)(K-2TQ-zT^2)+\{P-\lambda Q-T(y+\lambda z)\}^2=0,$$

agreeing with (2.91).

8. The contact quartics of the quintic Φ associated, in the way we have described, with the contact conic S are obtained from the quartic enveloping cones to sections of V by solids, drawn from the points in which the solids meet l. In order that a contact quartic may break up into two contact conics, the quartic enveloping cone must break up into two quadric cones. This happens if the section of V is a four-nodal cubic surface.

Now G. Fano* has shown that the general cubic V_3^3 has 495 quadritangent solids, each of which meets it in a four-nodal cubic surface.

Consider one such solid; it will meet the line l in a point P, and the tangent solid at P, which gives rise to a bitangent of the quartic Ψ in the

* Fano, *Annali di Mat.* (3), 10 (1904), 251–285; a summary of his results is given by Segre, *Encykl. Math. Wiss.*, III C 7, 947, footnote 544.

general case, here gives rise to a common tangent of the two conics into which Ψ breaks up.

In this case, we have, in fact, the identity

$$(x+2\lambda y+\lambda^2 z)\,\Phi \equiv \Gamma^2+S\,S_1\,S_2.$$

Thus we have the result:

Given any 5-tangent conic of the quintic curve, there are associated with it 495 pairs of 5-tangent conics, forming 495 triads. Each triad has a common tangent and there is a cubic curve passing through the fifteen points of contact of the conics with the quintic, and also through the three points of contact of the common tangent with the conics.

9. *The Abelian and theta-function theory.* The general plane quintic curve is of genus 6, but it is not the most general curve of its genus, since it depends only on $20-8=12$ moduli, instead of $3p-3=15$. The speciality arises from the fact that it possesses a linear series of grade 5 and freedom 2, a g_5^2, cut out on it by the straight lines of the plane. The canonical curve, which is of order 10 and is normal in space of five dimensions, has a doubly infinite system of 5-secant planes and the conics determined by the five points in each of these planes generate a Veronese quartic surface on which the curve lies, every quadric through the curve containing this Veronese surface*.

This means that the theta-functions associated with the plane quintic curve are special; we have, in fact, one of the exceptional cases discussed by Weber and Kraus†. The straight lines of the plane, each counted twice, form a doubly infinite system of contact conics, so that one of the odd theta-functions $\Theta(u;\ \frac{1}{2}\beta,\ \frac{1}{2}\alpha)$ not only vanishes for zero values of the argument, but its partial derivatives of the first and second order also vanish for zero values of the argument. The other 2015 [*i.e.* $2^{p-1}(2^p-1)-1$] odd theta-functions correspond to the non-degenerate contact conics; no one of the even theta-functions vanishes for zero values of the argument.

Using the notation of Baker‡, we have the result that, if $v_r^{x,\,a}$ ($r=1,\ 2,\ \ldots,\ 6$) denote the normal integrals of the first kind associated

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* See Enriques, *Lezioni sulla teoria geometrica delle equazioni*, 3 (1924), 103.

† H. Weber, *Math. Annalen*, 13 (1878), 35–48; L. Kraus, *ibid.*, 16 (1880), 245–259. See also Krazer and Wirtinger, "Abelsche Funktionen und allgemeine Thetafunktionen", *Encykl. Math. Wiss.*, II B 7, 723; Baker, *Abel's theorem* (1897), 270.

‡ *Abel's theorem* (1897), ch. 10.

with the curve, and $x_1, x_2, \ldots, x_5$ are the points of contact of a conic with the curve, then we have six equations

$$v^{m_6, m} - v^{x_1, m_1} - \ldots - v^{x_5, m_5} \equiv \tfrac{1}{2}\Omega_{\beta, \alpha}, \qquad (9.1)$$

where $\Omega_{\beta, \alpha}$ stands for a row of six quantities $\beta + \tau\alpha$, τ being the matrix of the periods of the v's at the b loops, and β, α being each a row of six numbers, either 0 or 1, such that $\beta\alpha (= \beta_1\alpha_1 + \ldots + \beta_6\alpha_6)$ is odd. Here m is an arbitrary point (or place) on the curve (or Riemann surface), and $m_1, m_2, \ldots, m_6$ are six points on the curve depending only on m (and the mode of dissection of the Riemann surface)*.

We shall write $[\epsilon] = \frac{1}{2}[\beta, \alpha]$ for two rows of six numbers $\frac{1}{2}\beta$, $\frac{1}{2}\alpha$, in which each β and each α is either 0 or 1. There are† $2^{2p} = 4096$ such *theta characteristics*, of which $2^{p-1}(2^p+1) = 2080$ are *even*, *i.e.* have $\beta\alpha$ even, and the remaining $2^{p-1}(2^p-1) = 2016$ are *odd*.

Now suppose that $n_1, n_2, \ldots, n_5$ are the intersections of the quintic with a straight line. Then we also have

$$v^{m_6, m} - v^{n_1, m_1} - \ldots - v^{n_5, m_5} \equiv \tfrac{1}{2}\Omega_{\beta_0, \alpha_0}, \qquad (9.2)$$

where $[\epsilon_0] = \frac{1}{2}[\beta_0, \alpha_0]$ is the particular odd theta-characteristic associated with the degenerate contact conics.

From equations (9.1), (9.2), we deduce

$$\begin{aligned} v^{x_1, n_1} + \ldots + v^{x_5, n_5} &\equiv \tfrac{1}{2}(\Omega_{\beta, \alpha} - \Omega_{\beta_0, \alpha_0}) \\ &\equiv \tfrac{1}{2}\Omega_{\beta+\beta_0, \alpha+\alpha_0}. \end{aligned} \qquad (9.3)$$

Hence the characteristic appearing in the equations connecting the points of contact of a conic with the points of intersection with a straight line is $[\epsilon + \epsilon_0]$, the sum of two odd characteristics, of which one is that associated with the degenerate contact conics.

Now take a cubic through $x_1, x_2, \ldots, x_5$ meeting the quintic again in $y_1, y_2, \ldots, y_{10}$. Using also the cubic consisting of the line taken three times, we have, from Abel's theorem,

$$v^{x_1, n_1} + \ldots + v^{x_5, n_5} + v^{y_1, n_1} + \ldots + v^{y_5, n_5} + v^{y_6, n_1} + \ldots + v^{y_{10}, n_5} \equiv 0,$$

and thus

$$v^{y_1, n_1} + \ldots + v^{y_5, n_5} + v^{y_6, n_1} + \ldots + v^{y_{10}, n_5} \equiv \tfrac{1}{2}\Omega_{\beta+\beta_0, \alpha+\alpha_0}. \qquad (9.4)$$

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* Baker, *loc. cit.*, 255.

† *Ibid.*, 251.

Thus the points $y_1, \ldots, y_{10}$ are the points of contact of a quartic curve, which we may thus associate with the characteristic $[\epsilon+\epsilon_0]$.

If this quartic breaks up into two conics, whose points of contact are respectively $y_1, \ldots, y_5$ and $y_6, \ldots, y_{10}$, equations (9.4) must result from the addition of two equations of the type (9.3); hence, if $[\eta]$, $[\xi]$ are the odd theta-characteristics associated with the two conics, $[\epsilon+\epsilon_0]$ must be the sum of $[\eta+\epsilon_0]$ and $[\xi+\epsilon_0]$. In this case we have

$$[\epsilon+\epsilon_0] = [\eta+\xi],$$

i.e. we have to express $[\epsilon+\epsilon_0]$ as the sum of two odd characteristics. Now this can be done* in $2^{p-2}(2^{p-1}-1) = 496$ ways, one of which, however, is $[\eta], [\xi] = [\epsilon], [\epsilon_0]$. Hence there are 495 quartics which break up into two conics. This is the result of § 8.

* Baker, *loc. cit.*, 381. Cf. Krazer, *Lehrbuch der Thetafunktionen* (1903), 258.

THE EQUATION $px = xq$ IN A LINEAR ASSOCIATIVE ALGEBRA

By R. WILSON.

[Received 8 March, 1929.—Read 14 March, 1929.]

1. *Introduction.* The solution of this equation has already been obtained in matrices*. Although every linear associative algebra can be exhibited as a matrix algebra†, and conversely, yet it is desirable, in practice, to have a solution which will be independent of the system of units employed. This is the more necessary as a solution for general matrices may be difficult to apply to special subsystems, some of which, such as quaternions, are important. A solution which is expressed in terms of invariant properties of the algebra does not present this difficulty.

In the solutions here given the scalar coefficients in p, q, and x, whether the latter are expressed as matrices or as numbers in a linear algebra, are taken over a field. It will be noted that the solution in cases where p and q belong to a subalgebra may sometimes be found in an extension of that subalgebra. Such cases are, however, readily classified from the solution here given. An important example occurs when p and q belong to a division algebra, a characteristic of the solutions of $px = xq$ in an algebra with a modulus being the prevalence among them of divisors of zero. It is clear that such solutions do not belong to the division algebra but to an extension of it (see § 7, Ex. 2). Likewise, in the case of an algebra lacking a modulus, solutions may be obtained in the augmented system obtained by adjoining a modulus.

2. *General Properties.* Let x be any solution of the equation $px = xq$. If $\Delta(p-\lambda)$ denote the left-hand characteristic determinant of p, and $\Delta'(q-\lambda)$ the right-hand characteristic determinant of q, the corresponding

* Cullis, *Matrices and determinoids*, 3 (1925), Ch. XXVII.

† Poincaré, see Dickson, *Algebras and their arithmetics* (1923), Ch. VII.

characteristic equations are $\Delta(p-\lambda)=0$, $\Delta'(q-\lambda)=0$. Each of these is of degree n, the order of the algebra, and has coefficients drawn from the field. Let $P(\lambda)=0$ and $Q(\lambda)=0$ denote the reduced equations of p and q respectively. Then* $\Delta(p-\lambda)$ and $P(\lambda)$ have the same zeros, as have also $\Delta'(q-\lambda)$ and $Q(\lambda)$.

It is clear that $x=0$ is a solution of $px=xq$ and that there may be others.

THEOREM 1. *Unless $P(\lambda)$, $Q(\lambda)$ have a common zero, the only solution of $px=xq$ is $x=0$.*

For, if x is a solution, the following relations hold:

$$x = x,$$
$$px = xq$$
$$p^2x = xq^2,$$
$$\cdots \quad \cdots$$
$$p^kx = xq^k.$$
$$\cdots \quad \cdots$$

Hence
$$S(p)\,x = xS(q), \tag{1}$$

where $S(\lambda)$ is a polynomial with coefficients from the field.

If $P(\lambda)$, $Q(\lambda)$ are prime to one another, $\phi(\lambda)$, $\psi(\lambda)$ can be found such that

$$P(\lambda)\,\psi(\lambda)+Q(\lambda)\,\phi(\lambda)=1.$$

Since $P(p)=0$, $Q(q)=0$,

$$P(q)\,\psi(q)=1; \quad Q(p)\,\phi(p)=1.$$

Putting $S(\lambda)=Q(\lambda)\,\phi(\lambda)$ in (1),

$$Q(p)\,\phi(p)\,x = xQ(q)\,\phi(q)$$

or
$$1.x=0.$$

THEOREM 2. *If $P(\lambda)$, $Q(\lambda)$ have the common zeros $\lambda_1, \ldots, \lambda_s$, then any solution x satisfies the relations*

$$(p-\lambda_1)\ldots(p-\lambda_s)\,x = 0 = x(q-\lambda_1)\ldots(q-\lambda_s). \tag{2}$$

For let
$$P(\lambda) \equiv (\lambda-\lambda_1)\ldots(\lambda-\lambda_s)(\lambda-\rho_{s+1})\ldots(\lambda-\rho_l),$$
$$Q(\lambda) \equiv (\lambda-\lambda_1)\ldots(\lambda-\lambda_s)(\lambda-\sigma_{s+1})\ldots(\lambda-\sigma_m).$$

* Dickson, *loc. cit.*

From (1), putting $S(\lambda) = P(\lambda)$, we have

$$P(p)\,x = xP(q) = x(q-\lambda_1)\dots(q-\lambda_s)(q-\rho_{s+1})\dots(q-\rho_l)$$

or

$$0 = x(q-\lambda_1)\dots(q-\lambda_s),$$

since, if $q-\rho_k$ $(k = s+1, \dots, l)$ were a right divisor of zero, then $\Delta'(q-\rho_k)$ would be zero, making ρ_k a root of $Q(\lambda) = 0$, contrary to hypothesis.

THEOREM 3. *Unless $P(\lambda)$ and $Q(\lambda)$ have the same zeros there can be no solutions other than $x = 0$ or x a divisor of zero.*

For from $px = xq$ it follows that

$$(p-\lambda)\,x = x(q-\lambda). \tag{3}$$

Now determinant $[(p-\lambda)\,x] = \Delta(p-\lambda)\,\Delta'(x)$,

and determinant $[x(q-\lambda)] = \Delta(x)\,\Delta'(q-\lambda)$.

Hence, choosing for λ a characteristic value of p such that $\Delta(p-\lambda) = 0$, it follows from (3) that

$$\Delta(x)\,\Delta'(q-\lambda) = \Delta(p-\lambda)\,\Delta'(x) = 0.$$

Therefore, unless $\Delta(x) = 0$, $\Delta'(q-\lambda) = 0$ for every value of λ which makes $\Delta(p-\lambda) = 0$.

Similarly, unless $\Delta'(x) = 0$, $\Delta(p-\lambda) = 0$ for every value of λ which makes $\Delta'(q-\lambda) = 0$.

3. *The Solution.* It is clear from (2) that the solutions must be sought from those values of x which satisfy at the same time the equations

$$(p-\lambda_1)\dots(p-\lambda_s)\,x = 0,$$

$$x(q-\lambda_1)\dots(q-\lambda_s) = 0.$$

If C is an arbitrary number of the algebra, general solutions to these equations are

$$(p-\rho_{s+1})\dots(p-\rho_l)\,C, \quad C(q-\sigma_{s+1})\dots(q-\sigma_m),$$

respectively, giving, as a general solution to the equation $px = xq$,

$$x = (p-\rho_{s+1})\dots(p-\rho_l)\,C(q-\sigma_{s+1})\dots(q-\sigma_m). \tag{4}$$

Although the solution *apparently* involves n arbitrary scalar coefficients, this number is, in fact, reduced by the left- and right-hand multipliers. In order to show the distribution of the constants we shall first consider the cases for which $\lambda_1, \dots, \lambda_s$ are non-repeated zeros of the characteristic equations.

If there is only one zero λ_1 common to $P(\lambda)$ and $Q(\lambda)$, then equations (2) become respectively

$$(p-\lambda_1)x = 0, \quad x(q-\lambda_1) = 0,$$

where $\Delta(p-\lambda_1)$, $\Delta'(q-\lambda_1)$ are each of rank $n-1$.

Hence C consists of a single arbitrary scalar c_1 and the solution may be written

$$x = c_1(p-\rho_2)\dots(p-\rho_l)(q-\sigma_2)\dots(q-\sigma_m).$$

If $P(\lambda)$, $Q(\lambda)$ have two zeros λ_1, λ_2 in common with the solution, it takes the form

$$\begin{aligned} x = {} & c_1(p-\lambda_2)(p-\rho_3)\dots(p-\rho_l)(q-\lambda_2)(q-\sigma_3)\dots(q-\sigma_m) \\ & +c_2(p-\lambda_1)(p-\rho_3)\dots(p-\rho_l)(q-\lambda_1)(q-\sigma_3)\dots(q-\sigma_m), \end{aligned}$$

for, writing $x = x_1+x_2$,

$$\begin{aligned} px &= p(x_1+x_2) = (p-\lambda_1)x_1+(p-\lambda_2)x_2+\lambda_1x_1+\lambda_2x_2 \\ &= \lambda_1x_1+\lambda_2x_2 = xq, \text{ similarly.} \end{aligned}$$

In the same way, if $P(\lambda)$, $Q(\lambda)$ have the zeros λ_1, λ_2, ... λ_s in common, the solution may be written

$$x = \sum_{t=1}^{s} c_t P_t(p)\, Q_t(q), \tag{5}$$

where $P_t(q)$, $Q_t(q)$ are $P(p)$, $Q(q)$ with the respective factors $(p-\lambda_t)$, $(q-\lambda_t)$ omitted, and the c_t are scalar arbitraries.

4. *The exceptional case.* If all the zeros of $P(\lambda)$ and $Q(\lambda)$ are the same the solution takes the form

$$x = \sum_{t=1}^{l} c_t P_t(p)\, Q_t(q). \tag{6}$$

Unless special values are given to the c_t, this is not, in general, a divisor of zero (§ 2, Th. 3), differing from the preceding cases in that $P_t(q)$, $Q_t(q)$, $t = 1, 2, \dots, l$ do not contain a common factor which is a divisor of zero. If, however, any c_t is zero, then x is a divisor of zero, while, if every c_t is zero, the corresponding solution is $x = 0$.

Note that in this case the solution is ignored by (4) since conditions (2) do not exhibit x as a divisor of zero, but merely state that

$$P(p)\,x = 0 = xQ(q),$$

these being identities, since $P(p) = 0$ and $Q(q) = 0$.

The equation $px = xp$ is of this type.

5. *Nilpotent elements.* If p is a nilpotent element, then $P(p) = p^m$. As in § 2, Ths. 1 and 3, unless q is a divisor of zero the equation has no solution other than $x = 0$, while, unless q is a nilpotent of the same order as p, the only solutions are divisors of zero and $x = 0$.

Assume first that p and q are both nilpotents of order 2, so that $p^2 = 0$, $q^2 = 0$. Then

$$0 = p^2x = p(px), \quad 0 = xq^2 = (xq)q.$$

Hence the solution is given by

$$px = pCq = xq,$$

where C is an arbitrary number of the algebra.

The general solution to $px = xq$ in this case is therefore

$$x = pC + Cq.$$

If p and q are both nilpotents of order r, it can be shown similarly that the solution is

$$x = p^{r-1}C + p^{r-2}Cq + \ldots + pCq^{r-2} + Cq^{r-1},$$

while, if $p^r = 0$, $q^s = 0$, $r > s$, the solution becomes

$$x = p^{r-1}C + p^{r-2}Cq + \ldots + p^{r-s}Cq^{s-1}.$$

Similarly for $s > r$.

6. *The solution in general terms.* In the more general case in which the common zeros $\lambda_1, \ldots, \lambda_s$ of $P(\lambda)$, $Q(\lambda)$ are repeated roots of the characteristic equations, solutions (5) and (6) take the respective forms

$$x = \sum_{t=1}^{s} P_t(p)\, C_t Q_t(q), \quad x = \sum_{t=1}^{l} P_t(p)\, C_t P_t(q) \tag{7}$$

where the C_t are arbitrary numbers of the algebra.

This is the most practicable form of the solution, as the surviving scalar arbitraries adjust themselves in any given case. It is, however, necessary to show that their number cannot exceed n. Corresponding to the root λ_t, suppose that $\Delta(p-\lambda)$, $\Delta'(q-\lambda_t)$ are of ranks $n-\tau$, $n-\tau'$, respectively. From (2) it is clear that the number of arbitrary coefficients occurring in C_t cannot exceed the smaller of τ and τ'. Further $\Delta(p-\lambda_t) = 0$ has a τ-repeated root λ_t, and $\Delta'(q-\lambda) = 0$ a τ'-repeated root λ_t, as follows from the ranks of $\Delta(p-\lambda_t)$, $\Delta'(q-\lambda_t)$. Consequently the total number of arbitrary coefficients occurring in the solution cannot exceed the degrees of each of $\Delta(p-\lambda_t) = 0$ or $\Delta'(q-\lambda_t) = 0$.

Solutions (7), although adequate when the reduced equations of p and q have only non-repeated roots in common, require further generalization.

Suppose that the common roots λ_r of $P(p)$, $Q(q)$ are repeated t_r times in the former and t'_r times in the latter ($r = 1, 2, \ldots, s$), so that equations (2) become

$$(p-\lambda_1)^{t_1}(p-\lambda_2)^{t_2}\ldots(p-\lambda_s)^{t_s}x = 0 = x(q-\lambda_1)^{t'_1}(q-\lambda_2)^{t'_2}\ldots(q-\lambda_s)^{t'_s}.$$

Then, as in § 5, the following and similar rearrangements

$$(p-\lambda_1)^{t_1}[(p-\lambda_2)^{t_2}\ldots(p-\lambda_s)^{t_s}x] = 0 = [x(q-\lambda_2)^{t'_2}\ldots(q-\lambda_s)^{t'_s}](q-\lambda_1)^{t'_1}$$

produce the solution

$$\begin{aligned} x = (p-\rho_{s+1})\ldots(p-\rho_l)\{&\Sigma(p-\lambda_2)^{t_2}\ldots(p-\lambda_s)^{t_s} \\ &\times[(p-\lambda_1)^{t_1-1}C_1+(p-\lambda_1)^{t_1-2}C_1(q-\lambda_1)+\ldots \\ &\ldots+(p-\lambda_1)^{t_1-t'_1}C_1(q-\lambda_1)^{t'_1-1})]\}(q-\sigma_{s+1})\ldots(q-\sigma_m) \quad (t_1 > t'_1), \end{aligned} \tag{8}$$

where C_1 is an arbitrary number of the algebra.

An artifice similar to that of § 3 suffices to prove it to be a solution.

7. *Applications.* The following examples are confined to matrices and quaternions. In both systems the left- and right-hand characteristic equations are the same, and in the case of n^2-matrices are the n-th powers of the so-called characteristic equation.

Example 1. Consider $\begin{pmatrix} a & b \\ c & d \end{pmatrix} x = x \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$,

where the equations

$$\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0, \quad \begin{pmatrix} \alpha-\lambda & \beta \\ \gamma & \delta-\lambda \end{pmatrix} = 0,$$

have only one root, λ_1, in common.

As above, the solution is

$$\begin{pmatrix} a-\lambda_2 & b \\ c & d-\lambda_2 \end{pmatrix}\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}\begin{pmatrix} \alpha-\lambda'_2 & \beta \\ \gamma & \delta-\lambda'_2 \end{pmatrix},$$

which, on writing $k_1 = (\alpha-\lambda'_2)c_1+\gamma c_2$, $k_2 = (\alpha-\lambda'_2)c_3+\gamma c_4$, becomes

$$\begin{pmatrix} \{(a-\lambda_2)k_1+bk_2\}, & \dfrac{\beta}{\alpha-\lambda'_2}\{(a-\lambda_2)k_1+bk_2\} \\ \{ck_1+(d-\lambda_2)k_2\}, & \dfrac{\beta}{\alpha-\lambda'_1}\{ck_1+(d-\lambda_2)k_2\} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} (\alpha-\lambda'_2)c'_1, & \beta c'_1 \\ (\alpha-\lambda'_2)c'_2, & \beta c'_2 \end{pmatrix},$$

a divisor of zero.

Although the algebra is of order 4, the left- and right-hand characteristic determinants are each of rank 2, giving two independent scalar arbitraries in the solution.

If there are two common roots λ_1, λ_2, we get the solution

$$\begin{pmatrix}(a-\lambda_2)c_1', & \beta c_1'\\ (a-\lambda_2)c_2', & \beta c_2'\end{pmatrix}+\begin{pmatrix}(a-\lambda_1)c_1'', & \beta c_1''\\ (a-\lambda_1)c_2'', & \beta c_2''\end{pmatrix}=\begin{pmatrix}(a-\lambda_2)c_1'+(a-\lambda_1)c_1'', & c_1'+c_1''\\ (a-\lambda_2)c_2'+(a-\lambda_1)c_2'', & c_2'+c_2''\end{pmatrix},$$

which is a divisor of zero only if $c_1'c_2''-c_1''c_1'=0$ or if $\lambda_1=\lambda_2$.

In the complex quaternion system the characteristic equation is the square of the reduced equation, while every divisor of zero is of rank 2.

Example 2. Consider the equation $px=xq$ in complex quaternions, where

$$p=p_0+p_1i+p_2j+p_3k,\quad q=q_0+q_1i+q_2j+q_3k.$$

Their reduced equations are

$$\lambda^2-2p_0\lambda+(p_0^2+p_1^2+p_2^2+p_3^2)=0 \quad \text{or} \quad (\lambda-\lambda_1)(\lambda-\lambda_2)=0,$$

$$\lambda^2-2q_0\lambda+(q_0^2+q_1^2+q_2^2+q_3^2)=0 \quad \text{or} \quad (\lambda-\lambda_1')(\lambda-\lambda_2')=0.$$

If they have a common root $\lambda=\lambda_1'$, the solution is

$$(p-\lambda_2)\,C_1(q-\lambda_2'),$$

a divisor of zero, while, if they have the common roots $\lambda_1=\lambda_1'$, $\lambda_2=\lambda_2'$, the solution is

$$(p-\lambda_2)\,C_1(q-\lambda_2)+(p-\lambda_1)\,C_2(q-\lambda_1),$$

where C_1, C_2 are arbitrary quaternions.

It is easy to see that the first involves two and the second four arbitraries. For the determinants of $(p-\lambda_2)$, $(q-\lambda_2)$ are each of rank 2. Hence the determinant of $[(p-\lambda_2)C_1]$ is of rank 2, while that of $[(p-\lambda_2)C_1(q-\lambda_2)]$ is of rank 2 at most.

When $\lambda_1=\lambda_1'=\lambda_2=\lambda_2'$, $(p-\lambda_1)^2=0$, $(q-\lambda_1)^2=0$, and thus $p-\lambda_1$ or p', $q-\lambda_1$ or q' are nilpotents.

Since the solutions of $px=xq$ are those of $p'x=xq'$, and conversely, the general solution is

$$x=p'C+Cq'=[(-s_1c_1-s_2c_2-s_3c_3)+(s_1c_0-d_3c_2+d_2c_3)i$$
$$+(s_2c_0+d_3c_1-d_1c_3)j+(s_3c_0-d_2c_1+d_1c_2)k], \quad (9)$$

where $s_r=p_r+q_r$, $d_r=p_r-q_r$, $r=1,2,3$, and $s_1d_1+s_2d_2+s_3d_3=0$.

Since $\begin{vmatrix} 0 & -s_1 & -s_2 & -s_3\\ s_1 & 0 & -d_3 & d_2\\ s_2 & d_3 & 0 & -d_1\\ s_3 & -d_2 & d_1 & 0\end{vmatrix}$

is of rank 2, the coefficients of $1, i, j, k$ in (9) are expressible linearly in terms of two arbitrary constants.

It should be noted that when p and q are real quaternions their latent roots are complex, and the solutions other than $x = 0$ are complex quaternions.

The number of independent scalar arbitraries occurring in the above examples is in each case the maximum possible; in special circumstances it may be diminished.

Example 3. Consider the equation in complex quaternions

$$\{2+\sqrt{(-1)}i\}x = x\{-1+2\sqrt{(-1)}i\}.$$

The corresponding reduced equations are

$$(\lambda-1)(\lambda-3) = 0, \quad (\lambda+1)(\lambda-3) = 0.$$

The solution is $c\{j+\sqrt{(-1)}k\}$, only one scalar constant arising, since the complex quaternions $a+bi$ involve only 2 units.

Example 4. Again

$$\{i+\sqrt{(-1)}j\}x = x\{-1+\sqrt{(-1)}i\},$$

with reduced equations $\lambda^2 = 0, \quad \lambda(\lambda+2) = 0,$

has the solution, as in (8),

$$x = \{i+\sqrt{(-1)}j\}\,C\{1+\sqrt{(-1)}i\} = c\{-\sqrt{(-1)}+i+\sqrt{(-1)}j+k\}.$$

Only one arbitrary scalar arises here as a result of the specially chosen coefficients in the equation.

I am indebted to Professor A. R. Richardson for the introduction to this problem and for much helpful criticism. In particular, part of the work of §§ 2 and 3 is due to him.

AN EXAMPLE IN THE THEORY OF SUMMATION OF SERIES BY RIESZ'S TYPICAL MEANS

By K. ANANDA-RAU.

[Received and read 14 February, 1929.]

1. The following theorems were given by Hardy*: —

THEOREM I.—*Suppose that*

$$0 < \lambda_1 < \lambda_2 < \ldots \quad (\lambda_n \to \infty),$$

and that the series $\sum_1^\infty c_n$ *is summable* (λ, κ) *for a* $\kappa > 0$. *Suppose further that*

$$c_n = O\left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right) \tag{1}$$

as $n \to \infty$. *Then the series* $\sum_1^\infty c_n$ *is convergent.*

THEOREM II.—*Suppose that* $\sum_1^\infty c_n$ *is summable* (λ, κ) *for a* $\kappa > 0$, *and that*

$$c_n > -H\frac{\lambda_n - \lambda_{n-1}}{\lambda_n} \quad (n = 2, 3, \ldots), \tag{2}$$

where $H > 0$ *is a constant. Then the series* $\sum_1^\infty c_n$ *is convergent.*

Hardy's proof of Theorem I was defective in one respect, and this was rectified in a note published about ten years ago†. There was also a similar defect in connexion with the proof of Theorem II, and it has not

* G. H. Hardy, "An extension of a theorem on oscillating series", *Proc. London Math. Soc.* (2), 12 (1913), 174–180.

† K. Ananda-Rau, "A note on a theorem of Mr. Hardy's", *Proc. London Math. Soc.* (2), 17 (1918), 334–336.

been found possible so far to give a satisfactory proof of Theorem II. Now it appears that this is not to be wondered at, since examples can be constructed to show that Theorem II, as it stands, is not true. The object of the present note is to give one such example*. If, however, to the hypotheses of Theorem II we add the further supposition that $\lambda_n \sim \lambda_{n-1}$ as $n \to \infty$, then the conclusion that Σc_n is convergent is valid. This can be either proved directly† or deduced as a consequence of known theorems‡.

The fact that, when we consider general sequences (λ_n) unrestricted as to their mode of increase, Theorem I is true while Theorem II is not true is of some theoretical interest. It brings out, for the first time I believe, in the case of summability-convergence theorems, the difference in effect of Tauberian hypotheses of the forms (1) and (2)§. We have methods of summation in which the summability of a series together with a two-sided condition like (1) are sufficient to ensure the convergence of the series, while the summability of a series and the corresponding one-sided condition like (2) do not necessarily imply the convergence of the series.

2. Adopting the notation usual in the theory of summation of series by typical means, let‖

$$C(\omega) = \sum_{\lambda_n \leqslant \omega} c_n,$$

so that

$$C(\omega) = c_1 + c_2 + \ldots + c_p,$$

if $\lambda_p \leqslant \omega < \lambda_{p+1}$; and let

$$C^\kappa(\omega) = \kappa \int_0^\omega C(\tau)(\omega - \tau)^{\kappa-1} d\tau,$$

* The principle underlying the example is sufficiently clear to enable one to construct other examples of a similar character.

† The direct proof is simpler.

‡ The theorems are Theorem 28 (p. 44) of the book, "The general theory of Dirichlet's series" (1915), by G. H. Hardy and M. Riesz, and Theorem F in "Some theorems concerning Dirichlet's series", by G. H. Hardy and J. E. Littlewood, published in the *Messenger of Math.*, 43 (1914), 134-147 (146-147).

If, instead of supposing that $\lambda_n \sim \lambda_{n-1}$, we make the hypothesis that $\lim_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} > 1$, the conclusion of Theorem II that Σc_n is convergent is also valid. In the latter case no condition whatever need be imposed on c_n. See G. H. Hardy, *loc. cit.*, 178, and G. H. Hardy and M. Riesz, *loc. cit.*, Theorem 21 (p. 36).

§ Such a difference has been recently observed in certain methods of summation, when the sum to which the series is summable is $+\infty$ or $-\infty$. See T. Vijayaraghavan, "Converse theorems on summability", *Journal London Math. Soc.*, 2 (1927), 215-222.

‖ $C(\omega) = 0$ for $0 \leqslant \omega < \lambda_1$.

where $\kappa > 0$. If $\omega^{-\kappa} C^{\kappa}(\omega) \to C$ as $\omega \to \infty$, we say, as is well known, that Σc_n is summable (λ, κ) to sum C.

The series considered here is

$$0-1+1-1+1-\ldots, \tag{3}$$

where $c_1 = 0$ and $c_n = (-1)^{n+1}$ for $n > 1$. The sequence $\lambda_1, \lambda_2, \ldots$, is defined as follows*:

$$\lambda_1 = 1,$$

$$\lambda_n = 2\lambda_{n-1} \quad \text{if } n \text{ is even}, \tag{4}$$

$$\lambda_n = \lambda_{n-1} + \frac{1}{2^n} \quad \text{if } n \text{ is odd and } > 1. \tag{5}$$

With these λ's we observe that the terms of the series (3) satisfy the conditions that, when n is odd and greater than 1,

$$c_n = 1 > -3\frac{\lambda_n - \lambda_{n-1}}{\lambda_n},$$

and that, when n is even,

$$c_n = -1 > -\tfrac{3}{2} = -3\frac{\lambda_n - \lambda_{n-1}}{\lambda_n},$$

using (4); so that, for all values of $n > 1$,

$$c_n > -3\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}. \tag{6}$$

3. We shall now prove that the series (3), which is obviously not convergent, is summable (λ, κ) for every $\kappa > 0$ (however small); so that, in view of (6), it will follow that Theorem II is not true as it stands. In proving the summability of (3) we shall suppose that $0 < \kappa < 1$. This is really no restriction, since the summability for $\kappa \geqslant 1$ will follow from the summability for $0 < \kappa < 1$ by making use of a known theorem†.

We observe, to begin with, that if $\lambda_n \leqslant \tau < \lambda_{n+1}$ and n is even, then $C(\tau) = -1$; while, if $\lambda_n \leqslant \tau < \lambda_{n+1}$ and n is odd, then $C(\tau) = 0$.

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* The mode of increase of the λ's may be described as alternately fast and slow. To find the values of the successive terms of the λ-sequence we have to apply formulae (4) and (5) alternately.

† G. H. Hardy and M. Riesz, *loc. cit.*, Theorem 16 (p. 29). The summability of the series (3) for $\kappa \geqslant 1$ can also be proved directly; the direct proof is preferable as the details are very much simpler for the case $\kappa \geqslant 1$ than for the case $0 < \kappa < 1$.

Let $\omega > \lambda_7$ and let $\lambda_p \leqslant \omega < \lambda_{p+1}$, so that $p \geqslant 7$.

First let p be even. Expressing the integral in

$$C^\kappa(\omega) = \kappa \int_0^\omega C(\tau)(\omega-\tau)^{\kappa-1}\, d\tau \tag{7}$$

as a sum of integrals taken over $(0, \lambda_1)$, (λ_1, λ_2), ..., $(\lambda_{p-1}, \lambda_p)$, (λ_p, ω), and inserting the values of $C(\tau)$ in these intervals, we easily see that

$$C^\kappa(\omega) = -\kappa \sum_r \int_{\lambda_{r-1}}^{\lambda_r} (\omega-\tau)^{\kappa-1}\, d\tau - \kappa \int_{\lambda_p}^{\omega} (\omega-\tau)^{\kappa-1}\, d\tau, \tag{8}$$

where the index of summation r on the right-hand side runs only through the odd integers 3, 5, ..., $p-1$.

Now $\omega \geqslant \lambda_p$, $\lambda_r \leqslant \lambda_{p-1}$, and so $\omega-\lambda_r \geqslant \lambda_p-\lambda_{p-1}$; further

$$\lambda_p-\lambda_{p-1} = \lambda_{p-1}$$

since p is even, and $\lambda_r-\lambda_{r-1} = 1/2^r$ since r is odd. Using these and $\kappa-1 < 0$, we get

$$\begin{aligned}\int_{\lambda_{r-1}}^{\lambda_r} (\omega-\tau)^{\kappa-1}\, d\tau &< (\lambda_r-\lambda_{r-1})(\omega-\lambda_r)^{\kappa-1} \\ &\leqslant \frac{1}{2^r}(\lambda_p-\lambda_{p-1})^{\kappa-1} \\ &= \frac{1}{2^r}\lambda_{p-1}^{\kappa-1} < \frac{1}{2^r}.\end{aligned}$$

Therefore*

$$\sum_r \int_{\lambda_{r-1}}^{\lambda_r} (\omega-\tau)^{\kappa-1}\, d\tau < \sum_r \frac{1}{2^r} < \sum_{n=0}^{\infty} \frac{1}{2^n} = 2. \tag{9}$$

Also

$$\int_{\lambda_p}^{\omega} (\omega-\tau)^{\kappa-1}\, d\tau = \frac{1}{\kappa}(\omega-\lambda_p)^\kappa < \frac{1}{\kappa}(\lambda_{p+1}-\lambda_p)^\kappa = \frac{1}{\kappa}\cdot\frac{1}{2^{\kappa(p+1)}} < \frac{1}{\kappa}. \tag{10}$$

Using (9) and (10) in (8), we see that

$$|C^\kappa(\omega)| < 2\kappa+1 < 3. \tag{11}$$

We now proceed to the case when p is odd. Expressing as before the integral in (7) as a sum of integrals over $(0, \lambda_1)$, (λ_1, λ_2), ..., $(\lambda_{p-1}, \lambda_p)$,

* The index of summation r in (9) takes only the odd values 3, 5, ..., $(p-1)$.

(λ_p, ω) and inserting the values of $C(\tau)$ in these intervals, we get

$$C^\kappa(\omega) = -\kappa \sum_r \int_{\lambda_{r-1}}^{\lambda_r} (\omega-\tau)^{\kappa-1}\, d\tau - \kappa \int_{\lambda_{p-1}}^{\lambda_p} (\omega-\tau)^{\kappa-1}\, d\tau, \tag{12}$$

where the index of summation r now runs through the odd integers $3, 5, \ldots, p-2$.

Observing that $\omega > \lambda_{p-1}$, $\lambda_r \leqslant \lambda_{p-2}$,

$$\omega-\lambda_r > \lambda_{p-1}-\lambda_{p-2},$$

$$\lambda_{p-1}-\lambda_{p-2} = \lambda_{p-2}$$

since $p-1$ is even, and $\lambda_r-\lambda_{r-1} = 1/2^r$ since r is odd, we get as before

$$\begin{aligned}\int_{\lambda_{r-1}}^{\lambda_r} (\omega-\tau)^{\kappa-1}\, d\tau &< (\lambda_r-\lambda_{r-1})(\omega-\lambda_r)^{\kappa-1} \\ &< \frac{1}{2^r}(\lambda_{p-1}-\lambda_{p-2})^{\kappa-1} \\ &= \frac{1}{2^r}\lambda_{p-2}^{\kappa-1} < \frac{1}{2^r};\end{aligned}$$

and so*

$$\sum_r \int_{\lambda_{r-1}}^{\lambda_r} (\omega-\tau)^{\kappa-1}\, d\tau < \sum_r \frac{1}{2^r} < 2. \tag{13}$$

To estimate the last integral on the right-hand side of (12), first suppose that $\omega \leqslant \lambda_p + 1$. Then

$$\begin{aligned}\int_{\lambda_{p-1}}^{\lambda_p} (\omega-\tau)^{\kappa-1}\, d\tau &= \frac{1}{\kappa}\{(\omega-\lambda_{p-1})^\kappa - (\omega-\lambda_p)^\kappa\} \\ &\leqslant \frac{1}{\kappa}(\omega-\lambda_{p-1})^\kappa \\ &\leqslant \frac{1}{\kappa}(\lambda_p-\lambda_{p-1}+1)^\kappa \\ &= \frac{1}{\kappa}\left(\frac{1}{2^p}+1\right)^\kappa < \frac{2}{\kappa}.\end{aligned} \tag{14}$$

* The index of summation r in (13) takes only the odd values $3, 5, \ldots, p-2$.

Next let $\omega > \lambda_p + 1$. Then

$$\int_{\lambda_{p-1}}^{\lambda_p} (\omega-\tau)^{\kappa-1}\, d\tau < (\lambda_p-\lambda_{p-1})(\omega-\lambda_p)^{\kappa-1} \tag{15}$$

$$< (\lambda_p-\lambda_{p-1}) = \frac{1}{2^p} < 1.$$

Combining (12), (13), (14), (15), we see that, when p is odd,

$$|C^\kappa(\omega)| < 3\kappa+2 < 5\,;$$

and combining this last result with (11), proved when p is even, it follows that, for all values of $\omega > \lambda_7$, we have

$$|C^\kappa(\omega)| < 5\,;$$

and therefore

$$\omega^{-\kappa} C^\kappa(\omega) \to 0$$

as $\omega \to \infty$. This completes the proof of the summability (λ, κ) of the series (3) to sum zero.

ON A GENERALIZATION OF WATSON'S EXPRESSIONS FOR LEGENDRE'S FUNCTIONS

By E. W. HOBSON.

[Received 3 April, 1929.—Read 25 April, 1929.]

If the plane of μ has a cross-cut on the real axis from $-\infty$ to 1, the value of $Q_n^m(\mu)$ is given* by the formula

$$Q_n^m(\mu) = e^{m\pi i} 2^m \Pi(m-\tfrac{1}{2})\,\Pi(-\tfrac{1}{2})\,\frac{\cos m\pi}{\pi}\,(\mu^2-1)^{\frac{1}{2}m}\int_0^{1/z}\frac{h^{n+m}\,dh}{(1-2\mu h+h^2)^{m+\frac{1}{2}}},$$

provided that $R(n+m+1)>0$ and $R(\tfrac{1}{2}-m)>0$; and where, in $1-2\mu h+h^2$, the phases of $h-z$, $h-1/z$ are taken to be zero at points at which the expressions are real and positive. If we change m into $-m$, and make use of the relation H (12), we have

$$Q_n^m(\mu) = \frac{e^{m\pi i}\,\Pi(-\tfrac{1}{2})\,\Pi(n+m)}{2^m\,\Pi(m-\tfrac{1}{2})\,\Pi(n-m)}\,(\mu^2-1)^{-\frac{1}{2}m}\int_0^{1/z} h^{n-m}(1-2\mu h+h^2)^{m-\frac{1}{2}}\,dh,$$

where $R(n-m+1)>0$ and $R(\tfrac{1}{2}+m)>0$.

We may transform this expression by taking τ as the independent variable, where $h=e^{-\tau}/z$. Let

$$\mu = \cosh\zeta = \cosh(\xi+i\eta);$$

then

$$z = \mu+(\mu^2-1)^{\frac{1}{2}} = e^{\zeta}.$$

We now have

$$Q_n^m(\cosh\zeta) = \frac{e^{m\pi i}\,\Pi(-\tfrac{1}{2})\,\Pi(n+m)}{2^m\,\Pi(m-\tfrac{1}{2})\,\Pi(n-m)}\sinh^{-m}\zeta\,.\,e^{(m-n-1)\zeta}$$

$$\times\int_0^{\infty} e^{-(n+1)\tau+m\tau}\left\{(1-e^{-\tau})(1-e^{-\tau-2\zeta})\right\}^{m-\frac{1}{2}}d\tau. \quad (1)$$

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* See Hobson, *Phil. Trans. Royal Soc.*, 187 (1896), 478, Formula (43). References to this memoir are denoted by H, with the number of the formula added.

If $m = 0$, this becomes

$$Q_n(\cosh\xi) = e^{-(n+1)\zeta}\int_0^\infty e^{-(n+1)\tau}\{(1-e^{-\tau})(1-e^{-\tau-2\zeta})\}^{-\frac{1}{2}}\,d\tau, \qquad (2)$$

which is a formula obtained* otherwise by Watson. Since

$$1-e^{-2\zeta-\tau} = e^{-\zeta}(e^{\zeta}-e^{-\zeta-\tau}) = e^{-\zeta}[\cosh\xi(1-e^{-\tau})+\sinh\xi\,(1+e^{-\tau})],$$

we have

$$Q_n^m(\cosh\xi) = \frac{e^{m\pi i}\,\Pi(-\frac{1}{2})\,\Pi(n+m)}{2^m\,\Pi(m-\frac{1}{2})\,\Pi(n-m)}\operatorname{cosech}^m\xi\,.\,e^{-(n+\frac{1}{2})\zeta}$$

$$\times\int_0^\infty e^{-(n+1)\tau+m\tau}(1-e^{-\tau})^{m-\frac{1}{2}}\{(1-e^{-\tau})\cosh\xi+(1+e^{-\tau})\sinh\xi\}^{m-\frac{1}{2}}\,d\tau.$$

This formula holds good for values of n and m such that

$$R(n-m+1) > 0 \quad \text{and} \quad R(m+\tfrac{1}{2}) > 0\,;$$

in particular, it is valid if n and m are real, n is positive and

$$-\tfrac{1}{2} < m < n+1.$$

Let us take $\xi = 0+i\theta$, where θ is between 0 and π; we have then $\cosh\xi = \cos\theta$ and $\sinh\xi = e^{\frac{1}{2}\pi i}\sin\theta$. We obtain thus

$$Q_n^m\{\cosh(0+i\theta)\} = \frac{e^{m\pi i}\,\Pi(-\frac{1}{2})\,\Pi(n+m)}{2^m\,\Pi(m-\frac{1}{2})\,\Pi(n-m)}e^{-\frac{1}{2}m\pi i}\operatorname{cosec}^m\theta\,.\,e^{-(n+\frac{1}{2})i\theta}e^{(m-\frac{1}{2})\frac{1}{2}\pi i}$$

$$\times\int_0^\infty e^{-(n+1)\tau+m\tau}(1-e^{-\tau})^{m-\frac{1}{2}}\{(1+e^{-\tau})\sin\theta-i(1-e^{-\tau})\cos\theta\}^{m-\frac{1}{2}}\,d\tau.$$

Similarly we find that

$$Q_n^m\{\cosh(0-i\theta)\} = \frac{e^{m\pi i}\,\Pi(-\frac{1}{2})\,\Pi(n+m)}{2^m\,\Pi(m-\frac{1}{2})\,\Pi(n-m)}e^{\frac{1}{2}m\pi i}\operatorname{cosec}^m\theta\,.\,e^{(n+\frac{1}{2})i\theta}e^{-(m-\frac{1}{2})\frac{1}{2}\pi i}$$

$$\times\int_0^\infty e^{-(n+1)\tau+m\tau}(1-e^{-\tau})^{m-\frac{1}{2}}\{(1+e^{-\tau})\sin\theta+i(1-e^{-\tau})\cos\theta\}^{m-\frac{1}{2}}\,d\tau.$$

From these expressions we can find $Q_n^m(\cos\theta)$ and $P_n^m(\cos\theta)$.

Employing the relation H(31),

$$e^{m\pi i}Q_n^m(\cos\theta) = \tfrac{1}{2}[e^{-\frac{1}{2}m\pi i}Q_n^m\{\cos(\theta-0i)\}+e^{\frac{1}{2}m\pi i}Q_n^m\{\cos(\theta+0i)\}],$$

* *Trans. Camb. Phil. Soc.*, 22 (1918), 294.

we have

$$Q_n^m(\cos\theta) = \frac{\Pi(-\frac{1}{2})\,\Pi(n+m)}{2^{m+1}\,\Pi(m-\frac{1}{2})\,\Pi(n-m)}\operatorname{cosec}^m\theta$$
$$\times\Big[e^{-(n+\frac{1}{2})i\theta-\frac{1}{4}\pi i-\frac{1}{2}m\pi i}\int_0^\infty e^{-(n+1)\tau+m\tau}(1-e^{-\tau})^{m-\frac{1}{2}}$$
$$\times\{(1+e^{-\tau})\sin\theta-i(1-e^{-\tau})\cos\theta\}^{m-\frac{1}{2}}\,d\tau$$
$$+e^{(n+\frac{1}{2})i\theta+\frac{1}{4}\pi i+\frac{1}{2}m\pi i}\int_0^\infty e^{-(n+1)\tau+m\tau}(1-e^{-\tau})^{m-\frac{1}{2}}$$
$$\times\{(1+e^{-\tau})\sin\theta+i(1-e^{-\tau})\cos\theta\}^{m-\frac{1}{2}}\,d\tau\Big].$$

Also, using the relation H(30),

$$-i\pi e^{m\pi i}P_n^m(\cos\theta) = e^{-\frac{1}{2}m\pi i}Q_n^m\{\cos(\theta+0i)\}-e^{\frac{1}{2}m\pi i}Q_n^m\{\cos(\theta-0i)\},$$

we have

$$\tfrac{1}{2}\pi P_n^m(\cos\theta) = \frac{\Pi(-\frac{1}{2})\,\Pi(n+m)}{2^{m+1}\,\Pi(m-\frac{1}{2})\,\Pi(n-m)}\operatorname{cosec}^m\theta\,.i$$
$$\times\Big[e^{-(n+\frac{1}{2})i\theta-\frac{1}{4}\pi i-\frac{1}{2}m\pi i}\int_0^\infty e^{-(n+1)\tau+m\tau}(1-e^{-\tau})^{m-\frac{1}{2}}$$
$$\times\{(1+e^{-\tau})\sin\theta-i(1-e^{-\tau})\cos\theta\}^{m-\frac{1}{2}}\,d\tau$$
$$-e^{(n+\frac{1}{2})i\theta+\frac{1}{4}\pi i+\frac{1}{2}m\pi i}\int_0^\infty e^{-(n+1)\tau+m\tau}(1-e^{-\tau})^{m-\frac{1}{2}}$$
$$\times\{(1+e^{-\tau})\sin\theta+i(1-e^{-\tau})\cos\theta\}^{m-\frac{1}{2}}\,d\tau\Big].$$

Henceforth we shall take m and n to be real. Let the phase of

$$\int_0^\infty e^{-(n+1)\tau+m\tau}(1-e^{-\tau})^{m-\frac{1}{2}}\{(1+e^{-\tau})\sin\theta+i(1-e^{-\tau})\cos\theta\}^{m-\frac{1}{2}}\,d\tau$$

be denoted by ω; then the integral is $\rho e^{i\omega}$, where ρ is real and positive. We thus find that

$$\tfrac{1}{2}\pi P_n^m(\cos\theta) = W\sin[(n+\tfrac{1}{2})\theta+\tfrac{1}{2}m\pi+\tfrac{1}{4}\pi+\omega], \tag{3}$$

$$Q_n^m(\cos\theta) = W\cos[(n+\tfrac{1}{2})\theta+\tfrac{1}{2}m\pi+\tfrac{1}{4}\pi+\omega], \tag{4}$$

where W is real and positive; it is assumed that n and m are real and that $m > -\frac{1}{2}$, $n+1 > m$. These are generalizations of results obtained by Watson for the case $m = 0$.

We have

$$\tan\left[(n+\tfrac{1}{2})\,\theta+\tfrac{1}{2}m\pi+\tfrac{1}{4}\pi+\omega\right]=\frac{\pi}{2}\,\frac{P_n^m(\cos\theta)}{Q_n^m(\cos\theta)},$$

hence

$$\frac{d}{d\theta}\left[(n+\tfrac{1}{2})\,\theta+\tfrac{1}{2}m\pi+\tfrac{1}{4}\pi+\omega\right]$$

has the same sign as

$$P_n^m(\mu)\frac{d}{d\mu}Q_n^m(\mu)-Q_n^m(\mu)\frac{d}{d\mu}P_n^m(\mu),$$

where μ denotes $\cos\theta$.

From the differential equation satisfied by $P_n^m(\mu)$ and $Q_n^m(\mu)$, it follows that this expression when multiplied by $1-\mu^2$ is constant for all values of μ between ± 1. Thus the sign of the expression is that of

$$P_n^m(0)\left[\frac{d}{d\mu}Q_n^m(\mu)\right]_{\mu=0}-Q_n^m(0)\left[\frac{d}{d\mu}P_n^m(\mu)\right]_{\mu=0}.$$

From formulae* H, pp. 470, 471,

$$Q_n^m(0)=-2^{m-1}\sin\tfrac{1}{2}(m+n)\,\pi\,\frac{\Pi(\frac{1}{2}n+\frac{1}{2}m-\frac{1}{2})\,\Pi(-\frac{1}{2})}{\Pi(\frac{1}{2}n-\frac{1}{2}m)},$$

$$\left[\frac{d}{d\mu}Q_n^m(\mu)\right]_{\mu=0}=2^m\cos\tfrac{1}{2}(m+n)\,\pi\,\frac{\Pi(\frac{1}{2}n+\frac{1}{2}m)\,\Pi(-\frac{1}{2})}{\Pi(\frac{1}{2}n+\frac{1}{2}m-\frac{1}{2})},$$

$$P_n^m(0)=2^m\cos\tfrac{1}{2}(m+n)\,\pi\,\frac{\Pi(\frac{1}{2}n+\frac{1}{2}m-\frac{1}{2})}{\Pi(\frac{1}{2}n-\frac{1}{2}m)\,\Pi(-\frac{1}{2})},$$

$$\left[\frac{d}{d\mu}P_\mu^m(\mu)\right]_{\mu=0}=2^{m+1}\sin\tfrac{1}{2}(m+n)\,\pi\,\frac{\Pi(\frac{1}{2}n+\frac{1}{2}m)}{\Pi(\frac{1}{2}n-\frac{1}{2}m-\frac{1}{2})\,\Pi(-\frac{1}{2})};$$

we thus find that

$$P_n^m(0)\left[\frac{d}{d\mu}Q_n^m(\mu)\right]_{\mu=0}-Q_n^m(0)\left[\frac{d}{d\mu}P_n^m(\mu)\right]_{\mu=0}=2^{2m}.$$

It follows that

$$(1-\mu^2)\left[P_n^m(\mu)\frac{d}{d\mu}Q_n^m(\mu)-Q_n^m(\mu)\frac{d}{d\mu}P_n^m(\mu)\right]=2^{2m},$$

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* On p. 471, in the expression for $Q_n^m(\mu)$, in the second part the factor 2^{m+1} should be 2^m.

and therefore

$$\frac{d}{d\theta}\{(n+\tfrac{1}{2})\theta+\tfrac{1}{2}m\pi+\tfrac{1}{4}\pi+\omega\} > 0,$$

whence the result follows that

$$n+\tfrac{1}{2}\theta+\tfrac{1}{2}m\pi+\tfrac{1}{4}\pi+\omega$$

is an increasing function of θ.

We see that, as τ increases from 0 to ∞,

$$\arg\{(1+e^{-\tau})\sin\theta+i(1-e^{-\tau})\cos\theta\}$$

changes monotonically from 0 to $\frac{1}{2}\pi-\theta$, and therefore

$$\arg\{(1+e^{-\tau})\sin\theta+i(1-e^{-\tau})\cos\theta\}^{m-\frac{1}{2}}$$

changes monotonically from 0 to $(m-\frac{1}{2})(\frac{1}{2}\pi-\theta)$. Let it be assumed that $-\frac{1}{2} < m < \frac{3}{2}$, then $(m-\frac{1}{2})(\frac{1}{2}\pi-\theta)$ is numerically less than $\frac{1}{2}\pi$ for all values of θ in the interval $(0, \pi)$.

We have then

$$\int_0^\infty e^{-(n+1)\tau+m\tau}(1-e^{-\tau})^{m-\frac{1}{2}}\{(1+e^{-\tau})\sin\theta-(1-e^{-\tau})\cos\theta\}^{m-\frac{1}{2}}\,d\tau$$

$$=\int_0^\infty e^{-(n+1)\tau+m\tau}(1-e^{-\tau})^{m-\frac{1}{2}}\rho^{m-\frac{1}{2}}(\cos\psi+i\sin\psi)\,d\tau,$$

where ψ is in the interval bounded by 0 and $(m-\frac{1}{2})(\frac{1}{2}\pi-\theta)$, and thus ψ is in one of the intervals $(0, \frac{1}{2}\pi)$, $(-\frac{1}{2}\pi, 0)$.

We have, by the first mean value theorem,

$$\int_0^\infty e^{-(n+1)\tau+m\tau}(1-e^{-\tau})^{m-\frac{1}{2}}\rho^{m-\frac{1}{2}}\cos\psi\,d\tau$$

$$=\cos\psi_1\int_0^\infty e^{-(n+1)\tau+m\tau}(1-e^{-\tau})^{m-\frac{1}{2}}\rho^{m-\frac{1}{2}}\,d\tau,$$

and $$\int_0^\infty e^{-(n+1)\tau+m\tau}(1-e^{-\tau})^{m-\frac{1}{2}}\rho^{m-\frac{1}{2}}\sin\psi\,d\tau$$

$$=\sin\psi_2\int_0^\infty e^{-(n+1)\tau+m\tau}(1-e^{-\tau})^{m-\frac{1}{2}}\rho^{m-\frac{1}{2}}\,d\tau.$$

where ψ_1 and ψ_2 are in the same interval as ψ, namely $(0, \frac{1}{2}\pi)$ or $(-\frac{1}{2}\pi, 0)$. We thus have

$$\tan\omega=\frac{\sin\psi_2}{\cos\psi_1};$$

now $$0 \leqslant |\sin \psi_2| \leqslant |\sin \{(m-\tfrac{1}{2})(\tfrac{1}{2}\pi-\theta)\}| < 1$$

and $$1 \geqslant |\cos \psi_1| \geqslant |\cos \{(m-\tfrac{1}{2})(\tfrac{1}{2}\pi-\theta)\}| > 0,$$

hence $$0 \leqslant |\tan \omega| \leqslant |\tan \{(m-\tfrac{1}{2})(\tfrac{1}{2}\pi-\theta)\}|,$$

and thus ω is in the interval bounded by 0 and $(m-\frac{1}{2})(\frac{1}{2}\pi-\theta)$.

This argument would not apply if

$$|(m-\tfrac{1}{2})(\tfrac{1}{2}\pi-\theta)| > \tfrac{1}{2}\pi.$$

We can now obtain limits for the zeros of $P_n^m(\cos\theta)$, when $n > 0$, $-\frac{1}{2} < m \leqslant \frac{3}{2}$ as Watson has done for the case $m = 0$. The case of $Q_n^m(\cos\theta)$ can be dealt with similarly.

PORISTIC CONFIGURATIONS ASSOCIATED WITH A RATIONAL NORMAL CURVE

By V. C. MORTON.

[Received 25 November, 1927.—Read 15 December, 1927.—Revised 10 April, 1929.]

Preface.

It is a well known result due to Darboux* that a curve of order $n-1$ will pass through the vertices of any two n-ads circumscribed to a conic and that there is an involution of such n-ads inscribed in the curve. Von Staudt† and Hurwitz‡ showed that a space cubic will pass through the vertices of any two tetrahedra whose faces osculate a space cubic. Cremona§ generalized further to the case of three n-ads on the cubic curve and the involution determined by them, showing that all their faces touch a surface of class $n-2$. Later Emil Weyr‖ proved that this surface contains $\frac{1}{2}(n-1)(n-2)$ chords of the cubic.

W. F. Meyer¶ proved the general result that all the vertices of the q k-ads formed by q groups of k osculating primes to a rational normal curve of order d in space of d dimensions lie on a variety of dimensions $q-1$ and order $\binom{k-q+1}{d-q+1}$ and that the vertices of the k-ads of the involution determined by the q k-ads also lie on this variety.

Kubota** has extended Cremona's result for the particular case $n=5$ as follows:

Three pentads formed by three groups of osculating planes to a space cubic determine a cubic surface; four pentads, taken in threes, determine

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* Darboux, *Sur une classe remarquable de courbes et de surfaces algébriques* (1896), 195.

† Von Staudt, *Beiträge zur Geometrie der Lage* (1860), 378.

‡ *Math. Ann.*, 20 (1882), 135.

§ Cremona, *Rend. Lombardo* (2), 12 (1879), 347–52.

‖ Weyr, *Bull. de l'acad. roy. de Belgique* (3), 3 (1882), 472–85.

¶ Meyer, *Apolarität und Rationale Curven* (1883), 387.

** *Math. Zeitschrift*, 26 (1927), 450–56.

four such surfaces which pass through the same cubic curve; five pentads, taken in fours, determine five such cubic curves which meet in a point.

F. P. White* has recently used projective methods to obtain several of the above results and, in particular, has given a generalization of Kubota's theorem for the case of $(2r-1)$-ads and for the case of $2r$-ads on a rational normal curve of order r in space of r dimensions.

I wish to investigate, in particular, the poristic nature of the theory and for this purpose I use a modification of the algebraic method of Meyer without, however, making use of the idea of apolarity. Incidentally, I establish generalizations of the results of Weyr, Kubota, and White.

In the second part of the paper I apply a particularization of the theory to obtain some properties of the values of the parameters at multiple points on a rational plane curve and give certain equivalent geometric theorems.

PART I.

1. Meyer† uses a result due to Garbieri‡ to express the alternant $|\phi_1(\theta_1), \phi_2(\theta_2), \ldots, \phi_r(\theta_r)|$ as the product of $\zeta^{\frac{1}{2}}(\theta_1, \theta_2, \ldots, \theta_r)$, the difference product of the parameters θ, and a determinant whose elements are composed of the elementary symmetric functions of these parameters and of the coefficients of the functions ϕ, where the general ϕ is a polynomial in θ of degree n $(n \geqslant r)$.

I shall use a different form for the second factor leading to a more symmetric equation for the surface associated with r n-ads in the subsequent theory. For this purpose the following interpolation theorem, due to Borchardt§, may be used:

If $\phi_1(\theta), \phi_2(\theta), \ldots, \phi_r(\theta)$ *are polynomials, each of degree* n $(n \geqslant r)$, *then the quotient of* $|\phi_1(\theta_1), \phi_2(\theta_2), \ldots, \phi_r(\theta_r)|$ *by* $\zeta^{\frac{1}{2}}(\theta_1, \theta_2, \ldots, \theta_r)$ *can be expressed in the form*

$$\sum \frac{|\phi_1(\alpha_1), \phi_2(\alpha_2), \ldots, \phi_r(\alpha_r)|}{\zeta^{\frac{1}{2}}(\alpha_1, \alpha_2, \ldots, \alpha_r)} \cdot \frac{R(\theta_1, \theta_2, \ldots, \theta_r; \alpha_{r+1}, \ldots, \alpha_{n+1})}{R(\alpha_1, \alpha_2, \ldots, \alpha_r; \alpha_{r+1}, \ldots, \alpha_{n+1})},$$

the summation being for all ways of choosing r *different quantities* α *from* $\alpha_1, \alpha_2, \ldots, \alpha_{n+1}$.

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* *Proc. Camb. Phil. Soc.*, 23 (1927), 886–87. I must acknowledge my indebtedness to this paper for the references used above.

† Meyer, *loc. cit.*, 26.

Giornale di Mat., 16 (1878), 1–17.

§ *Ges. Werke*, 153. I am indebted to Prof. H. W Turnbull for this reference.

If now $\alpha_1, \alpha_2, \ldots, \alpha_n$ are chosen to be the n roots of one of the polynomials $\phi_r(\theta)$, the following result can be deduced by easy algebra:

Let $\phi_1(\theta), \phi_2(\theta), \ldots, \phi_r(\theta)$ be r given polynomials, with leading coefficients unity, each of degree n $(n \geqslant r)$ and with no repeated roots, and let the roots of any chosen one, $\phi_r(\theta)$, be $\alpha_1, \alpha_2, \ldots, \alpha_n$; let L_m be written for $\alpha_m^r - p_1\alpha_m^{r-1} + \ldots + (-1)^r p_r$, where $p_1, p_2, \ldots, p_r$ are the elementary symmetric functions of $\theta_1, \theta_2, \ldots, \theta_r$; then the quotient P of

$$|\phi_1(\theta_1), \phi_2(\theta_2), \ldots, \phi_r(\theta_r)|$$

by $\xi^{\frac{1}{2}}(\theta_1, \theta_2, \ldots, \theta_r)$ can be expressed as a polynomial of degree $n-r+1$ in the expressions L in the form

$$L_1 L_2 \ldots L_n \Sigma \frac{|\phi_1(\alpha_1), \phi_2(\alpha_2), \ldots, \phi_{r-1}(\alpha_{r-1})| \cdot \xi^{\frac{1}{2}}(\alpha_1, \alpha_2, \ldots, \alpha_{r-1})}{\phi_r'(\alpha_1)\, \phi_r'(\alpha_2) \ldots \phi_r'(\alpha_{r-1})\, L_1 L_2 \ldots L_{r-1}},$$

where the summation is for all groups of $r-1$ different roots chosen from $\alpha_1, \alpha_2, \ldots, \alpha_n$.

If roots $\alpha_s, \ldots$ exist common to all the polynomials, then the quotient is $L_s \ldots$ multiplied into a sum similar to P but formed with the expressions $L_1, L_2, \ldots, L_n$, except for $L_s \ldots$.

2. If we now put

$$p_0 = x_0 = 1,\ p_1 = rx_1,\ \ldots,\ p_s = \binom{r}{s} x_s,\ \ldots,\ p_r = x_r,$$

and consider $x_0, x_1, \ldots, x_r$ as homogeneous point coordinates in $[r]$, then the equation $P = 0$ represents a variety V of dimensions $r-1$ and of order $n-r+1$ in this space. Also the osculating prime at the point $\theta = \alpha_s$ on the rational normal curve C, whose equations are

$$x_0 = 1,\ x_1 = \theta,\ \ldots,\ x_r = \theta^r,$$

has the equation $L_s = 0$. Finally, the point $(x_0, x_1, \ldots, x_r)$ is the point of intersection of the osculating primes at the points $\theta_1, \theta_2, \ldots, \theta_r$ on this curve.

If $\theta_1, \theta_2, \ldots, \theta_r$ are put equal to any r roots of any one of the polynomials ϕ, then every element in one column of the numerator of P vanishes. It follows that the variety V passes through all the $r\binom{n}{r}$ ver-

tices of the r n-ads formed by the r groups of n osculating primes at the r groups of n points on C whose parameters are the roots of the polynomials ϕ.

This result is a generalization of that of Cremona, but a special case of that of Meyer. The involution property is given at once by the fact that every group of r roots of the general polynomial

$$\lambda_1\phi_1+\lambda_2\phi_2+\ldots+\lambda_r\phi_r$$

also satisfies the equation $P=0$.

We can thus write down either the point equation or the θ equation of the variety V when the r n-ads are given, the former equation having the natural form

$$\Sigma\frac{\lambda}{L_1L_2\ldots L_{r-1}}=0,$$

where the coefficients λ are independent of the variables x.

3. Two direct generalizations of Weyr's result are given by the following theorems:

THEOREM I. *The variety V passes through $\frac{1}{2}(n-r+1)(n-r+2)$ straight lines on the solid of intersection of any given $r-3$ osculating primes of C, each straight line lying on two other osculating primes.*

THEOREM II. *If r is odd and equal to $2p+1$, then the variety V passes through just $\binom{n-p}{p+1}$ linear varieties (in the whole of space) formed by the intersection of $p+1$ osculating primes of C.*

These results are included in the following still more general theorem:

THEOREM III. *The variety V passes through $\binom{n-r+p+1}{p+1}$ linear varieties of dimensions p in the S_{2p+1} of intersection of any $r-2p-1$ osculating primes of C, each such variety lying in $p+1$ other osculating primes.*

The proof depends on the properties of a set of array equations which will also be used later (Theorem V).

LEMMA. *If $r-2p-1$ of the variables θ are fixed in value, then there are just $\binom{n-r+p+1}{p+1}$ sets of values for the remaining variables which*

will make to vanish every determinant of the array

$$\left| \begin{array}{cccc} \phi_1(\theta_1), & \phi_2(\theta_1), & \ldots, & \phi_r(\theta_1) \\ \ldots & \ldots & \ldots & \ldots \\ \phi_1(\theta_{r-p}), & \phi_2(\theta_{r-p}), & \ldots, & \phi_r(\theta_{r-p}) \end{array} \right|.$$

Consider the rational curve of order n in $[r-1]$ whose equations in homogeneous coordinates are

$$\xi_1 = \phi_1(\theta), \ \ldots, \ \xi_r = \phi_r(\theta).$$

The number of sets of solutions of the array equations, when $r-2p-1$ of the variables θ are fixed, is the number of $(r-p)$-secant linear varieties of dimensions $r-p-2$ of the above curve which pass through $r-2p-1$ fixed points on the curve.

If the curve is projected on an S_{2p} by means of a variable S_{r-2p-1} through the $r-2p-1$ fixed points, a rational curve of order $n-r+2p+1$ is obtained. Also every S_{r-p-2} through the fixed points will project into an S_{p-1} in the new space.

Thus the number required is the number of $(p+1)$-secant spaces of dimensions $p-1$ of a rational curve of order $n-r+2p+1$ in S_{2p}.

This number is known* to be $\binom{n-r+p+1}{p+1}$ and the lemma is proved.

Now consider the θ form of the equation $P=0$ of the variety V. Whatever the values of $\theta_{r-p+1}, \ldots, \theta_r$, this equation will be satisfied for all groups of the remaining variables which satisfy the array equations of the lemma.

It follows that there will be just $\binom{n-r+p+1}{p+1}$ sets of values for $p+1$ of the variables which, together with fixed values for $r-2p-1$ of the variables and any values for the remaining p variables, will satisfy the equation of the variety V. The theorem stated follows at once.

4. A consideration of the array equations formed by the vanishing of every determinant of the array

$$\left| \begin{array}{ccc} \phi_1(\theta_1), & \ldots, & \phi_p(\theta_1) \\ \ldots & \ldots & \ldots \\ \phi_1(\theta_r), & \ldots, & \phi_p(\theta_r) \end{array} \right| \qquad (r > p)$$

* Castelnuovo, *Rend. di Palermo*, 3 (1889), 27.

(where in this case the number of variables is greater than the number of functions ϕ) leads to the general result due to Meyer given in the preface. A simple method of obtaining the result is as follows:

The vanishing of the array determinants gives the locus of all the vertices of the n-ad formed by the n osculating primes at points on C whose parameters are the roots of the equation

$$\lambda_1\phi_1(\theta)+\ldots+\lambda_p\phi_p(\theta)=0,$$

where the functions ϕ are fixed but the numbers λ vary.

Now consider the rational curve A of order n in $[p-1]$ whose equations in homogeneous coordinates are

$$\xi_1=\phi_1(\theta),\ \ldots,\ \xi_p=\phi_p(\theta).$$

The vanishing of the array determinants expresses the fact that the r points on this curve, $\theta_1, \theta_2, \ldots, \theta_r$, lie on a prime.

If $p-1$ of these points are fixed, the prime is also fixed and the remaining $r-p+1$ points of the group of r points can then be chosen in $\binom{n-p+1}{r-p+1}$ ways.

If we consider a parametric correspondence between the curves A and C in which points on A correspond to osculating primes of C, the fixing of the $p-1$ points $\theta_1, \theta_2, \ldots, \theta_{p-1}$ on A is equivalent to the fixing of $p-1$ osculating primes of C. The S_{r-p+1} of intersection of these primes must then meet the locus in the space of C given by the matrix equations in $\binom{n-p+1}{r-p+1}$ *points*, each of which will have, as θ coordinates,

$$\theta_1,\ \theta_2,\ \ldots,\ \theta_{p-1}$$

and a set of $r-p+1$ values of θ chosen from the group of remaining intersections with A of a constructed prime.

The locus must thus be a variety of dimensions $p-1$ and order $\binom{n-p+1}{r-p+1}$, which is Meyer's result. The involution property is obvious.

Meyer's theorem gives a complete account of the loci generated by any number p of n-ads from $p=2$ to $p=r$. In the next paragraph I give a succession of theorems, generalizations of the results of Kubota and White for p greater than r.

5. It will be sufficient to give the first, the general, and the last results of the set, the first and last being included for the sake of clearness.

THEOREM IV. *The $r+1$ varieties of order $n-r+1$, each through the vertices of all but one of any $r+1$ osculating n-ads of the curve C, meet in a common V_{r-2} of order $\frac{1}{2}(n-r)(n-r+1)$, each pair meeting in a residual V_{r-2} of order $\frac{1}{2}(n-r+1)(n-r+2)$.*

THEOREM V. *The $\binom{r+k}{r}$ varieties of order $n-r+1$, each through the vertices of a group of r chosen from any $r+k$ osculating n-ads of the curve C, meet in a common V_{r-k-1} of order $\binom{n-r+1}{k+1}$.*

THEOREM VI. *The $\binom{2r-1}{r}$ varieties of order $n-r+1$, each through the vertices of a group of r chosen from any $2r-1$ osculating n-ads of the curve C, meet in a group of $\binom{n-r+1}{r}$ common points.*

The proof of the general result V is given by means of the lemma of Theorem III as follows:

All points whose θ coordinates satisfy the equations obtained by the vanishing of all determinants of the array

$$\begin{vmatrix} \phi_1(\theta_1), & \dots, & \phi_{r+k}(\theta_1) \\ \dots & \dots & \dots \\ \vdots & & \vdots \\ \phi_1(\theta_r), & \dots, & \phi_{r+k}(\theta_r) \end{vmatrix}$$

will be common to all the $\binom{r+k}{r}$ varieties defined in the statement of the theorem. Suppose that such points describe a variety V.

Using the lemma, it is clear that the S_{k+1} of intersection of any $r-k-1$ osculating primes meets V in just $\binom{n-r+1}{k+1}$ points.

The variety V is thus of dimensions $r-k-1$ and of order $\binom{n-r+1}{k+1}$.

Theorems IV and VI are obtained by putting 1 and $r-1$ respectively for k.

No result is obtained for k greater than $r-1$. The structure of the configurations is thus as follows:

The $(r+1)$ V_{r-1}^{n-r+1} determined by sets of r of $(r+1)$ n-ads meet in a common $V_{r-2}^{\binom{n-r+1}{2}}$.

The $r+2$ such $V_{r-2}^{\binom{n-r+1}{2}}$ determined by sets of $r+1$ of $r+2$ n-ads meet in a common $V_{r-3}^{\binom{n-r+1}{3}}$, and so on until, finally,

The $2r-1$ such $V_1^{\binom{n-r+1}{r-1}}$ determined by sets of $2r-2$ of $2r-1$ n-ads meet in a group of $\binom{n-r+1}{r}$ common points.

Kubota's result is obtained by putting $r=3$, $n=5$. White's two results are obtained by putting in turn $n=2r-1$, $n=2r$; the former of these is of particular interest in that it leads to a single final point. A special case worthy of note of Theorem VI is for $r=2$. The result is then:

The $n(n-1)$ vertices of two polygons of n sides circumscribed to a conic lie on a curve of order $n-1$; the three curves so formed by pairs of three such polygons intersect in a group of $\frac{1}{2}(n-1)(n-2)$ points*. It will be shown later that this group of points has a close connection with the double points of a certain unicursal curve of order n.

6. The results of the last paragraph suggest a further generalization of paragraph 3.

THEOREM VII. *The* $V_{r-k-1}^{\binom{n-r+1}{k+1}}$ *derived from* $r+k$ *osculating* n*-ads of* C *passes through* $\binom{n-r+h+1}{h+k+1}$ *linear varieties of dimensions* h *in the* S_{2h+k+1} *of intersection of any* $r-2h-k-1$ *osculating primes of* C*, each such variety lying in* $h+k+1$ *other osculating primes.*

The proof is obtained by applying to the matrix

$$\begin{vmatrix} \phi_1(\theta_1), & \dots, & \phi_{r+k}(\theta_1) \\ \cdots & \cdots & \cdots \\ \vdots & & \vdots \\ \phi_1(\theta_{r-h}), & \dots, & \phi_{r+k}(\theta_{r-h}) \end{vmatrix}$$

the result of the lemma of Theorem III. It is then found that the $V_{r-k-1}^{\binom{n-r+1}{k+1}}$ is satisfied for $\binom{n-r+h+1}{h+k+1}$ sets of values for $h+k+1$ of the variables when fixed values are given to $r-2h-k-1$ of the variables and any values are assigned to the remaining h variables. The result follows.

* Cf. Darboux, *loc. cit.*, 195.

Again, the most interesting special cases are for $h = 1$ and for

$$2h+k+1 = r.$$

7. It is evident that a variety V_{r-1} through the vertices of r osculating n-ads of C is of a special type (*e.g.* a conic through the six vertices of two triangles circumscribed to a given conic).

The following succession of results indicates the poristic nature of the theory and forms a generalization of the known theorem due to Darboux that, if a curve of order $n-1$ passes through the vertices of a complete polygon of n sides circumscribed to a conic and also through two vertices of a circumscribed triangle, then it must pass through the third vertex.

THEOREM VIII. *If a* V_{r-1}^{n-r+1} *passes through the* $\binom{n}{r}$ *vertices of one osculating n-ad of C and also through any r vertices (in one osculating prime) of an osculating simplex of C, then it must pass through the remaining vertex*.*

Let the parameters of the faces of the simplex be $\beta_1, \beta_2, \ldots, \beta_{r+1}$ and those of the n-ad be $a_1, a_2, \ldots, a_n$. Then the equation of any V_{r-1}^{n-r+1}, which we denote by σ, through the vertices of the n-ad is

$$\Sigma \frac{A_{1,2,\ldots,r-1}}{L_1 L_2 \ldots L_{r-1}} = 0,$$

where L_m has the same significance as before, except for sign, and so can be written $\prod_{s=1}^{r} (\theta_s - a_m)$, and where the summation is for all ways of choosing $r-1$ suffixes from $1, 2, \ldots, n$.

We have to show that, using θ coordinates, if the r points

$$(\beta_1, \beta_3, \ldots \beta_{r+1}),\ (\beta_1, \beta_2, \beta_4, \ldots, \beta_{r+1}),\ \ldots,\ (\beta_1, \beta_2, \ldots, \beta_r)$$

lie on σ, then so does the point $(\beta_2, \beta_3, \ldots, \beta_{r+1})$.

Expressing the conditions that the r given points lie on σ and re-grouping the factors in the denominators, we have a set of r equations

$$\Sigma \frac{A_{1,2,\ldots,r-1} \prod_{m=1}^{r-1} (\beta_2 - a_m)}{\prod_{m=1}^{r-1} (\beta_1 - a_m) \prod_{m=1}^{r-1} (\beta_2 - a_m) \ldots \prod_{m=1}^{r-1} (\beta_{r+1} - a_m)} = 0$$

* We except the special case when the given prime of the simplex is itself a face of the n-ad. In this case the given conditions themselves can be shown to be poristic. (Cf. the proof of Theorem XI.)

with $r-1$ other equations derived, from the one given, by writing $\beta_3, \beta_4, \ldots, \beta_{r+1}$ in turn for β_2 in the numerators only. The leading term only is written down, and the summation is for all groups of $r-1$ different quantities α chosen from $\alpha_1, \alpha_2, \ldots, \alpha_n$. The quantities β remain unaltered throughout all the terms of each equation, and denominators in corresponding terms are the same for all the r equations.

Now we can choose (uniquely) multiplying factors $P_2, P_3, \ldots, P_{r+1}$ *independent of the quantities* α, so that

$$\sum_{s=2}^{r+1} P_s \prod_{m=1}^{r-1} (\beta_s - \alpha_m) = \prod_{m=1}^{r-1} (\beta_1 - \alpha_m),$$

for we have only to arrange for the coefficients of the elementary symmetric functions of the $r-1$ quantities α to be the same on each side, and the determinant of the equations so obtained cannot vanish, since all the quantities β are different.

So, multiplying the r equations in turn by $P_2, P_3, \ldots, P_{r+1}$ and adding, we have the equation

$$\Sigma \frac{A_{1,2,\ldots,r-1}}{\prod_{m=1}^{r-1} (\beta_2 - \alpha_m) \ldots \prod_{m=1}^{r-1} (\beta_{r+1} - \alpha_m)} = 0$$

showing that the point $(\beta_2, \ldots, \beta_{r+1})$ lies on σ^*.

The result can now be extended to the following:

THEOREM IX. *If a* V_{r-1}^{n-r+1} *passes through the* $\binom{n}{r}$ *vertices of one* n*-ad and also through the* $\binom{n-1}{r-1}$ *vertices in one osculating prime of a second* n*-ad, then it must pass through all the remaining vertices*†.

For taking any r vertices, in the given prime, whose coordinates are α_1 and $r-1$ parameters chosen from any r of $\alpha_2, \alpha_3, \ldots, \alpha_n$, then σ must pass through the $(r+1)$-th point associated with them to form a simplex, and by choosing all possible groups of r such points the theorem follows.

---

* This result gives an example of a group of $\binom{n}{r} + r + 1$ points which are not independent for the purpose of determining a V_{r-1}^{n-r+1} through them, although their number is short by $\binom{n}{r-1} - (r+2)$ of the number of independent conditions necessary to determine such a variety.

† This theorem will evidently hold if an m-ad is used instead of the second n-ad so long as $n \geqslant m \geqslant r+1$.

THEOREM X. *Any* V_{r-1}^{n-r+1} *through the vertices of two n-ads must contain the vertices of the single infinity of n-ads in the involution determined by them, and so must contain a fixed curve.*

Let the parameters of the two n-ads be respectively $\beta_1, \beta_2, \ldots, \beta_n$ and $\alpha_1, \alpha_2, \ldots, \alpha_n$, and let the two sets be the roots of the polynomials $\phi_1(\theta)$ and $\phi_2(\theta)$.

Using the same equation as before for the variety σ through the n-ad ϕ_2, a set of equations, expressing the conditions that σ also passes through the n-ad ϕ_1, will be obtained by putting $\theta_1 = \beta_1$, $\theta_2 = \beta_{p_2}$, $\ldots$, $\theta_r = \beta_{p_r}$ in the equation of σ for all ways of choosing $p_2, \ldots, p_r$ from 2, 3, $\ldots$, n.

The leading term of the typical equation will then have for numerator $A_{1,2,\ldots,r-1}$ and for denominator

$$\prod_{s=1}^{r-1}(\alpha_s-\beta_1)\prod_{s=1}^{r-1}(\alpha_s-\beta_{p_2})\ldots\prod_{s=1}^{r-1}(\alpha_s-\beta_{p_r}),$$

that is, will be

$$\text{(i)}\qquad \frac{A_{1,2,\ldots,r-1}\prod_{s=1}^{r-1}(\alpha_s-\beta_{q_1})\ldots\prod_{s=1}^{r-1}(\alpha_s-\beta_{q_t})}{\prod_{s=1}^{r-1}(\alpha_s-\beta_1)\prod_{s=1}^{r-1}(\alpha_s-\beta_2)\ldots\prod_{s=1}^{r-1}(\alpha_s-\beta_n)},$$

where t is written for $n-r$ and $\beta_{q_1}, \ldots, \beta_{q_t}$ are just the numbers chosen from 2, 3, $\ldots$, n different from $p_2, \ldots, p_r$.

Re-grouping the factors in the denominator of (1), this denominator is seen to be the product $\phi_1(\alpha_1)\ldots\phi_1(\alpha_{r-1})$, which, since $\phi_2(\alpha_s) = 0$ for all s, may be written

$$\text{(ii)}\qquad \prod_{s=1}^{r-1}\{\phi_1(\alpha_s)+\lambda\phi_2(\alpha_s)\}.$$

Now multipliers $P_{q_1,\ldots,q_t}$, *independent of* $\alpha_1, \alpha_2, \ldots, \alpha_{r-1}$, can be chosen so that

$$\text{(iii)}\qquad \Sigma P_{q_1\ldots q_t}\prod_{s=1}^{r-1}(\alpha_s-\beta_{q_1})\ldots\prod_{s=1}^{r-1}(\alpha_s-\beta_{q_t}) = \prod_{s=1}^{r-1}(\alpha_s-\gamma_{r+1})\ldots\prod_{s=1}^{r-1}(\alpha_s-\gamma_n),$$

the summation being for all ways of choosing $n-r$ numbers $q_1, \ldots, q_t$ from 2, 3, $\ldots$, n, and $\gamma_{r+1}, \ldots, \gamma_n$ being any chosen quantities. For there is a homogeneous linear relation, independent of the quantities α,

between any $r+1$ of the expressions

$$\prod_{s=1}^{r-1}(a_s-\xi_p) \quad (p=1, 2, \ldots),$$

where ξ is any number; so, in particular, the expression on the right of (iii) is a homogeneous polynomial of degree $n-r$ in any r of the products on the left. The right-hand side of (iii) can thus be expressed (uniquely) as a linear product sum, $n-r$ at a time, of just the $n-1$ products

$$\prod_{s=1}^{r-1}(a_s-\beta_m) \quad (m=2, 3, \ldots, n),$$

which do occur on the left*.

In fact, the typical multiplier can be determined at once by putting

$$a_1=\beta_{p_2},\ a_2=\beta_{p_3},\ \ldots,\ a_{r-1}=\beta_{p_r},$$

in which case all the terms on the left vanish except the typical term written.

We now choose $\gamma_{r+1}, \ldots, \gamma_n$ to be any $n-r$ roots of the equation $\phi_1(\theta)+\lambda\phi_2(\theta)=0$, the remaining roots being $\gamma_1, \ldots, \gamma_r$.

Then, multiplying the equations corresponding to (i) by the appropriate multipliers and adding, we obtain, by the use of (ii) and (iii), the equation

$$\Sigma \frac{A_{1, 2, \ldots, r-1}}{\prod_{s=1}^{r-1}(a_s-\gamma_1)\ldots\prod_{s=1}^{r-1}(a_s-\gamma_r)}=0,$$

which, when written in the form

$$\Sigma \frac{A_{1, 2, \ldots, r-1}}{\prod_{p=1}^{r}(\gamma_p-\alpha_1)\ldots\prod_{p=1}^{r}(\gamma_p-\alpha_{r-1})}=0,$$

shows that the point $(\gamma_1, \ldots, \gamma_r)$ lies on σ, where $\gamma_1, \ldots, \gamma_r$ are any r roots of the equation

$$\phi_1(\theta)+\lambda\phi_2(\theta)=0.$$

The last part of the theorem follows from Meyer's result given in the preface.

There is a set of results of the type of Theorems VIII, IX, and X relating to V_{r-1}^{n-r+1} through two, three, four, etc., n-ads. I give the general theorem of the set.

* A. L. Dixon, *Journal London Math. Soc.*, 2 (1927), 25, 26.

THEOREM XI. *If a* V_{r-1}^{n-r+1}, σ, *passes through all the vertices of* p *n-ads,* $\phi_1, \ldots, \phi_p$ $(p < r)$ *and through the* $r-p+1$ *vertices of a simplex which lie in the* S_{r-p} *of intersection of any* p *faces of the simplex, then* σ *must pass through the remaining* p *vertices.*

If σ *passes through all the vertices of the* p *n-ads, and through all the* $\binom{n-p}{r-p}$ *vertices of a* $(p+1)$*-th n-ad* ϕ_{p+1} *which lie in the* S_{r-p} *of intersection of* p *of its faces, then* σ *must pass through the remaining* $\binom{n}{r} - \binom{n-p}{r-p}$ *vertices;* σ *will then contain the vertices of all n-ads* $\phi_1+\lambda_2\phi_2+\ldots+\lambda_{p+1}\phi_{p+1}$, *and will thus contain a fixed* $V_p^{\binom{n-p}{r-p}}$.

The proof is by induction as follows.

Let us assume that, if σ passes through the vertices of p n-ads $\phi_1, \ldots, \phi_p$, then it passes through all the vertices of any n-ad

$$\phi_1+\lambda_2\phi_2+\ldots+\lambda_p\phi_p.$$

Let σ pass, in addition, through the $r-p+1$ points (of a simplex)

$$(a_1, a_2, \ldots, a_p, a_{p+2}, \ldots, a_{r+1}),$$
$$(a_1, a_2, \ldots, a_p, a_{p+1}, a_{p+3}, \ldots, a_{r+1}),$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$(a_1, a_2, \ldots, a_p, a_{p+1}, \ldots, a_r).$$

Then, by a proper choice of $\lambda_2, \ldots, \lambda_p$, we can, by supposition, choose an n-ad to lie in σ and to have $a_1, a_2, \ldots, a_{p-1}$ as $p-1$ of its parameters. Let the remaining roots of the equation

$$\phi_1+\lambda_2\phi_2+\ldots+\lambda_p\phi_p = 0$$

be $\beta_p, \beta_{p+1}, \ldots, \beta_n$. Then the equation of σ can be written

$$L_1 L_2 \ldots L_n \Sigma \frac{A_{1,2,\ldots,r-1}}{L_1 L_2 \ldots L_{r-1}} = 0,$$

where

$$L_s = \prod_{m=1}^{r} (\theta_m - a_s) \quad (s = 1, 2, \ldots, p-1),$$

$$L_s = \prod_{m=1}^{r} (\theta_m - \beta_s) \quad (s = p, p+1, n).$$

Now L_s vanishes when a_s is substituted for any θ $(s = 1, 2, \ldots, p-1)$; so the conditions that the $r-p+1$ given points should lie on σ reduce to the equations

(iv)
$$\Sigma \frac{A_{1,2,\ldots,p-1,q_1,\ldots,q_t} (\beta_{q_1} - a_{p+1}) \ldots (\beta_{q_t} - a_{p+1})}{\prod_{s=1}^{r+1} (\beta_{q_1} - a_s) \ldots \prod_{s=1}^{r+1} (\beta_{q_t} - a_s)} = 0$$

(where t is written for $r-p$ and the summation is for all ways of choosing $q_1, \ldots, q_t$ from $p, p+1, \ldots, n$) together with $r-p$ further equations similar to (iv), with $\alpha_{p+2}, \ldots, \alpha_{r+1}$ in turn for α_{p+1} in the numerators.

Multipliers, independent of the quantities β, can be so chosen that an equation may be derived of the form (iv) with α_p for α_{p+1} in the numerators. This equation is the condition that the point $(\alpha_1, \alpha_2, \ldots, \alpha_{p-1}, \alpha_{p+1}, \ldots, \alpha_{r+1})$ should lie on σ.

Similarly, by choosing any $p-1$ of $\alpha_1, \alpha_2, \ldots, \alpha_p$ as roots of the equation $\phi_1+\lambda_2\phi_2+\ldots+\lambda_p\phi_p=0$, we can prove that all the vertices of the simplex formed by the osculating primes at $\alpha_1, \alpha_2, \ldots, \alpha_{r+1}$ lie on σ.

The second part of the theorem follows as in Theorem IX.

It has now been shown that σ passes through the vertices of the $p+1$ n-ads, $\phi_1, \phi_2, \ldots, \phi_p, \phi_{p+1}$, so, by the assumption, any n-ad

$$\phi_1+\mu_2\phi_2+\ldots+\mu_p\phi_p,$$

and any n-ad $\quad \phi_1+\nu_2\phi_2+\ldots+\nu_{p-1}\phi_{p-1}+\nu_{p+1}\phi_{p+1}$

will both lie in σ. Thus, by Theorem X, the n-ad

$$\phi_1+\lambda_2\phi_2+\ldots+\lambda_p\phi_p+\lambda_{p+1}\phi_{p+1}$$

must lie in σ for all values of $\lambda_2, \ldots, \lambda_{p+1}$.

Using Theorem X as an initial result, the induction is completed. The last part of the theorem again follows from Meyer's result.

The last result of the set is that, if σ passes through $r-1$ n-ads and through any two vertices of a simplex, then it must pass through the remaining $r-1$ vertices; if it passes through $n-r+1$ vertices, of an r-th n-ad, which lie on a straight line, then it must pass through all the remaining vertices.

The total number of independent points which define a V_{r-1}^{n-r+1} through the vertices of r osculating n-ads is thus

$$\binom{n}{r}+\binom{n-1}{r-1}+\ldots+\binom{n-r+1}{1}.$$

This number is $\binom{n+1}{r}-1$, which is just the number necessary to determine a general V_{r-1}^{n-r+1}.

8. For $r=2$ the foregoing results lead to the theory of complete poristic polygons circumscribed to a conic.

For $r = 3$ we have the following properties of the space cubic curve:

If a surface of order $n-2$ passes through the vertices of an osculating n-ad of a space cubic, and also through three vertices of an osculating tetrahedron, then it must pass through the fourth. If the surface passes through the vertices of two such n-ads, then it passes through a fixed curve of order $\frac{1}{2}(n-1)(n-2)$; if it also passes through two vertices of an osculating tetrahedron, then it must pass through the remaining two and is fixed in position.

A variety of results may be obtained by taking special cases in the general theory. For example, it is easy to prove, by Theorems X and XI, the following particular applications to the space cubic:

The centroids of all tetrahedra, whose faces osculate a space cubic and whose vertices lie in a space cubic, lie on a straight line; the circumscribing spheres of such tetrahedra form a coaxal system.

The centroids of all tetrahedra, whose faces osculate a space cubic and whose vertices lie on a quadric, lie on a plane; the centres of the circumscribing spheres of such tetrahedra lie in a plane and the spheres pass through two fixed points.

PART II.

Application to a rational plane curve.

1. The auxiliary curve A used in the lemma of § 3, Part I, will be the subject of investigation for the case $r = 3$, $p = 1$, and we recall the particular result in § 5 that the three curves of order $n-1$, each through the vertices of two of three n-ads ϕ, ψ, f, circumscribed to the conic C, $x = \theta^2$, $y = 2\theta$, $z = 1$, meet in a common group of $\frac{1}{2}(n-1)(n-2)$ points. In this case the point coordinates of any point in the plane, whose Darboux coordinates are θ_1 and θ_2, are $x = \theta_1\theta_2$, $y = \theta_1+\theta_2$, $z = 1$.

If ϕ, ψ, and f are taken to be of degree three in θ and so chosen that each pair has one common root, the three circumscribed triangles form a hexagon, and the above result gives Brianchon's theorem as a special case. So in the general case the $\frac{1}{2}(n-1)(n-2)$ common points of the three curves may be called the Brianchon group of the three n-ads.

Consider the rational plane curve A whose coordinates are given by the equations

$$x = \phi(\theta), \quad y = \psi(\theta), \quad z = f(\theta).$$

Then the double points of A have parameters θ_1 and θ_2 given by the equations

$$\phi(\theta_1)/\phi(\theta_2) = \psi(\theta_1)/\psi(\theta_2) = f(\theta_1)/f(\theta_2).$$

These three equations (taking pairs)

$$P_1 = 0, \quad P_2 = 0, \quad P_3 = 0$$

are the equations in Darboux coordinates of the three curves of order $n-1$ each through all the vertices of two of the three polygons ϕ, ψ, and f circumscribed to the conic C. We thus have the result:

THEOREM I. *The Brianchon group of three n-ads ϕ, ψ, and f circumscribed to the conic C gives, by the Darboux coordinates θ_1 and θ_2 of the points of the group, the pairs of parameters at the double points of the rational curve of order n, $x = \phi(\theta)$, $y = \psi(\theta)$, $z = f(\theta)$.*

If $\frac{1}{2}r(r-1)$ of the points of such a Brianchon group themselves form a polygon of r sides circumscribed to C, then the corresponding $\frac{1}{2}r(r-1)$ double points of A coalesce into a point of multiplicity r; for in that case, if the corresponding tangents to C touch at the points $\theta_1, \theta_2, \ldots, \theta_r$, then the ratios $\phi(\theta_p) : \psi(\theta_p) : f(\theta_p)$ are equal to the ratios $\phi(\theta_q) : \psi(\theta_q) : f(\theta_q)$, where p and q are any two suffixes chosen from $1, 2, \ldots, r$

If any one of the points of the Brianchon group lies on C itself, then the corresponding double point is evidently a cusp.

Finally the parameters of the points of intersection with C of the curve $P_1 = 0$, through the vertices of the polygons ψ and f, are the parameters of the points of contact of the tangents from the point $y = 0, z = 0$ to A, thus verifying that the class of a rational curve is $2n-2-\kappa$.

2. *Notation.* A curve A of order $n-1$ will be said to be special for a conic C if complete polygons of n sides can be inscribed in the curve and circumscribed to the conic.

The point $(\theta_1\theta_2, \theta_1+\theta_2, 1)$ in the plane of C will be said to be the representation of that double point on A at which the parameters are θ_1 and θ_2.

Let the roots of the three polynomials ϕ, ψ, f be

$$\alpha_1, \alpha_2, \ldots, \alpha_n; \quad \alpha_{n+1}, \alpha_{n+2}, \ldots, \alpha_{2n}; \quad \alpha_{2n+1}, \alpha_{2n+2}, \ldots, \alpha_{3n}.$$

Let the polygons ϕ, ψ, f be then denoted by

$$(1, 2, \ldots, n), \quad (n+1, n+2, \ldots, 2n), \quad (2n+1, 2n+2, \ldots, 3n).$$

Let the curve $P_3 = 0$ through the vertices of the polygons ϕ and ψ be represented by

$$\{1, 2, \ldots, n; \; n+1, n+2, \ldots, 2n\}$$

with corresponding symbols for the curves $P_1 = 0$ and $P_2 = 0$.

If a_r and a_{n+s} are equal, in which case r may be written for $n+s$, then the equation $P_3 = 0$ is satisfied by $\theta_1 = a_r$ for all values of θ_2. It follows that the curve

$$\{1, 2, \ldots, r, \ldots, n;\ n+1, \ldots, n+s-1, r, n+s+1, \ldots, 2n\}$$

must break up into the tangent at the point a_r to the conic and a special curve of order $n-2$,

$$\{1, 2, \ldots, r-1, r+1, \ldots, n;\ n+1, \ldots, n+s-1, n+s+1, \ldots, 2n\}.$$

In general, if the two sets in the brackets contain m numbers in common, then the curve breaks up into m tangents to C, each corresponding to one of these numbers, and a residual special curve of order $n-m-1$ obtained by cancelling the common numbers.

Finally let the curve A,

$$x = \phi(\theta), \quad y = \psi(\theta), \quad z = f(\theta),$$

be represented by

$$x = [1, 2, \ldots, n], \quad y = [n+1, n+2, \ldots, 2n], \quad z = [2n+1, 2n+2, \ldots, 3n]$$

with appropriate modification by repetition of numbers if roots exist common to any pair of the polynomials ϕ, ψ, f.

3. If the curve A has $\frac{1}{2}m(m+3)-1$ $(= M)$ points of multiplicities $m_1, m_2, \ldots, m_M$, then it cannot have another point of multiplicity $r+1$ where $m_1+m_2+\ldots+m_M+r \geqslant mn$, for otherwise a curve of order m would cut A in more points than mn. This fact suggests, by the statement immediately following Theorem I, that, if A has M points with the given multiplicities, then the representations of all the remaining double points lie on a curve of order $r-1$, where $r = mn-m_1-m_2-\ldots-m_M$. It can be shown by somewhat detailed working that, if m lies between 2 and $n-3$, the equation

$$m_1+m_2+\ldots+m_M+r = mn \quad (r \geqslant 2)$$

leads to the inequality

$$\tfrac{1}{2}(n-1)(n-2)-\tfrac{1}{2}\sum_{s=1}^{M} m_s(m_s-1) < \tfrac{1}{2}(r-1)(r+2).$$

So for such values of m the above statement is certainly true, since the number of remaining double points is less than the number necessary to determine a curve of degree $r-1$. If m is equal to or greater than $n-3$, it can again be shown that the proposed theorem gives no information.

I shall prove the results for the two remaining cases $m=1$ and $m=2$ and give the corresponding geometrical theorems on curves special for a conic.

4. THEOREM II. *If the curve A has a point of multiplicity p, then the representations of all the remaining double points lie on a curve of order $n-p-1$ special for the conic C.*

Any linear transformation of coordinates in the plane of A will lead to a rational curve with multiple points having the same parameters as the multiple points on the original curve A. By such a transformation three of the multiple points can be made the vertices of the triangle of reference. In this case let the point of multiplicity p be $x=0,\ y=0$.

Then the curve is

$$x=[1, 2, \ldots, p, p+1, \ldots, n],$$
$$y=[1, 2, \ldots, p, n+1, \ldots, 2n-p],$$
$$z=[2n+1, 2n+2, \ldots, 3n].$$

Thus the representations of all the double points lie on the special curve

$$\{1, 2, \ldots, p, p+1, \ldots, n;\ 1, 2, \ldots, p, n+1, \ldots, 2n-p\},$$

i.e. on the tangents 1, 2, ..., p to C, and on the curve

$$\{p+1, \ldots, n;\ n+1, \ldots, 2n-p\}.$$

They also lie on the curve of order $n-1$

$$\{1, 2, \ldots, n;\ 2n+1, \ldots, 3n\}.$$

Now the intersections of the tangents 1, 2, ..., p with this last curve are all accounted for in the polygon $(1, 2, \ldots, n)$ which contains the representations of the double points of the given multiple point. Thus the representations of the remaining $\frac{1}{2}(n-1)(n-2)-\frac{1}{2}p(p-1)$ double points must lie on the special curve of order $n-p-1$

$$\{p+1, p+2, \ldots, n;\ n+1, n+2, \ldots, 2n-p\}.$$

The corresponding geometrical theorem is:

THEOREM II *a*. *If two curves of order $n-1$ pass through the vertices of a circumscribed n-ad and also of a circumscribed p-ad of a conic, then all the remaining intersections of the two curves lie on a curve of order $n-p-1$ special for the conic**.

* Since

$$(n-1)^2-\tfrac{1}{2}n(n-1)-\tfrac{1}{2}p(p-1)-\tfrac{1}{2}(n-p-1)(n-p+2)=(p-2)(n-p-1),$$

the theorem will certainly give information when $n>p+1$.

There are two special cases of interest, for $p = 2$ and $p = n-2$ respectively, giving the following results:

THEOREM II *b*. *The representations of all the double points but one of a rational plane curve of order n lie on a curve of order* $n-3$ *special for the conic* C^*.

THEOREM II *c*. *If a rational plane curve of order n has a point of multiplicity* $n-2$, *then the representations of all the remaining* $n-2$ *double points lie on a straight line.*

This last result is of particular interest, since it gives the complete algebraic relations between the parameters at the $n-2$ double points—namely, the equations $|\theta_1\theta_2,\ \theta_3+\theta_4,\ 1| = 0$ for the pairs of parameters $\theta_1, \theta_2;\ \theta_3, \theta_4;\ \theta_5, \theta_6$ at any three of these points.

In fact, it can be shown that, given any $n-2$ values of θ to be the parameters at the point of multiplicity $n-2$ and $n-2$ pairs of values of θ, such that each trio of pairs satisfies the above line relation, to be the pairs of parameters at the $n-2$ double points, then a rational curve of order n can be constructed, unique except for magnification of coordinates, to have the multiple point and two of the double points at three given points.

The geometrical result corresponding to II *c* is also interesting.

If two curves of order $n-1$ pass through the vertices of a circumscribed n-ad and also of a circumscribed $(n-2)$-ad of a conic, then their remaining $n-2$ intersections lie on a straight line.

The following result is a deduction from the general case:

THEOREM II *d*. *If a rational plane curve of order n has a point of multiplicity* $n-p$ *and a number of points of multiplicities* $m_1, m_2, \ldots$ (*each less than* $p+1$) *then, between the parameters at any three of these latter points, there exists an identity in* θ *of the form*

$$\lambda_1 f_1(\theta)\, M_1(\theta)+\lambda_2 f_2(\theta)\, M_2(\theta)+\lambda_3 f_3(\theta)\, M_3(\theta) \equiv 0,$$

where the roots of the polynomial $M_1(\theta)$ *are the parameters at the point* m_1, *the polynomial* $f_1(\theta)$ *is of degree* $p-m_1$ *in* θ *and similarly for the second and third terms of the identity*†.

* For example, since two special conics can in general be constructed through four points in the plane of C, then two of the representations of the six double points of a rational quintic curve will lie one on each of the two special conics through the other four.

† For $p = 2, m_1 = m_2 = \ldots = 2$, this result reduces, on elimination of $\lambda_1, \lambda_2, \lambda_3$, to Theorem II*c*.

For the representations of all the multiple points except the first must lie on a curve of order $p-1$ special for C. Now a p-ad can be inscribed to this curve to contain the m_1-ad representing the point of multiplicity m_1 and similarly for each of the remaining given multiple points. These polygons are the p-ads $f_1(\theta)\,M_1(\theta)$, $f_2(\theta)\,M_2(\theta)$, etc., and, since they all lie in the same special curve of order $p-1$, the result follows from Theorem X of Part I.

Relations, by the elimination of the arbitrary quantities introduced as coefficients in the polynomials $f_1(\theta)$, etc., will be obtained only when

$$m_i+m_j+m_k \geqslant 2p+2.$$

As an example of this result it is easily seen that, if a sextic curve is to have three triple points and one double point, then the parameters at the four points cannot be taken arbitrarily.

5. The following results are obtained by considering the case for $m = 2$.

THEOREM III. *If the curve A has four points of multiplicities p_1, p_2, p_3, p_4 respectively, then the representations of the remaining double points lie on a curve of order $p-1$ special for the conic C, where p is given by the equation*

$$p_1+p_2+p_3+p_4 = 2n-p.$$

Taking the first three multiple points at the vertices of the triangle of reference, the curve A is given by the equations

$$x = [1, 2, \ldots, p_1, p_1+1, \ldots, p_1+p_2, p_1+p_2+1, \ldots, n],$$

$$y = [1, 2, \ldots, p_1, n+1, \ldots, n+p_3, n+p_3+1, \ldots, 2n-p_1],$$

$$z = [p_1+1, \ldots, p_1+p_2, n+1, \ldots, n+p_3, 2n-p_1+1, \ldots, 3n-p_1-p_2-p_3].$$

The representations of all the remaining double points are just the points common to the three curves S_1, S_2, S_3:

$$\{1, 2, \ldots, p_1, n+p_3+1, \ldots, 2n-p_1;$$
$$p_1+1, \ldots, p_1+p_2, 2n-p_1+1, \ldots, 3n-p_1-p_2-p_3\},$$

$$\{1, 2, \ldots, p_1, p_1+p_2+1, \ldots, n;$$
$$n+1, \ldots, n+p_3, 2n-p_1+1, \ldots, 3n-p_1-p_2-p_3\},$$

$$\{p_1+1, \ldots, p_1+p_2, p_1+p_2+1, \ldots, n;$$
$$n+1, \ldots, n+p_3, n+p_3+1, \ldots, 2n-p_1\}.$$

Now consider the rational curve B given by the equations

$$x = [n+1, \ldots, n+p_3, n+p_3+1, \ldots, 2n-p_1, 2n-p_1+1, \ldots, 3n-p_1-p_2-p_3],$$

$$y = [p_1+1, \ldots, p_1+p_2, \ldots, n, 2n-p_1+1, \ldots, 3n-p_1-p_2-p_3],$$

$$z = [1, 2, \ldots, p_1, p_1+p_2+1, \ldots, n, n+p_3+1, \ldots, 2n-p_1]^*.$$

In each bracket the number of numbers is $2n-p_1-p_2-p_3$, which reduces to p_4+p. The curve B is thus of order p_4+p.

Also all the representations of its double points, except those of the obvious multiple points at the vertices of the triangle of reference, are just the points common to the three curves obtained, by the usual cancelling of common numbers, from the numbers in the three pairs of coordinates. These curves are easily seen to be S_1, S_2, S_3.

Thus all the double points of B, except the multiple points at the vertices of the triangle of reference, have the same representations as all the double points of A except the multiple points at the reference vertices.

Now A also has a point of multiplicity p_4. Thus B must have a point of multiplicity p_4, and since the order of B is p_4+p, it follows from Theorem II that the representations of the double points now remaining on B, and so on A, must lie on a curve of order $p-1$ special for the conic C.

The corresponding geometrical theorem is:

THEOREM III*a*. *If two curves of order $n-1$ pass through the vertices of a circumscribed n-ad, a p_1-ad, a p_2-ad, a p_3-ad, and a p_4-ad of a conic (where the sum of no two of p_1, p_2, p_3, p_4 is greater than n), then all the remaining points of intersection of the two curves lie on a curve of order $p-1$ special for the conic where $p = 2n-p_1-p_2-p_3-p_4$.*

Again the most interesting special case of III and III*a* is for $p = 2$. In this case III gives a line relation between the representations, but it is easily seen that, under the necessary restrictions $\sum_{i=1}^{4} p_i = 2n-2$, $p_i+p_\kappa \leqslant n$, there are only five possible cases:

(*a*) n odd and equal to $2r+1$.

* Use is made of the fact that the sum of no two of the three numbers p_1, p_2, p_3 can be greater than n, since the line joining the corresponding multiple points cannot cut A in more than n points.

The possible multiplicities of the four points are

(i) r, r, r, r; (ii) $r, r, r+1, r-1$.

(*b*) n even and equal to $2r$.

The possible multiplicities are

(i) $r, r, r, r-2$; (ii) $r, r, r-1, r-1$; (iii) $r+1, r-1, r-1, r-1$.

Theorem III*a* then gives the result:

If two curves of order $n-1$ pass through the vertices of a circumscribed n-ad, a p_1-ad, a p_2-ad, a p_3-ad, and a p_4-ad of a conic, where n, p_1, p_2, p_3, p_4 have any one of the above five sets of values, then the remaining r or $r-1$ or $r-2$* points of intersection lie on a straight line.

The following result is a deduction from the general case:

THEOREM III*b*. *If the curve A has four points of multiplicities* p_1, p_2, p_3, p_4, *such that* $p_1+p_2+p_3+p_4 = 2n-p$, *and a number of points of multiplicities* $m_1, m_2, \ldots$ (*each less than* $p+1$), *then the parameters at any three of these latter points satisfy an identity in* θ *of the form*

$$\lambda_1 f_1(\theta)\, M_1(\theta)+\lambda_2 f_2(\theta)\, M_2(\theta)+\lambda_3 f_3(\theta)\, M_3(\theta) = 0$$

(*with the same interpretations as in Theorem II d*).

The proof is similar to that of Theorem II *d*.

Conditions will be obtained only when

$$m_i+m_j+m_k \geqslant 2p+2.$$

* The five cases give in turn r, $r-1$, $r-2$, $r-1$, $r-2$ points.

A SET OF DIFFERENTIAL EQUATIONS WHICH CAN BE SOLVED BY POLYNOMIALS

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[Received and read 14 February, 1929.]

Although the results which we present here actually came to our notice in studying the theory of Commutative Differential Operators, it seems preferable to develop them from more elementary considerations, showing in the sequel the connection with the commutative-operator theory.

1. *Condition that certain integrals be rational.*

We begin by asking what conditions must be satisfied by two polynomials $\phi(x)$, $y(x)$ in order that the indefinite integral

$$\int \frac{\phi(x)}{\{y(x)\}^2}\,dx \tag{1}$$

may be rational, *i.e.* expressible without logarithms. We suppose that $\phi(x)$, $y(x)$ have no common factor and that $y(x)$ has no repeated factor. We can then express $\phi(x)/\{y(x)\}^2$ in partial fractions in the form

$$\frac{\phi(x)}{\{y(x)\}^2} = P(x)+\Sigma\frac{A_r}{x-a_r}+\Sigma\frac{B_r}{(x-a_r)^2}, \tag{2}$$

where $P(x)$ is some polynomial and where the summation Σ is taken over every zero a_r of $y(x)$. The integral (1) is rational if, and only if, every A_r vanishes. Writing

$$y(x) \equiv (x-a_r)\,\psi_r(x), \tag{3}$$

we see that

$$A_r = \left[\frac{\partial}{\partial x}\,\frac{\phi(x)}{\{\psi_r(x)\}^2}\right]_{x=a_r}.$$

Hence $A_r = 0$ gives

$$\phi'(a_r)\,\psi_r(a_r) = 2\phi(a_r)\,\psi_r'(a_r).$$

From (3)

$$\psi_r(a_r) = y'(a_r), \qquad 2\psi_r'(a_r) = y''(a_r).$$

Thus the condition is that, for every a_r,

$$y''(a_r)\,\phi(a_r) - y'(a_r)\,\phi'(a_r) = 0.$$

Now suppose that

$$\phi(x) \equiv \{z(x)\}^p, \tag{4}$$

where p is a positive integer greater than unity. Then we must have for every a_r

$$y''(a_r)\,z(a_r) - py'(a_r)\,z'(a_r) = 0.$$

In other words, $$y''(x)\,z(x) - py'(x)\,z'(x)$$

must contain $y(x)$ as a factor. This condition is certainly satisfied if $y(x)$, $z(x)$ are connected by the differential equation

$$y''(x)\,z(x) - py'(x)\,z'(x) + y(x)\,z''(x) = 0. \tag{5}$$

This is the differential equation which we propose to discuss, confining ourselves strictly to polynomial functions. The symmetry of the equation evidently allows us to select either of $y(x)$, $z(x)$ as the unknown, regarding the other of the two as a given function. We rely much on this possibility in our discussion of the equation.

2. *Characteristic property of the differential equation.*

It follows from the argument which led to equation (5) that

(6) if $y(x)$, $z(x)$ are polynomials having neither (i) a common factor nor (ii) a repeated factor, and satisfying the differential equation

$$y''(x)\,z(x) - py'(x)\,z'(x) + y(x)\,z''(x) = 0,$$

then both the integrals

$$\int \frac{\{y(x)\}^p}{\{z(x)\}^2}\,dx, \qquad \int \frac{\{z(x)\}^p}{\{y(x)\}^2}\,dx$$

are rational.

We remark that the conditions (i), (ii) of (6) are equivalent for polynomials which satisfy the differential equation.

For suppose that y, z contain a common factor $x-a$ occurring in them respectively to degrees r, s, *i.e.* suppose that

$$y \equiv (x-a)^r u, \qquad z \equiv (x-a)^s v,$$

where u, v are polynomials not divisible by $x-a$. Substitution in the differential equation gives, after division by $(x-a)^{r+s-2}$,

$$(x-a)^2(u''v-pu'v'+uv'')+(x-a)\{(2r-ps)u'v+(2s-pr)uv'\}$$
$$+\{r(r-1)-prs+s(s-1)\}uv = 0.$$

Since, by hypothesis, uv is not divisible by $x-a$, we must have

$$r(r-1)-prs+s(s-1) = 0. \tag{7}$$

This equation is not satisfied by $r=s=1$, and therefore a common root of y, z is a repeated root of at least one of them.

Again, if $r=0$, then $s=0$ or 1, and therefore a repeated root of z must also be a root of y.

We may also remark that, if, for given z, the differential equation in y is known to have two polynomial solutions y_1, y_2, then we also know, without needing to invoke the conditions (i), (ii) of (6), that

$$\int \frac{\{z(x)\}^p}{\{y_1(x)\}^2}\,dx, \qquad \int \frac{\{z(x)\}^p}{\{y_2(x)\}^2}\,dx$$

are both rational. This, of course, follows immediately from the relation

$$y_1'y_2-y_2'y_1 = Cz^p \quad (C \text{ some constant})$$

connecting the two solutions. As a rule, however, the condition of knowing two polynomial solutions is a more exacting condition than (i) or (ii) of (6), and this line of approach is therefore unpromising.

3. *"Consecutive" solutions of the differential equation.*

We content ourselves then with a single pair of polynomials $y(x)$, $z(x)$ having neither common nor repeated factor and connected by the relation (5). We have thus a single solution of (5), regarded as an equation in y with given z. By elementary theory another solution is

$$y(x)\int \frac{\{z(x)\}^p}{\{y(x)\}^2}\,dx.$$

By (6) the integral is rational, and in the notation of (2) can be written in the form

$$\int P(x)\,dx+\Sigma\int\frac{B_r\,dx}{(x-a_r)^2},$$

so that it is of the form $y_1(x)/y(x)$, where $y_1(x)$ is some polynomial in x. It follows that this polynomial $y_1(x)$ is a second solution of the differential equation in y. More generally, if k is any constant, the two polynomials $y_1(x)+ky(x)$, $z(x)$ are connected by the relation (5). They have common factors for at most a finite set of values of k, and so, by (6), for all other values of k, we have the result that the integral

$$\int\frac{(y_1+ky)^p\,dx}{z^2}$$

is rational. By picking out the coefficients of the various powers of k we see that

(8) the $p+1$ integrals

$$\int\frac{y_1^s y^{p-s}}{z^2}\,dx \quad (s=0,\ 1,\ \ldots,\ p)$$

are all rational.

More precisely, our argument would show that they are fractions having z as denominator. For the present it is more important to notice that in particular we have been able to prove that

(9) $$z_1\equiv z\int\frac{y_1^p}{z^2}\,dx$$

is a polynomial.

If, for brevity, we denote the differential equation

$$y''z-py'z'+yz''=0$$

by the symbol $$\{y,\ z\}=0,$$

we may restate what we have just proved in the form:

if $\{y,\ z\}=0$, then

$$\{y_1,\ z\}=0 \quad \text{where} \quad y_1\equiv y\int\frac{z^p}{y^2}\,dx,$$

$$\{y_1,\ z_1\}=0 \quad \text{where} \quad z_1\equiv z\int\frac{y_1^p}{z^2}\,dx.$$

Proceeding in this way, we may next prove that

$$\{y_2,\ z_1\}=0 \quad \text{where} \quad y_2\equiv y_1\int\frac{z_1^p}{y_1^2}\,dx,$$

and so on. The induction is seen better if we change the notation and write y_r, y_{r+1} for y, z, and define inductively

$$y_{s+1} \equiv y_{s-1} \int \frac{y_s^p}{y_{s-1}^2}\, dx \quad \text{if} \quad s > r. \tag{10}$$

Then $\{y_r, y_{r+1}\}$ leads to $\{y_s, y_{s+1}\} = 0$, where $s > r$.

But again, since $\{y_r, y_{r+1}\}$ is symmetrical in y_r, y_{r+1}, we may pursue the induction in the opposite direction, defining, for $s \leqslant r$,

$$y_{s-1} \equiv -y_{s+1} \int \frac{y_s^p}{y_{s+1}^2}\, dx. \tag{11}$$

4. *Dimensional considerations. The sequence of polynomial solutions.*

To determine whether the induction is terminated in either direction we consider the degrees m_s, m_{s+1} of the polynomials y_s, y_{s+1}. Suppose, then, that

$$y_s \equiv a_s x^{m_s} + b_s x^{m_s - 1} + \ldots,$$

$$y_{s+1} \equiv a_{s+1} x^{m_{s+1}} + b_{s+1} x^{m_{s+1} - 1} + \ldots,$$

and substitute into the differential equation $\{y_s, y_{s+1}\} = 0$. The coefficient of the highest term (of degree $m_s + m_{s+1} - 2$) must vanish. This gives the Diophantine equation

$$m_s(m_s - 1) - p m_s m_{s+1} + m_{s+1}(m_{s+1} - 1) = 0 \tag{12}$$

which is a recurrence-formula connecting the degrees of consecutive polynomials. It will be observed that this is essentially (7), the equation connecting the multiplicity of repeated factors of y_s, y_{s+1}.

From (10) it is seen that the degrees of three consecutive polynomials, y_{s+1}, y_s, y_{s-1}, are connected by the recurrence-formula

$$m_{s+1} = p m_s - m_{s-1} + 1. \tag{13}$$

It is clear from the symmetry of (12) that, if we fix m_s and regard the equation as a quadratic in m_{s+1}, the two roots are m_{s-1}, m_{s+1}. The formula for the sum of the two roots again gives (13), and we find that actual substitution of (13) in (12) gives the corresponding equation in m_s, m_{s-1}.

If, in (12), we have m_s, m_{s+1} equal, then either

$$\text{(i)} \qquad m_s = 0,$$

or

$$\text{(ii)} \quad (2 - p)\, m_s = 2.$$

The alternative (ii) is impossible, since $p \geqslant 2$, and therefore we have

$$m_{s+1} \gtrless m_s \quad (m_s \neq 0).$$

From the symmetry of (12) we are at liberty to choose between these two inequalities. Take

$$m_{s+1} > m_s \quad (m_s \neq 0).$$

Then from (13)

$$m_{s+2} > (p-1)\, m_{s+1} > m_{s+1},$$

since $p \geqslant 2$. Thus

(14) the degrees of the polynomials y_s steadily increase as s increases.

Consequently, also, m_s decreases as s decreases, and the induction terminates in this direction, since the degree of y_s is never negative. There is, therefore, a first term m_0, say, which must be zero, for otherwise by the arguments above we could have an $m_{-1} < m_0$. From (12), $m_1 = 1$, and successive values of m_s can then be determined from (13).

It is sufficient to take

$$y_0 = 1, \quad y_1 = x + a_1,$$

and we can then determine successive polynomials $y_r(x)$ by use of (10). At each passage from y_r to y_{r+1} integration introduces an arbitrary constant. There are thus r arbitrary constants in the general $y_r(x)$. One of these may be removed by writing x for $x + a_1$, which leaves the fundamental differential equation invariant.

The results at which we have now arrived may be summed up as follows :—

(15) The recurrence-formula

$$y_{r+1} = y_{r-1} \int \frac{y_r^p}{y_{r-1}^2}\, dx$$

and the initial values

$$y_0 = 1, \quad y_1 = x + a_1$$

define a sequence of polynomials y_r which satisfy the differential equation

$$y''_{r+1} y_r - p y'_{r+1} y'_r + y_{r+1} y''_r = 0$$

and are such that, for $s = 0, 1, \ldots, p$,

$$y_r \int \frac{y_{r+1}^s y_{r-1}^{p-s}}{y_r^2}\, dx$$

is a polynomial.

From the recurrence-formula of (15) we can also derive the two equations :

(16) $$\int \frac{y_r^p}{y_{r+1}^m y_{r-1}^n}\,dx = \frac{y_{r+1}^{1-m} y_{r-1}^{1-n}}{1-m},$$

where $$m+n = 2, \quad m \neq n,$$

and $$\int \frac{y_r^p}{y_{r+1} y_{r-1}}\,dx = \log \frac{y_{r+1}}{y_{r-1}},$$

the integral in the first case being purely rational, if m, n are integers, and in the second case being purely logarithmic

5. *Formula for y_r when $p = 2$.*

The actual successive calculation of the polynomials y_r soon becomes burdensome for arbitrary p, and we have not succeeded in obtaining any general formula for y_r, except in the special case $p = 2$. This is perhaps not surprising when we consider the dimensions of y_r. Elementary methods give the solution of the recurrence-formula (13) subject to the initial values $m_0 = 0$, $m_1 = 1$ in any of the equivalent forms :

(i) $$m_r = \left(\frac{\alpha^{r+1}-1}{\alpha-1} - \frac{\beta^{r+1}-1}{\beta-1}\right) \Big/ (\alpha-\beta),$$

where α, β are the roots of $t^2-pt+1 = 0$;

(ii) $$m_r = \Sigma \alpha^i \beta^j,$$

where Σ is taken over the integral values

$$0 \leqslant i, j \leqslant r-1;$$

(iii) $$m_r = \frac{(\alpha^{r+1}-1)(\alpha^r-1)}{(p-2)(\alpha^{r+1}+\alpha^r)};$$

(iv) $m_r =$ the coefficient of t^{r-1} in $(1-t)^{-1}(1-pt+t^2)^{-1}$;

(v) $$m_r = \frac{1}{p-2}\left\{-1+p^r-p^{r-1}-(r-1)p^{r-2}+(r-2)p^{r-3} + \frac{(r-2)(r-3)}{2!}p^{r-4} - \frac{(r-3)(r-4)}{2!}p^{r-5} - \cdots\right\}.$$

It can be verified without much difficulty that any of these equivalent expressions for m_r also satisfies the Diophantine equation (12). It evidently needs a certain optimism to set out to find a polynomial of degree m_r given by the above formulae.

In the special case $p=2$, the formulae (i), (iii), (v) become indeterminate. The other two agree in giving

(17) $$m_r = \tfrac{1}{2}r(r+1),$$

a formula which comes at once from (12), when written in the form

$$(m_{r+1}-m_r)^2 = m_{r+1}+m_r.$$

For we may write

$$m_{r+1}+m_r = t^2, \quad m_{r+1}-m_r = t,$$

i.e. $$m_{r+1} = \tfrac{1}{2}t(t+1), \quad m_r = \tfrac{1}{2}t(t-1),$$

whence, by consideration of $r=0$, we see that $t=r+1$. In this special case we are able, as we have already said, to give a general formula for y_r. To do this most conveniently we introduce the following notation :

(18) (r) denotes a polynomial in x of degree r defined by the initial value

$$(0) = \text{an arbitrary constant}$$

together with the recurrence-formula

$$(r+1) = \int (r)\, dx,$$

where at each integration we are careful to add the arbitrary constant. So defined, the polynomial (r) has $r+1$ arbitrary coefficients, and is therefore merely the general polynomial of degree r. The point of the notation is, however, that $(r-1)$, $(r-2)$, ... are not also independent, arbitrary polynomials, but are precisely the successive derivatives of (r).

In this notation we assert that

(19) in the special case $p=2$,

$$y_r = \begin{vmatrix} (2r-1) & (2r-3) & \dots & (3) & (1) \\ (2r-2) & (2r-4) & \dots & (2) & (0) \\ \dots & \dots & \dots & & \dots \\ (r+1) & (r-1) & \dots & 0 & 0 \\ (r) & (r-2) & \dots & 0 & 0 \end{vmatrix},$$

the determinant being of order r.

Alternatively, if X_r denote the general polynomial in x of degree $2r-1$, we may write y_r as the Wronskian of

$$X_r,\ D^2X_r,\ \dots,\ D^{2r-2}X_r,$$

i.e. in a convenient symbolism as

$$(20) \qquad (1, D)^{r-1} \left| X_r,\ D^2 X_r,\ \ldots,\ D^{2r-2} X_r \right|.$$

6. *Quadratic operators derived from D^2.*

It can be verified by direct substitution that the expression (19) satisfies the recurrence-formula*

$$y'_{r-1} y_{r+1} - y'_{r+1} y_{r-1} = y_r^2.$$

We find the verification, however, somewhat cumbersome, especially in print, and we prefer to prove (19) by a more roundabout method, which also brings out the connection with the commutative-operator theory. For this purpose it is convenient to quote the recurrence-formulae (10), (11) in the two equivalent forms

$$(21) \qquad \frac{y_{r+1}^2}{y_r^2} = D \frac{y_{r+2}}{y_r}, \qquad \frac{y_r^2}{y_{r+1}^2} = -D \frac{y_{r-1}}{y_{r+1}}.$$

Now y_1 is a linear expression, and so is annihilated by the operator D^2. We may put this in evidence by writing D^2 with the end-factor

$$y_1 D \frac{1}{y_1}, \quad i.e. \quad D - \frac{y'_1}{y_1}.$$

The other factor is clearly

$$D + \frac{y'_1}{y_1} \quad i.e. \quad \frac{1}{y_1} D y_1,$$

and we may write

$$P_0 \equiv D^2 = \left(\frac{1}{y_1} D y_1\right)\left(y_1 D \frac{1}{y_1}\right).$$

Interchange the linear operators and define the new quadratic operator

$$(22) \qquad P_1 \equiv \left(y_1 D \frac{1}{y_1}\right)\left(\frac{1}{y_1} D y_1\right).$$

It is evident, if we write P_1 as

$$\left(D - \frac{y'_1}{y_1}\right)\left(D + \frac{y'_1}{y_1}\right),$$

* Presumably it is also possible to verify that it satisfies the other recurrence-formula $y''_{r+1} y_r - 2y'_{r+1} y'_r + y_{r+1} y_r = 0$, but this we have not been bold enough to attempt. See, however, the postscript at the end of this paper.

that it can be expanded in the form

$$D^2+\theta, \tag{23}$$

where θ is some function of x.

It is rather more convenient to introduce y_0, a mere constant, and write (22) as

$$P_1 \equiv \frac{y_1}{y_0} D \left(\frac{y_0}{y_1}\right)^2 D \frac{y_1}{y_0}. \tag{24}$$

Operate with this operator on y_2/y_1, and use brackets [ ] to indicate the presence of an operand. We have

$$P_1\left[\frac{y_2}{y_1}\right] = \frac{y_1}{y_0} D \left(\frac{y_0}{y_1}\right)^2 D\left[\frac{y_2}{y_0}\right]$$

$$= \frac{y_1}{y_0} D\ [\text{constant}],$$

since $$D\left[\frac{y_2}{y_0}\right] = \left(\frac{y_1}{y_0}\right)^2 \text{ by (21).}$$

Thus P_1 annihilates y_2/y_1, and we put this in evidence by writing P_1 with the end-factor

$$\frac{y_2}{y_1} D \frac{y_1}{y_2}.$$

Since P_1 is of the form (23), the other factor must be

$$\frac{y_1}{y_2} D \frac{y_2}{y_1}$$

and we have

$$P_1 = \left(\frac{y_1}{y_2} D \frac{y_2}{y_1}\right)\left(\frac{y_2}{y_1} D \frac{y_1}{y_2}\right).$$

We interchange these factors, and define

$$P_2 \equiv \frac{y_2}{y_1} D \left(\frac{y_1}{y_2}\right)^2 D \frac{y_2}{y_1}, \tag{25}$$

and we can show in similar fashion that P_2 annihilates y_3/y_2, and so on. Thus at length, defining, for $r \geqslant 1$,

$$L_r \equiv \frac{y_r}{y_{r-1}} D \frac{y_{r-1}}{y_r}, \qquad M_r \equiv \frac{y_{r-1}}{y_r} D \frac{y_r}{y_{r-1}}, \tag{26}$$

we have, for $r \geqslant 1$,

$$L_{r-1} M_{r-1} = P_{r-1} = M_r L_r. \tag{27}$$

Moreover, we have successively

$$L_r P_{r-1} = P_r L_r, \quad L_{r-1} P_{r-2} = P_{r-1} L_{r-1}, \quad \ldots, \quad L_1 D^2 = P_1 L_1.$$

Thus

$$S_r D^2 = P_r S_r, \tag{28}$$

where $$S_r \equiv L_r L_{r-1} \ldots L_1.$$

7. *Solutions of the differential equation* $S_r y = 0$.

We shall prove that

(29) (i) every solution of the differential equation

$$S_r y = 0$$

is annihilated by D^{2r};

(ii) not every solution of the differential equation is annihilated by D^{2r-2}.

To prove (i) we observe that, by (27),

$$L_{r-1} M_{r-1} = M_r L_r.$$

Hence $$M_{r-1} M_r L_r L_{r-1} = (M_{r-1} L_{r-1})^2 = (L_{r-2} M_{r-2})^2.$$

Similarly $$M_{r-2} M_{r-1} M_r L_r L_{r-1} L_{r-2} = (L_{r-3} M_{r-3})^3$$

and so on. Thus at length

$$M_1 \ldots M_r L_r \ldots L_1 = (L_0 M_0)^r = D^{2r}, \tag{30}$$

i.e. $$M_1 \ldots M_r S_r = D^{2r}$$

and so $D^{2r} y = 0$ if $S_r y = 0$. This establishes (29) (i).

To prove (29)(ii), suppose the contrary, namely that D^{2r-2} annihilates every solution of $S_r y = 0$. Then we must have an identity of operators

$$D^{2r-2} = T S_r,$$

where T is some operator.

Now, from (30), writing $r-1$ for r, we have

$$M_1 \ldots M_{r-1} L_{r-1} \ldots L_1 = D^{2r-2}$$
$$= T L_r L_{r-1} \ldots L_1 \quad \text{by hypothesis.}$$

Thus $$M_1 \ldots M_{r-1} = T L_r,$$

i.e. $$M_1 \dots M_{r-1}\left[\frac{y_r}{y_{r-1}}\right] = 0.$$

But $$M_{r-1}\left[\frac{y_r}{y_{r-1}}\right] = \frac{y_{r-2}}{y_{r-1}} D\left[\frac{y_r}{y_{r-2}}\right]$$

$$= \frac{y_{r-1}}{y_{r-2}},$$

on substitution from (21). Hence

$$0 = M_1 \dots M_{r-1}\left[\frac{y_r}{y_{r-1}}\right] = M_1 \dots M_{r-2}\left[\frac{y_{r-1}}{y_{r-2}}\right],$$

and so on inductively, until at length we obtain

$$0 = M_1\left[\frac{y_2}{y_1}\right] = \frac{y_1}{y_0},$$

which is untrue. The hypothesis $D^{2r-2} = TS_r$ is therefore false, and Theorem (29) (ii) is accordingly established.

There is thus a function X_r annihilated by S_r, and therefore by D^{2r}, but not by D^{2r-2}. Such a function X_r must be a polynomial of degree $2r-1$ or $2r-2$: suppose the former, since the greater includes the less. Since no such polynomial is annihilated by an operator

$$a_0 D^{2r-2} + a_1 D^{2r-4} + \dots + a_{r-1}$$

(where the a's denote constants), the r polynomials

(31) $$D^{2r-2}X_r,\ D^{2r-4}X_r,\ \dots,\ X_r$$

are linearly independent.

By (28), if S_r annihilates X_r, it also annihilates D^2X_r, and so on. The r functions (31) are therefore a complete set of solutions of the differential equation

$$S_r y = 0.$$

8. *Determination of* y_r.

The differential equation may therefore be written in the Wronskian form

$$(D, 1)^r \,|\, y,\ X_r,\ D^2X_r,\ \dots,\ D^{2r-2}X_r\,| = 0.$$

The coefficient of $D^r y$ on the left is

$$(D, 1)^{r-1} \,|\, X_r,\ D^2X_r,\ \dots,\ D^{2r-2}X_r\,|.$$

Now in every linear operator L_r the coefficient of D is unity, and therefore in S_r the coefficient of D^r is also unity.

Thus we have exactly

$$S_r y = \frac{(D, 1)^r \,|\, y,\ X_r,\ D^2 X_r,\ \ldots,\ D^{2r-2} X_r \,|}{(D, 1)^{r-1} \,|\, X_r,\ D^2 X_r,\ \ldots,\ D^{2r-2} X_r \,|}. \tag{32}$$

Now define X_{r+1} by the relation

$$D^2[X_{r+1}] \equiv X_r. \tag{33}$$

Then

$$S_r D^2[X_{r+1}] = 0,$$

and again, by (28),

$$P_r S_r[X_{r+1}] = 0.$$

From (32), (33) we have

$$S_r[X_{r+1}] = \frac{(D, 1)^r \,|\, X_{r+1},\ D^2 X_{r+1},\ \ldots,\ D^{2r} X_{r+1} \,|}{(D, 1)^{r-1} \,|\, X_r,\ D^2 X_r,\ \ldots,\ D^{2r-2} X_r \,|}.$$

But, by (27), the roots of the differential equation

$$P_r y = 0$$

are

$$\frac{y_{r-1}}{y_r}, \quad \frac{y_{r+1}}{y_r}.$$

By considering the suffixes we must evidently identify

$$y_r = (D, 1)^{r-1} \,|\, X_r,\ D^2 X_r,\ \ldots,\ D^{2r-2} X_r \,|,$$

which may be confirmed inductively or by comparing degrees. But this is the identification (19) or (20) which we set out to prove.

In conclusion, we may sum up what we have achieved and what we have failed to achieve. We have been able to develop a theory of a sequence of polynomial solutions of the differential equation

$$y'' z - p y' z' + y z'' = 0,$$

and we have been able to show that any given pair of polynomial solutions must belong to this sequence, *provided that they have neither common nor repeated factors.* But in the absence of this provision we are immobilized, because we cannot then deduce other polynomial solutions from the given pair. Nevertheless, the dimensional considerations still hold, and the degrees of the given polynomials must be some m_r

and m_{r+1}, the degrees of two consecutive polynomials of the sequence. This fact makes it most unlikely that there can be any pairs of polynomial solutions lying outside the sequence. Nevertheless, the problem, though seemingly elementary, defeats us.

[*Postscript.* Added 7 May, 1929.]

A referee has pointed out that our y_r, as determined in § 8, is inexact in respect of a constant multiplier. This is immaterial as regards the two-term recurrence-formula

$$y''_{r+1}y_r - 2y'_{r+1}y'_r + y_{r+1}y''_r = 0,$$

but we have to make the constant precise, when we assert that y_r, and, in particular, the y_r of (19), satisfies the three-term recurrence-formula

$$y'_{r-1}y_{r+1} - y_{r-1}y'_{r+1} = y_r^2.$$

We can do this fairly simply by determining the coefficient of the leading term $x^{\frac{1}{2}r(r+1)}$ in y_r. It follows, by successive integration, that the leading coefficient in (r) is $1/r!$, and hence the leading coefficient of y_r in (19) may be obtained at once from the expression for y_r by interpreting every (s) as $1/s!$ Call the determinant, so interpreted, a_r. If we reduce it by subtracting the first column from the second column for a new second column, twice this new second column from the third column for a new third column, three times this new third column from the fourth column for a new fourth column, and so on, the determinant we finally obtain is merely

$$\frac{(-)^{r-1}a_{r-1}}{(2r-1)(2r-3)\ldots 3\,.\,1}.$$

Thus a_r is determined by the recurrence-formula

$$a_r = \frac{(-)^{r-1}a_{r-1}}{(2r-1)(2r-3)\ldots 3\,.\,1},$$

and is explicitly

$$a_r = \frac{(-)^{\frac{1}{2}r(r-1)}}{(2r-1)(2r-3)^2\ldots 3^{r-1}\,.\,1^r}.$$

If actually it is $k_r y_r$ (where k_r is some constant) which satisfies the three-term recurrence-formula, we have, on substituting and picking out the coefficients of the highest term,

$$k_{r+1}k_{r-1} = k_r^2.$$

We are thus at liberty to take $k_r = 1$ for every r.

SOME DEFINITE INTEGRALS ALLIED TO AN INTEGRAL OF JACOBI

By W. N. BAILEY.

[Received 14 May, 1929.—Read 16 May, 1929.]

1. One of the integrals given by Jacobi* is

$$(1.1)\quad \int_0^{\frac{1}{2}\pi} \{F(-n,\ \mu+\nu+n+1\,;\ \nu+1\,;\ \sin^2\phi)\}^2 \cos^{2\mu+1}\phi \sin^{2\nu+1}\phi\, d\phi = \frac{n!\,\Gamma(\mu+n+1)\{\Gamma(\nu+1)\}^2}{2(\mu+\nu+2n+1)\,\Gamma(\mu+\nu+n+1)\,\Gamma(\nu+n+1)},$$

where $R(\mu)$ and $R(\nu)$ both exceed -1, the function F being the hypergeometric function.

This formula can be used in conjunction with Bateman's expansion† to obtain some integrals involving Bessel functions. I give here some other definite integrals of a type similar to (1.1), and in particular a generalization of (1.1), and from these I deduce some integrals involving Bessel functions.

2. By expansion and integration we easily find that

$$(2.1)\quad \int_0^{\frac{1}{2}\pi} F(-n,\ \alpha\,;\ \beta\,;\ \sin^2\phi) \cos^{2\mu+1}\phi \sin^{2\nu+1}\phi\, d\phi = \frac{\Gamma(\mu+1)\,\Gamma(\nu+1)}{2\Gamma(\mu+\nu+2)}\ {}_3F_2\left[\begin{matrix}-n,\ \alpha,\ \nu+1\,;\\ \beta,\ \mu+\nu+2\end{matrix}\right].$$

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* *Journal für Math.*, 56 (1859), 149–175. *Werke*, 6 (1891), 184–202, § 6.

† G. N. Watson, *Theory of Bessel functions*, § 12.2.

The series ${}_3F_2$ can be summed by Saalschütz's formula* if $\alpha = \mu+\beta+n$, and we derive the formula

$$(2.2)\quad \int_0^{\frac{1}{2}\pi} F(-n,\ \mu+\beta+n;\ \beta;\ \sin^2\phi)\cos^{2\mu+1}\phi\sin^{2\nu+1}\phi\,d\phi = \frac{\Gamma(\mu+n+1)\,\Gamma(\nu+1)(\beta-\nu-1)_n}{2\Gamma(\mu+\nu+n+2)(\beta)_n}.$$

On the other hand, if $\beta = \nu+1$, the ${}_3F_2$ reduces to an ordinary hypergeometric series which can be summed by Gauss's formula, and we get

$$(2.3)\quad \int_0^{\frac{1}{2}\pi} F(-n,\ \alpha;\ \nu+1;\ \sin^2\phi)\cos^{2\mu+1}\phi\sin^{2\nu+1}\phi\,d\phi = \frac{\Gamma(\mu+1)\,\Gamma(\nu+1)(\mu+\nu-\alpha+2)_n}{2\Gamma(\mu+\nu+n+2)}.$$

From (2.2) we immediately derive the relation

$$(2.4)\quad \int_0^{\frac{1}{2}\pi} F(-n,\ \mu+\beta+n;\ \beta;\ \sin^2\phi)\,F(-m,\ a;\ b;\ \sin^2\phi)\times\cos^{2\mu+1}\phi\sin^{2\nu+1}\phi\,d\phi$$
$$= \sum_{r=0}^{m}\int_0^{\frac{1}{2}\pi}\frac{(-m)_r(a)_r}{r!\,(b)_r}F(-n,\ \mu+\beta+n;\ \beta;\ \sin^2\phi)\cos^{2\mu+1}\phi\sin^{2\nu+2r+1}\phi\,d\phi$$
$$= \frac{\Gamma(\mu+n+1)\,\Gamma(\nu+1)(\beta-\nu-1)_n}{2\Gamma(\mu+\nu+n+2)(\beta)_n}\,{}_4F_3\left[\begin{matrix}-m,\ a,\ \nu+1,\ \nu-\beta+2;\\ b,\ \mu+\nu+n+2,\ \nu-\beta-n+2\end{matrix}\right].$$

The series ${}_4F_3$ is Saalschützian if $a = \mu+b+m$. It can therefore be summed by Saalschütz's formula, when it reduces to a ${}_3F_2$, and this occurs when

(i) $\nu = b-1$,

(ii) $\nu = b+\beta-2$,

(iii) $\nu = b+m-n-2$,

(iv) $\nu = \mu+b+\beta+m+n-2$.

* L. Saalschütz, "Eine Summationsformel", *Zeitschrift für Math. u. Phys.*, 35 (1890), 186–188.

We are thus led to the formulae*

$$(2.5)\quad \int_0^{\frac{1}{2}\pi} F(-n,\ \mu+\beta+n;\ \beta;\ \sin^2\phi)\, F(-m,\ \mu+b+m;\ b;\ \sin^2\phi) \times \cos^{2\mu+1}\phi \sin^{2b-1}\phi\, d\phi = \frac{(-1)^m \Gamma(b)\,\Gamma(\mu+n+1)\,(\mu+n+\beta)_m (-n)_m (b-\beta)_{n-m}}{2\Gamma(\mu+b+m+n+1)\,(\beta)_n},$$

$$(2.6)\quad \int_0^{\frac{1}{2}\pi} F(-n,\ \mu+\beta+n;\ \beta;\ \sin^2\phi)\, F(-m,\ \mu+b+m;\ b;\ \sin^2\phi) \times \cos^{2\mu+1}\phi \sin^{2b+2\beta-3}\phi\, d\phi = \frac{\Gamma(\mu+m+n+1)\,\Gamma(b+\beta-1)\,(1-b)_{n-m}}{2\Gamma(\mu+b+\beta+m+n)\,(\beta)_{n-m}},$$

$$(2.7)\quad \int_0^{\frac{1}{2}\pi} F(-n,\ \mu+\beta+n;\ \beta;\ \sin^2\phi)\, F(-m,\ \mu+b+m;\ b;\ \sin^2\phi) \times \cos^{2\mu+1}\phi \sin^{2b+2m-2n-3}\phi\, d\phi = \frac{(-1)^m \Gamma(\mu+n+1)\,\Gamma(b+m-n-1)\,(-n)_m (\beta-b+n-m+1)_{n-m}}{2\Gamma(\mu+b+m)\,(b)_m (\beta)_{n-m}},$$

$$(2.8)\quad \int_0^{\frac{1}{2}\pi} F(-n,\ \mu+\beta+n;\ \beta;\ \sin^2\phi)\, F(-m,\ \mu+b+m;\ b;\ \sin^2\phi) \times \cos^{2\mu+1}\phi \sin^{2\mu+2b+2\beta+2m+2n-3}\phi\, d\phi = \frac{(-1)^{m+n} \Gamma(\mu+m+n+1)\,\Gamma(\mu+b+\beta+m+n-1)\,(\mu+b+m)_n (\mu+\beta+n)_m}{2\Gamma(2\mu+b+\beta+2m+2n)\,(\beta)_n (b)_m}.$$

3. Turning now to the formula (2.3), we find in the same way that

$$(3.1)\quad \int_0^{\frac{1}{2}\pi} F(-n,\ a;\ \nu+1;\ \sin^2\phi)\, F(-m,\ a;\ b;\ \cos^2\phi) \cos^{2\mu+1}\phi \sin^{2\nu+1}\phi\, d\phi = \frac{\Gamma(\mu+1)\,\Gamma(\nu+1)\,(\mu+\nu-a+2)_n}{2\Gamma(\mu+\nu+n+2)} \times {}_4F_3\left[\begin{matrix} -m,\ a,\ \mu+1,\ \mu+\nu-a+n+2 \\ b,\ \mu+\nu+n+2,\ \mu+\nu-a+2 \end{matrix};\right].$$

The series ${}_4F_3$ reduces to a Saalschützian ${}_3F_2$ if $a = b+\nu+m$ and also (i) $\mu = b-1$, or (ii) $\mu = b-\nu+a-n-2$, or (iii) $\mu = m-n+b-2$, or (iv) $\mu = a+b+m-2$.

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* The formulae are given in a form suitable when $n \geqslant m$. If $n < m$, $(a)_{n-m}$ must be replaced by $(-1)^{m-n}/(1-a)_{m-n}$. In the latter case the expressions on the right of (2.5) and (2.7) reduce to zero, because of the factor $(-n)_m$.

We are thus led to the formulae

$$(3.2)\quad \int_0^{\frac{1}{2}\pi} F(-n,\, a;\, \nu+1;\, \sin^2\phi)\, F(-m,\, b+\nu+m;\, b;\, \cos^2\phi) \times \cos^{2b-1}\phi \sin^{2\nu+1}\phi\, d\phi = \frac{\Gamma(b)\,\Gamma(\nu+1)\,(\nu+b-a+m+1)_{n-m}(-n)_m(a)_m}{2\Gamma(b+\nu+m+n+1)},$$

$$(3.3)\quad \int_0^{\frac{1}{2}\pi} F(-n,\, a;\, \nu+1;\, \sin^2\phi)\, F(-m,\, b+\nu+m;\, b;\, \cos^2\phi) \times \cos^{2b-2\nu+2a-2n-3}\phi \sin^{2\nu+1}\phi\, d\phi = \frac{(-1)^{n-m}\,\Gamma(b-\nu+a-n-1)\,\Gamma(\nu+1)\,(\nu+n+1)_m\,(\nu-a+1)_m(1-b)_{n-m}}{2\Gamma(b+a+m)},$$

$$(3.4)\quad \int_0^{\frac{1}{2}\pi} F(-n,\, a;\, \nu+1;\, \sin^2\phi)\, F(-m,\, b+\nu+m;\, b;\, \cos^2\phi) \times \cos^{2m-2n+2b-3}\phi \sin^{2\nu+1}\phi\, d\phi = \frac{(-1)^m\,\Gamma(b+m-n-1)\,\Gamma(\nu+1)\,(-n)_m(\nu-a+1)_m}{2\Gamma(b+\nu+m)(b)_m(b+\nu-a+m)_{m-n}},$$

$$(3.5)\quad \int_0^{\frac{1}{2}\pi} F(-n,\, a;\, \nu+1;\, \sin^2\phi)\, F(-m,\, b+\nu+m;\, b;\, \cos^2\phi) \times \cos^{2a+2b+2m-3}\phi \sin^{2\nu+1}\phi\, d\phi = \frac{(-1)^m\,\Gamma(a+b+m-1)\,\Gamma(\nu+1)\,(\nu+b+m)_n\,(\nu+n+1)_m\,(a)_m}{2\Gamma(\nu+a+b+2m+n)\,(b)_m}.$$

4. Jacobi's formula is the particular case of (2.5) when $\beta = b = \nu+1$, $m = n$. The corresponding particular cases of (2.6), (2.7), and (2.8) are

$$(4.1)\quad \int_0^{\frac{1}{2}\pi} \{F(-n,\, \mu+\nu+n+1;\, \nu+1;\, \sin^2\phi)\}^2 \cos^{2\mu+1}\phi \sin^{2\nu+1}\phi\, d\phi = \frac{\Gamma(\mu+2n+1)\,\Gamma(2\nu+1)}{2\Gamma(\mu+2\nu+2n+2)},$$

$$(4.2)\quad \int_0^{\frac{1}{2}\pi} \{F(-n,\, \mu+\nu+n+1;\, \nu+1;\, \sin^2\phi)\}^2 \cos^{2\mu+1}\phi \sin^{2\nu-1}\phi\, d\phi = \frac{\Gamma(\mu+n+1)\,\{\Gamma(\nu)\}^2\, n!}{2\Gamma(\mu+\nu+n+1)\,\Gamma(\nu+n+1)},$$

$$(4.3)\quad \int_0^{\frac{1}{2}\pi} \{F(-n,\, \mu+\nu+n+1;\, \nu+1;\, \sin^2\phi)\}^2 \cos^{2\mu+1}\phi \sin^{2\mu+4\nu+4n+1}\phi\, d\phi = \frac{\Gamma(\mu+2n+1)\,\Gamma(\mu+2\nu+2n+1)\,\{(\mu+\nu+n+1)_n\}^2}{2\Gamma(2\mu+2\nu+4n+2)\,\{(\nu+1)_n\}^2}.$$

Just as Jacobi's formula can be used in conjunction with Bateman's expansion, so the formulae (4 . 1) and (4 . 2) can be used to derive integrals involving Bessel functions. The results are

$$(4.4)\quad z\int_0^{\frac{1}{2}\pi} J_\mu(z\cos^2\phi)\,J_\nu(z\sin^2\phi)\cos\phi\sin^{2\nu+1}\phi\,d\phi$$

$$=\sum_{n=0}^{\infty}(-1)^n(\mu+\nu+2n+1)\,J_{\mu+\nu+2n+1}(z)$$

$$\times\frac{\Gamma(\mu+\nu+n+1)\,\Gamma(\nu+n+1)\,\Gamma(\mu+2n+1)\,\Gamma(2\nu+1)}{n!\,\Gamma(\mu+n+1)\,\{\Gamma(\nu+1)\}^2\,\Gamma(\mu+2\nu+2n+2)}$$

and

$$(4.5)\quad z\int_0^{\frac{1}{2}\pi} J_\mu(z\cos^2\phi)\,J_\nu(z\sin^2\phi)\cot\phi\,d\phi$$

$$=\frac{1}{\nu}\sum_{n=0}^{\infty}(-1)^n(\mu+\nu+2n+1)\,J_{\mu+\nu+2n+1}(z).$$

These formulae can be written in the forms

$$(4.6)\quad \int_0^z J_\mu(z-t)\,J_\nu(t)\,t^\nu\,dt$$

$$=2z^\nu\sum_{n=0}^{\infty}\frac{(-1)^n(\mu+\nu+2n+1)\,\Gamma(\mu+\nu+n+1)\,\Gamma(\nu+n+1)}{n!\;\Gamma(\mu+n+1)\,\{\Gamma(\nu+1)\}^2}$$

$$\times\frac{\Gamma(\mu+2n+1)\,\Gamma(2\nu+1)}{\Gamma(\mu+2\nu+2n+2)}\,J_{\mu+\nu+2n+1}(z),$$

where $R(\mu)>-1$, $R(\nu)>-\frac{1}{2}$, and*

$$(4.7)\quad \int_0^z J_\mu(z-t)\,J_\nu(t)\,\frac{dt}{t}=\frac{2}{\nu z}\sum_{n=0}^{\infty}(-1)^n(\mu+\nu+2n+1)\,J_{\mu+\nu+2n+1}(z),$$

where $R(\mu)>-1$, $R(\nu)>0$.

5. The formulae (3 . 2), (3 . 3), and (3 . 4) give particular cases which can be used with Bateman's expansion when $b=\nu+1$, $a=2\nu+n+1$,

* Cf. Watson, *loc. cit.*, § 12.2, formula (3).

$m = n$. They then become

$$(5.1)\quad \int_0^{\frac{1}{2}\pi} F(-n,\ 2\nu+n+1\,;\ \nu+1\,;\ \sin^2\phi)\, F(-n,\ 2\nu+n+1\,;\ \nu+1\,;\ \cos^2\phi) \times \cos^{2\nu+1}\phi \sin^{2\nu+1}\phi\, d\phi = \frac{(-1)^n \{\Gamma(\nu+1)\}^2\ n!}{2(2\nu+2n+1)\,\Gamma(2\nu+n+1)},$$

$$(5.2)\quad \int_0^{\frac{1}{2}\pi} F(-n,\ 2\nu+n+1;\ \nu+1\,;\ \sin^2\phi)\, F(-n,\ 2\nu+n+1\,;\ \nu+1\,;\ \cos^2\phi) \times \cos^{4\nu+1}\phi \sin^{2\nu+1}\phi\, d\phi = \frac{(-1)^n\,\Gamma(2\nu+1)\,\Gamma(\nu+2n+1)}{2\Gamma(3\nu+2n+2)},$$

$$(5.3)\quad \int_0^{\frac{1}{2}\pi} F(-n.\ 2\nu+n+1\,;\ \nu+1\,;\ \sin^2\phi)\, F(-n,\ 2\nu+n+1\,;\ \nu+1\,;\ \cos^2\phi) \times \cos^{2\nu-1}\phi \sin^{2\nu+1}\phi\, d\phi = \frac{\Gamma(\nu)\,\Gamma(\nu+1)\,n!}{2\Gamma(2\nu+n+1)}.$$

Now, in Bateman's expansion

$$(5.4)\quad \tfrac{1}{2}z J_\mu(z\cos\phi\cos\Phi) J_\nu(z\sin\phi\sin\Phi) = \cos^\mu\phi\cos^\mu\Phi\sin^\nu\phi\sin^\nu\Phi \sum_{n=0}^{\infty} (-1)^n(\mu+\nu+2n+1) J_{\mu+\nu+2n+1}(z) \times \frac{\Gamma(\mu+\nu+n+1)\,\Gamma(\nu+n+1)}{n!\,\Gamma(\mu+n+1)\{\Gamma(\nu+1)\}^2} F(-n,\ \mu+\nu+n+1\,;\ \nu+1\,;\ \sin^2\phi) \times F(-n,\ \mu+\nu+n+1\,;\ \nu+1\,;\ \sin^2\Phi),$$

take $\mu = \nu$, $\Phi = \frac{1}{2}\pi - \phi$, multiply by appropriate powers of $\sin\phi$ and $\cos\phi$ and integrate, using (5.1), (5.2), and (5.3). The resulting formulae are

$$(5.5)\quad z\int_0^{\frac{1}{2}\pi} J_\nu^2(z\sin\phi\cos\phi)\sin\phi\cos\phi\, d\phi = \sum_{n=0}^{\infty} J_{2\nu+2n+1}(z),$$

where $R(\nu) > -1$;

$$(5.6)\quad z\int_0^{\frac{1}{2}\pi} J_\nu^2(z\sin\phi\cos\phi)\sin\phi\cos^{2\nu+1}\phi\, d\phi = \sum_{n=0}^{\infty} \frac{\Gamma(2\nu+n+1)\,\Gamma(2\nu+1)\,\Gamma(\nu+2n+1)}{n!\ \{\Gamma(\nu+1)\}^2\,\Gamma(3\nu+2n+2)}\,(2\nu+2n+1) J_{2\nu+2n+1}(z),$$

where $R(\nu) > -\frac{1}{2}$;

$$(5.7) \quad z\int_0^{\frac{1}{2}\pi} J_\nu^2(z\sin\phi\cos\phi)\tan\phi\,d\phi = \frac{1}{\nu}\sum_{n=0}^{\infty}(2\nu+2n+1)J_{2\nu+2n+1}(z),$$

where $R(\nu) > 0$.

The relation (5.5) can be written in the equivalent form

$$(5.8) \quad \int_0^z tJ_\nu^2(t)\frac{dt}{\sqrt{(z^2-t^2)}} = \sum_{n=0}^{\infty} J_{2\nu+2n+1}(2z),$$

where again $R(\nu) > -1$.

6. In conclusion we note the form which (2.8) assumes when $b = \beta = 1$, $\mu = 0$, viz.

$$(6.1) \quad \int_{-1}^{1}(1+z)^{m+n}P_m(z)\,P_n(z)\,dz = \frac{2^{m+n+1}\{(m+n)!\}^4}{(m!\,n!)^2(2m+2n+1)!},$$

a result which has been obtained by Titchmarsh*. More generally, if in (2.8) we put $b = \beta = \nu+1$, $\mu = \nu$, we get

$$(6.2) \quad \int_{-1}^{1}(1+z)^{3\nu+m+n}(1-z)^\nu C_m^{\nu+\frac{1}{2}}(z)\,C_n^{\nu+\frac{1}{2}}(z)\,dz$$

$$= \frac{2^{4\nu+m+n+1}\Gamma(\nu+m+n+1)\Gamma(3\nu+m+n+1)\{\Gamma(\nu+1)\}^2\{\Gamma(2\nu+m+n+1)\}^2}{\Gamma(2\nu+m+1)\Gamma(2\nu+n+1)\Gamma(\nu+m+1)\Gamma(\nu+n+1)\Gamma(4\nu+2m+2n+2)},$$

where $C_n^\nu(z)$ is Gegenbauer's function. When $\nu = 0$, (6.2) reduces to Titchmarsh's formula.

* See R. G. Cooke, *Proc. London Math. Soc.* (2), 23 (1924), (*Records* for 13 December, 1923), also E. C. Titchmarsh, *Journal London Math. Soc.*, 2 (1927), 97–99.

SOME INTEGRALS OF KAPTEYN'S TYPE INVOLVING BESSEL FUNCTIONS

By W. N. BAILEY.

[Received 5 June, 1929.—Read 13 June, 1929.]

The trigonometrical integrals of Kapteyn*

$$\int_0^z \cos(z-t)\, J_0(t)\, dt = zJ_0(z), \tag{1}$$

$$\int_0^z \sin(z-t)\, J_0(t)\, dt = zJ_1(z), \tag{2}$$

$$\int_0^z \sin(z-t)\, J_1(t)\, dt = \sin z - zJ_0(z), \tag{3}$$

do not appear to have been generalized. I give here the corresponding formulae when $J_0(t)$ or $J_1(t)$ in the integrand is replaced by $J_\mu(t)$, where μ is any number for which the integral in question converges. It will be seen that, when μ is a positive integer, the values of the integrals can be obtained in finite form in terms of Bessel functions.

We begin with the formula†

$$\int_0^z J_\mu(z-t)\, J_\nu(t)\, t^\nu\, dt = 2z^\nu \sum_{n=0}^{\infty} \frac{(-1)^n(\mu+\nu+2n+1)\,\Gamma(\mu+\nu+n+1)\,\Gamma(\nu+n+1)}{n!\,\Gamma(\mu+n+1)\{\Gamma(\nu+1)\}^2} \times \frac{\Gamma(\mu+2n+1)\,\Gamma(2\nu+1)}{\Gamma(\mu+2\nu+2n+2)} J_{\mu+\nu+2n+1}(z), \tag{4}$$

which is valid when $R(\mu) > -1$, $R(\nu) > -\frac{1}{2}$.

* See G. N. Watson, *Theory of Bessel functions*, § 12.21.

† W. N. Bailey, "Some definite integrals allied to an integral of Jacobi", *Proc. London Math. Soc.* (2), 30 (1930), 415–421, formula (4.6).

When $\nu = \frac{1}{2}$, this becomes

$$(5)\quad \int_0^z J_\mu(z-t)\sin t\,dt = 4\left(\frac{2z}{\pi}\right)^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n(\mu+2n+\frac{3}{2})\,\Gamma(n+\frac{3}{2})\,\Gamma(\mu+n+\frac{3}{2})}{n!\,(\mu+2n+1)(\mu+2n+2)\,\Gamma(\mu+n+1)} J_{\mu+2n+\frac{3}{2}}(z)$$

But*

$$(\tfrac{1}{2}z)^{-\frac{1}{2}} J_{\mu+2n+\frac{3}{2}}(z) = \sum_{s=0}^{\infty} \frac{\Gamma(\mu+2n+s+1)\,\Gamma(\frac{3}{2})(\mu+2n+2s+1)}{s!\,\Gamma(\frac{3}{2}-s)\,\Gamma(\mu+2n+s+\frac{5}{2})} J_{\mu+2n+2s+1}(z).$$

Substituting in (5), and putting $s = t-n$, we find after some reduction that

$$\int_0^z J_\mu(z-t)\sin t\,dt = 2z \sum_{t=0}^{\infty} \frac{(-1)^t\,\Gamma(\mu+\frac{5}{2})\,\Gamma(\mu+t+1)(-\frac{1}{2})_t(\mu+2t+1)}{\Gamma(\mu+3)\,\Gamma(\mu+t+\frac{5}{2})\,t!} J_{\mu+2t+1}(z)$$
$$\times {}_7F_6\left[\begin{matrix}\mu+\frac{3}{2},\ 1+\frac{1}{2}(\mu+\frac{3}{2}),\ \frac{1}{2}(\mu+1),\ \frac{3}{2},\ \frac{1}{2}\mu+1,\ \mu+t+1,\ -t;\\ \frac{1}{2}(\mu+\frac{3}{2}),\ \frac{1}{2}\mu+2,\ \mu+1,\ \frac{1}{2}(\mu+3),\ \frac{3}{2}-t,\ \mu+t+\frac{5}{2}\end{matrix}\ 1\right].$$

The series ${}_7F_6$ can be summed by Dougall's Theorem†, and we are led to

$$\int_0^z J_\mu(t)\sin(z-t)\,dt = 2\mu z \sum_{t=0}^{\infty} \frac{(-1)^t J_{\mu+2t+1}(z)}{(\mu+2t)(\mu+2t+2)}$$
$$= zJ_{\mu+1}(z) + \mu z \sum_{n=1}^{\infty} \frac{(-1)^n}{\mu+2n}\{J_{\mu+2n-1}(z)+J_{\mu+2n+1}(z)\},$$

so that

$$(6)\quad \int_0^z J_\mu(t)\sin(z-t)\,dt = zJ_{\mu+1}(z) + 2\mu \sum_{n=1}^{\infty} (-1)^n J_{\mu+2n}(z)$$
$$= -zJ_{\mu-1}(z) + 2\mu \sum_{n=0}^{\infty} (-1)^n J_{\mu+2n}(z).$$

If μ is a positive integer, the series can be put into a finite form by using the relations‡

$$\cos z = J_0(z) + 2\sum_{n=1}^{\infty} (-1)^n J_{2n}(z),$$

$$\sin z = 2\sum_{n=0}^{\infty} (-1)^n J_{2n+1}(z),$$

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* G. N. Watson, *loc. cit.*, § 5.21.

† J. Dougall, "On Vandermonde's theorem, and some more general expansions", *Proc. Edinburgh Math. Soc.*, 25 (1907), 114–132, (6).

‡ G. N. Watson, *loc. cit.*, § 2.22.

and we find that, if μ is even,

$$(7)\quad \int_0^z J_\mu(t)\sin(z-t)\,dt$$
$$= -zJ_{\mu-1}(z)+(-1)^{\frac{1}{2}\mu}\mu\left[\cos z-J_0(z)+2J_2(z)-\ldots+(-1)^{\frac{1}{2}\mu}\,2J_{\mu-2}(z)\right]$$
$$= zJ_{\mu+1}(z)+(-1)^{\frac{1}{2}\mu}\mu\left[\cos z-J_0(z)+2J_2(z)-\ldots-(-1)^{\frac{1}{2}\mu}\,2J_\mu(z)\right],$$

while, if μ is odd,

$$(8)\quad \int_0^z J_\mu(t)\sin(z-t)\,dt$$
$$= -zJ_{\mu-1}(z)+(-1)^{\frac{1}{2}(\mu-1)}\mu\left[\sin z-2J_1(z)+2J_3(z)-\ldots\right.$$
$$\left.+(-1)^{\frac{1}{2}(\mu-1)}\,2J_{\mu-2}(z)\right]$$
$$= zJ_{\mu+1}(z)+(-1)^{\frac{1}{2}(\mu-1)}\mu\left[\sin z-2J_1(z)+2J_3(z)-\ldots\right.$$
$$\left.+(-1)^{\frac{1}{2}(\mu+1)}\,2J_\mu(z)\right].$$

If we differentiate (6) with regard to z, we easily find that

$$(9)\qquad \int_0^z J_\mu(t)\cos(z-t)\,dt = zJ_\mu(z)-2\mu\sum_{n=0}^{\infty}(-1)^n J_{\mu+2n+1}(z).$$

Hence, if μ is an even integer,

$$(10)\quad \int_0^z J_\mu(t)\cos(z-t)\,dt = zJ_\mu(z)-(-1)^{\frac{1}{2}\mu}\mu\left[\sin z-2J_1(z)+2J_3(z)-\ldots\right.$$
$$\left.+(-1)^{\frac{1}{2}\mu}\,2J_{\mu-1}(z)\right],$$

while, if μ is odd,

$$(11)\quad \int_0^z J_\mu(t)\cos(z-t)\,dt = zJ_\mu(z)-(-1)^{\frac{1}{2}(\mu+1)}\mu\left[\cos z-2J_0(z)+2J_2(z)-\ldots\right.$$
$$\left.+(-1)^{\frac{1}{2}(\mu+1)}\,2J_{\mu-1}(z)\right].$$

Finally we note the alternative form of (4), namely

$$(12)\quad \int_0^z J_\mu(z-t)\,J_\nu(t)\,t^\nu\,dt$$
$$= \frac{2^{2\nu+\frac{1}{2}}z^{\nu+\frac{1}{2}}\mu}{\sqrt{\pi}}\sum_{t=0}^{\infty}\frac{(-1)^t\,\Gamma(\mu+\nu+t+\frac{1}{2})\,\Gamma(\nu+t+\frac{1}{2})\,\Gamma(\mu+2t)}{t!\,\Gamma(\mu+t+1)\,\Gamma(\mu+2\nu+2t+2)}$$
$$\times(\mu+\nu+2t+\tfrac{1}{2})\,J_{\mu+\nu+2t+\frac{1}{2}}(z),$$

which can be derived from (4) in the way in which (6) was derived from (5).

THE TRITANGENT PLANES OF THE QUINTIC SYMMETROID

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[Received 6 February, 1929.—Read 14 March, 1929.]

1.1. Hesse's work* on the bitangents of the plane quartic has initiated several attempts† to solve the corresponding problem for canonical curves of higher genera. In the present paper a different analogy is considered, which follows the original method of Hesse. Some of the results, in fact, are immediately extensible to the general case in space $[n]$, but for several reasons the restricted problem presents more interesting details. The tritangent planes of the surface considered form only one of many systems; but the present discussion is complete in that it seems impossible to do anything further by this method (see § 4.1)

In the case of the plane quartic, a birational correspondence is set up between the points of a plane σ_2 and the quadrics of a sheaf Σ_2 in ordinary space. Then to the cones of Σ_2 or their vertices, which lie on the Jacobian of Σ_2, correspond the points of a general quartic in σ_2. The twenty-eight bitangents of this curve correspond to the joins in pairs of the eight base points of Σ_2, and the properties of their configuration follow from the three-dimensional figure, with the help of the bifid transformation.

1.2. We now consider the analogous problem one dimension higher. A birational correspondence is established between the points of a prime ω and the quadric forms (hereafter called "quadrics") of a net Σ_3 in $[4]$, having sixteen base points. To the points of a line σ_1 correspond the quadrics of a pencil Σ_1 having for base locus a quartic surface; to the points of a plane σ_2 correspond the quadrics of a sheaf Σ_2 having an octavic curve in common. The vertices of the cones of Σ_3 lie on the

* *Journal für Math.*, 49 (1855), 283.

† An account of these is given by F. P. White, *Science Progress*, 1927.

Jacobian Θ, a surface* of order 10, which contains the twenty axes of the line-cones of Σ_3.

The surface $\mathfrak{F}$ corresponding to this in ω is evidently a quintic, since an arbitrary pencil Σ_1 in [4] contains five cones. A plane section of $\mathfrak{F}$ corresponds to the Jacobian of the corresponding sheaf Σ_2, a decimic of genus 6. It is clear that the equation of $\mathfrak{F}$ may be expressed as $|u_{rs}| = 0$, where $|u_{rs}|$ is a symmetrical determinant of order 5, whose elements are linear in the coordinates. From this and other resemblances to the quartic symmetroid of Cayley†, we may call $\mathfrak{F}$ the "quintic symmetroid".

2.1. *Nodes of* $\mathfrak{F}$. A node of $\mathfrak{F}$ corresponds to a cone which counts twice in an arbitrary pencil containing it. Hence, either its vertex lies on all quadrics of Σ_3 (*i.e.* at a base point) or it must be a line-cone. If Σ_3 is of general character, the first alternative is impossible; thus the nodes of $\mathfrak{F}$ are twenty in number, corresponding to the line-cones contained in Σ_3.

2.2. We may note here some further properties of the correspondence, some of which will be needed later. The tangent plane at a point p cf $\mathfrak{F}$ corresponds to the sheaf Σ_2, containing the cone P, in any pencil of which P counts twice. Thus the vertex P_0 of P must be at a base point of Σ_2, which may be denoted by $\Sigma_2(P_0)$. The tangents at a node p have three-point intersection there, and thus correspond to pencils in which the line-cone P counts thrice; the cone of tangents therefore corresponds to the quadrics of Σ_3 which touch the axis S_1 of P, and thus, in the notation of § 3.1, satisfy the relation $\Pi(S_1)$.

2.3. The tangent cone to $\mathfrak{F}$ from an arbitrary point p meets $\mathfrak{F}$ in points q corresponding to cones Q such that Q counts twice in the pencil formed by P and Q. Thus the curve of contact of the tangent cone corresponds to the cones whose vertices lie on P, *i.e.* the intersection of P and Θ, a curve of order 20. The order of the tangent cone is the number of points in which P meets the Jacobian of an arbitrary sheaf; hence $\mathfrak{F}$ is of rank 20. The above reasoning shows that the points of contact of tangent planes drawn through a line pq correspond to the intersections with Θ of the quartic surface common to P and Q; thus $\mathfrak{F}$ is of class 40.

2.4. The first polar of an arbitrary point p is a quartic surface passing through the nodes and containing the curve of contact of the tangent

* See *Proc. London Math. Soc.* (2), 30 (1930), 118.

† *Collected papers*, VII, 136, 279.

cone from p. Hence it corresponds to those quadrics of Σ_3 which are inpolar to P. For four of these are contained in an arbitrary pencil, every line-cone is inpolar to P, and so is every cone whose vertex lies on P; thus all the conditions are satisfied. Further, there are ∞^3 such surfaces, as there should be. The second, third, ... polars of p are similarly obtained from the quadrics of Σ_3 which are apolar to P in respect of the second, third, ... invariants of a pencil of quadrics.

3.1. *The contact surfaces.* Let S_r $(r=1, 2, 3)$ be any space of [4], and consider the system of quadrics of Σ_3 which touch S_r. This system, which we denote by $\Pi(S_r)$, corresponds to a surface $\pi(S_r)$ of order $r+1$, since $r+1$ quadrics of an arbitrary pencil satisfy this condition. Now any quadric which touches S_r intersects it in a cone whose vertex is at the point of contact. Hence $\pi(S_3)$ and $\pi(S_2)$ are generated by the same correspondence as $\mathfrak{F}$, in [3] and [2] respectively, so that $\pi(S_3)$ is a quartic symmetroid, and $\pi(S_2)$ a four-nodal cubic surface*. Evidently $\pi(S_1)$ is a quadric cone, since there is a unique quadric of Σ_3 which contains an arbitrary S_1, and this corresponds to a node of $\pi(S_1)$. In the particular case where S_1 is the axis of a line-cone, $\pi(S_1)$ is the nodal cone at the corresponding point. Similarly, the nodes of $\pi(S_2)$ and $\pi(S_3)$ correspond to those quadrics which meet S_2, S_3 in a line. If a quadric can be found to contain S_2, then $\pi(S_2)$ has a triple point, and is therefore a cubic cone; but since the quadric must in this case be a cone, it follows that the vertex of $\pi(S_2)$ lies on $\mathfrak{F}$. If a quadric (cone) touches S_3 in a plane, then $\pi(S_3)$ has a triple point, which lies on $\mathfrak{F}$.

3.2. Suppose that S_r meets Θ in a point P_0, vertex of the cone P. Any pencil of the sheaf $\Sigma_2(P_0)$ which contains P is such that P counts twice in the system $\Pi(S_r)$; this is obvious by considering the section of the pencil by S_r. Hence this sheaf represents the tangent plane to $\pi(S_r)$ at the point p; but, by § 2.2, the sheaf also represents the tangent plane to $\mathfrak{F}$ at that point. Thus $\pi(S_r)$ touches $\mathfrak{F}$ at all points corresponding to the intersection of S_r and Θ. It follows that, if S_2, S_3 are arbitrary, $\pi(S_2)$ touches $\mathfrak{F}$ at ten points, and $\pi(S_3)$ touches $\mathfrak{F}$ along a decimic curve which passes through the nodes, since every line-cone satisfies $\Pi(S_3)$.

3.3. *The residual property.* We now show that, if S_r, S_r' $(r=2, 3)$ are any two spaces, the points of contact of $\pi(S_r)$, $\pi(S_r')$ with $\mathfrak{F}$ lie on

* In the light of the present methods this surface might appropriately be called "the cubic symmetroid."

a surface of the same order as $\pi(S_r)$. Consider the system of quadrics of Σ_3, denoted by $\Pi(S_r, S_r')$, with regard to which S_r and S' are conjugate; since $r+1$ of these are contained in an arbitrary pencil, the corresponding surface $\pi(S_r, S_r')$ is of order $r+1$. Now any cone whose vertex lies in S_r or S_r' belongs to $\Pi(S_r, S_r')$; hence the corresponding point lies on $\pi(S_r, S_r')$, which proves the required property. It follows that any two decimic curves of contact mentioned in § 3.2 lie on a quartic surface, whence it follows that these curves are of genus 11; so is an arbitrary prime section of Θ.

If S_1 is a secant of Θ, $\pi(S_1)$ touches $\mathfrak{F}$ at the corresponding point; it may be shown* that Θ has ∞^2 quadrisecants, which generate a form of order 15; these give rise to quadritangent cones of $\mathfrak{F}$. It is evident that the residual property proved for the above surfaces holds in these special cases for the quadric $\pi(S_1, S_1')$. Although this result is, in any case, trivial, special quadrics of this type have interesting properties which will be noted later.

4.1. *The tritangent planes.* As remarked in § 2.2, the tangent plane at a point p of $\mathfrak{F}$ corresponds to the sheaf $\Sigma_2(P_0)$, where P_0 is the vertex of the cone P corresponding to p. Evidently the base curve of the sheaf has a node at P_0, since any prime section through P_0 meets the sheaf in quadrics having two intersections at P_0. Conversely, if a base curve has a node P_0, then this is a point of Θ; for the tangent primes at P_0 to three linearly independent quadrics of $\Sigma_2(P_0)$ must have a common plane, so that one member of the sheaf will be a cone with vertex P_0.

It will now be apparent that the determination of the bitangential developable and the tritangent planes presents enumerative problems of great difficulty. Thus, to find the number of bitangent planes through p, we require the number of base curves of Σ_3, lying on P, which meet Θ in two points, and therefore have two double intersections with the c_{20} common to P and Θ. If P is projected on to ordinary space from a point of itself, the difficult nature of the problem is made more evident.

4.2. The tritangent planes correspond to those base curves of Σ_3 which meet Θ in three points. In this case, however, we can at once specify a set of base curves fulfilling this condition; for if l is the join of any two base points of Σ_3, the sheaf containing l has a base curve consisting of l and a septimic which meets l in three points—a fact first noted by Veronese†. This may be proved as follows. Any prime section

* As in the paper referred to in the note to § 1.2.

† *Math. Annalen*, 19 (1882), 206.

through l meets the sheaf in a sheaf of quadric surfaces with a common generator, and these meet in four points elsewhere; whence the septimic, which, with l, forms the base curve of the sheaf, meets l in three points. Hence the 120 base curves obtained in this way from the joins in pairs of the sixteen base points of Σ_3, correspond to a set of tritangent planes to $\mathfrak{S}$. Analytically, this result shows that the ten planes $u_{rs} = 0$ $(r \neq s)$ are tritangent to the surface $|u_{rs}| = 0$, a fact which it would be difficult to verify directly.

4.3. Following Hesse we denote the base points of Σ_3 by the numbers 1 to 16 and the tritangent plane corresponding to the join rs by the same symbol. This notation is free from ambiguity if we make the convention that, if there are more than two figures in the number pair, the first two refer to one base point. Thus the join of the points 11 and 6 is denoted by 116, and the join of 1 and 16 by 161. It may be noted that the sixteen base points are such that the joins of any one to the rest form the fifteen trisecants which can be drawn to Θ from that point. No other trisecant can be drawn from a base point, or we should have a pencil of Σ_3 containing this line and corresponding to a tritangent line to $\mathfrak{S}$, which is impossible.

5.1. *Contact quadrics.* By § 3.1, we know that, if S_2 is an arbitrary plane of [4], $\pi(S_2)$ is a four-nodal cubic surface which touches $\mathfrak{S}$ at ten points. If S_2 contains two base points, 12 say, then $\pi(S_2)$ breaks up into the tritangent plane 12 and a quadric Q, which therefore touches $\mathfrak{S}$ at seven points. The sheaf of planes through 12 corresponds to a sheaf of contact quadrics, one of which will touch $\mathfrak{S}$ at a given point, since one plane of the sheaf passes through the corresponding point of Θ; moreover, the remaining six points of contact are thereby determined.

If S_2, S_2' are any two planes of this system, the surface $\pi(S_2, S_2')$ likewise breaks up into the plane 12 and a quadric. Hence it follows from the residual property (§ 3.3) that *the fourteen points of contact of any two contact quadrics of the system lie on another quadric.*

5.2. Consider the quadric $\pi(S_1, S_1')$, where S_1, S_1' are the joins of 12, 34. Let Q be any quadric of Σ_3 which contains the lines 13, 14; thus the quadrics Q form a pencil corresponding to the line of intersection of the planes 13, 14. Now, since Q meets the plane 134 in a line-pair, it follows that this plane touches Q at the point 1. Hence the polar of the line 34 passes through 1, so that 12, 34 are conjugate with regard to Q, whence Q belongs to $\Pi(S_1, S_1')$. Thus any point q on the inter-

section of the planes 13, 14 lies on $\pi(S_1, S_1')$. Similar reasoning applies to the planes 23, 24; 31, 32; 41, 42. Since the quadric $\pi(S_1, S_1')$ passes through the points of contact of 12, 34, we see that :—

If to three pairs of tritangent planes correspond, in [4], *the pairs of opposite edges of a tetrahedron, the points of contact of one pair and the four lines of intersection of the other two pairs lie on a quadric.*

5.3. This is an extension of a result obtained analytically by Hesse. If we take S_2, S_2' to be the planes 123, 456, and consider the surface $\pi(S_2, S_2')$ we obtain, by similar methods, the result that :—

The points of contact of the tritangent planes 12, 23, 31, 45, 56, 64 *lie on a cubic surface which contains the six points* (14, 24, 34), (15, 25, 35), (16, 26, 36), (41, 51, 61), (42, 52, 62), (43, 53, 63).

5.31. If we take a pair of planes such as 123, 345, the result is obtained from the preceding by writing 3 for 6 and interpreting 33 as arbitrary. The cubic surface then contains the four points (14, 24, 34), (15, 25, 35), (31, 41, 51), (32, 42, 52), and the two lines (13, 23), (34, 35).

5.4. *Associated threes.* Consider the case where S_2 contains the points 1, 2, 3. S_2 meets Θ in ten points, which are the vertices of cones which pass through 1, 2, 3; hence these cones have their vertices on one of the lines 12, 23, 31, or else they contain S_2 entirely. Now each of the three lines contains three of the vertices (§ 4.2), so that the remaining vertex is that of the cone which contains S_2 and corresponds to the meet of the planes 12, 23, 31; it represents, in fact, the vertex of the degenerate cubic cone $\pi(S_2)$, (§ 3.1). But since this point corresponds to a cone, it must lie on $\mathfrak{F}$. Hence, *any three tritangent planes of the type* 12, 23, 31 *intersect on* $\mathfrak{F}$. We have considered in the previous paragraph the surface $\pi(S_2, S_2')$, where S_2, S_2' are planes 123, 456, and have obtained twenty-four points on it; it is now evident that the surface also contains the vertices of the associated threes, 123, 456.

It is clear that the unique quadric of Σ_3 which contains three joins of base points will not necessarily be a cone unless they are of the type above mentioned. Hence there are no associated threes among the tritangent planes other than those of this type.

5.5. If we take S_3 to pass through the points 1234, the contact quartic $\pi(S_3)$ touches each of the six planes 12, etc., at their points of contact with $\mathfrak{F}$. Hence :—

Any six planes of the type 12, 23, 31, 14, 24, 34 *are tritangent to a quartic symmetroid at their points of contact with* $\mathfrak{F}$. *This surface*

touches $\mathfrak{F}$ *along a decimic curve through the twenty nodes; it also contains the vertices of the associated threes which may be formed from the six planes.*

If S_3' passes through 5678, the residual property applied to $\pi(S_3, S_3')$ shows that the eighteen points of contact of the planes given above, and a similar set of eighteen formed from 5678, lie on a quartic surface passing through the twenty nodes; the surface contains the vertices of all associated threes formed from the twelve planes.

6.1. *Quintic symmetroids with triple points**. If $\mathfrak{F}$ possesses a triple point t, the corresponding quadric T must count thrice in any pencil containing it, and is therefore a space-pair with double plane σ. T will then count twice in any system $\Pi(S_3)$ and once in any system $\Pi(S_2)$; hence t is a node on all the contact quartics $\pi(S_3)$, and an ordinary point on the contact cubics $\pi(S_2)$. Eight base points of Σ_3 lie in each space of T; these give rise to fifty-six improper tritangent planes, corresponding to the joins in pairs of base points which lie in the same space.

The Jacobian Θ breaks up into σ and a surface F_9 which, since the Jacobian is the tac-locus of Σ_3, meets σ in a cubic curve. Hence the curve of contact of $\mathfrak{F}$ with $\pi(S_3)$ has a triple point at t, since S_3 will meet this cubic in three points. An arbitrary plane section through t corresponds to a septimic of genus 3, which meets σ in three points. The nodes of $\mathfrak{F}$ correspond to the line-cones of Σ_3; the number of such cones has been shown† to be sixteen.

6.2. If $\mathfrak{F}$ has two triple points t_1, t_2, corresponding to space-pairs with double planes σ_1, σ_2, the line t_1t_2 lies on $\mathfrak{F}$; this corresponds to the singular pencil of cones with a common vertex at the intersection $\sigma_1\sigma_2$ of σ_1 and σ_2. Since an arbitrary plane section of $\mathfrak{F}$ meets t_1t_2, the corresponding Jacobian curve passes through the point $\sigma_1\sigma_2$, which therefore lies on the surface F_8, forming, with σ_1 and σ_2, the surface Θ. A plane section through t_1 and t_2 consists of t_1t_2 and a quartic with nodes at t_1 and t_2; the corresponding curve on F_8 is an elliptic quintic which meets σ_1 and σ_2 in two points. There are thirteen nodes on $\mathfrak{F}$, corresponding to lines on F_8.

6.3. In the case of three triple points t_r, Θ consists of three planes σ_r and a surface F_7 meeting them in cubic curves. $\mathfrak{F}$ contains three lines t_rt_s, corresponding to the intersections $\sigma_r\sigma_s$, which lie on F_7, of the

* Note added January, 1930.

† *Proc. Camb. Phil. Soc.*, 25 (1929), 268.

three planes. A plane section of $\mathfrak{Z}$ corresponds to a Jacobian curve passing through $\sigma_r\sigma_s$. The plane section through $t_1t_2t_3$ corresponds to a rational quartic on F_7, meeting each plane σ_r. $\mathfrak{Z}$ has eleven nodes, corresponding to lines on F_7.

6.4. If $\mathfrak{Z}$ has four triple points t_r, it is rational and is represented on the surface F_6 which, with σ_r, forms the Jacobian Θ. F_6 is the polo-surface* in [4]; it contains ten lines, corresponding to nodes of $\mathfrak{Z}$, and four plane cubics, corresponding to triple points. There are six lines on $\mathfrak{Z}$, which correspond to the points $\sigma_r\sigma_s$, and any plane section of $\mathfrak{Z}$ is represented by a Jacobian curve through these points; two such curves must meet elsewhere in five points. A space section C_6^3 of F_6 corresponds to a C_{10}^3 on $\mathfrak{Z}$, with triple points at t_r; this curve passes through the nodes, since the corresponding space section must meet the lines on F_6; two such curves meet elsewhere in six points.

6.5. The number of tritangent planes in each case is the number of joins of base points which are trisecants of F_r and which do not lie in a space-pair; these numbers are easily seen to be 64, 32, 16, and 8 in the four cases ($r = 9, 8, 7, 6$). Thus, for F_6, there are four space-pairs (A_r, B_r) in Σ_3, and the eight trisecants are the joins of pairs of points such as $A_1B_2B_3A_4$ and $B_1A_2A_3B_4$, which have no symbol A_r or B_r in common. This is in accord with conclusions obtained in a different manner in previous work.

6.6. It is known from the plane representation of F_6 that the surface contains ten plane cubics; only four of these arise from space-pairs of Σ_3. Hence it would appear that there are ${}^{10}C_4 = 210$ systems Σ_3, each containing four space-pairs, for which F_6 is the residual Jacobian, and that these systems have in common the axes of their ten line-cones. Corresponding to these systems are 210 quintic symmetroids, each with four triple points.

* See the reference in §6.1.

THE CONSTRUCTION OF SOLVING NUCLEI FOR CERTAIN SYSTEMS OF INTEGRAL EQUATIONS

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[Received 31 December, 1928.—Read 10 January, 1929.]

In a former paper*, which will here be referred to as S.N., certain integral equations were discussed in which the nuclei were homogeneous and of degree —1 in the two variables, the lower limit of the integration being zero. In the present paper the same methods are largely followed, and the results are extended.

§ 1 is occupied with an important transformation of the equation (13 . 4) of S.N., which shows that the nuclei involved may be two rational algebraic fractions, homogeneous of degree —1, and having no repeated factors in the denominators.

In the present extension the restriction as to repeated factors in the denominators is removed, and, moreover, the theory is extended to include systems of integral equations with n unknown functions, any such system depending on four matrices of functions (S_1, S_2, S_3, S_4 of § 20) of order n, whereas the equation of S.N. depended only on two matrices of the first order, that is, two single functions, and even these contained constant coefficients where now there may be quantics† in the variable.

The problem of §§ 2–13 is to construct rows of functions satisfying two given functional equations in the two halves of the plane separated by the lateral axis. In § 2 the two functional equations are taken to be the same.

§§ 4–8 are concerned with properties of a pair of matrices whose constituents are analytic functions; the exponential function of a matrix and its logarithm are treated in § 9, and in § 10 a matrix is constructed

* *Proc. London Math. Soc.* (2), 27 (1926), 233–272.

† It is this change that removes the restriction as to repeated factors just referred to.

combining the singularities of two given matrices in the two halves of the plane. To simplify the work in § 10 it is assumed that the Riemann surface on which the multipliers are branches of a uniform function is simply connected. It is quite likely that this assumption is not a necessary one.

In § 11 rows of functions are constructed satisfying two different functional equations of a certain type in the two halves of the plane respectively.

Other rows having prescribed singularities and satisfying the same functional equations are constructed in §§ 12, 13, and further discussed in §§ 14, 15. In §§ 16–19 the method of contour integration is applied, and gives further properties leading in § 20 rather suddenly to the integral equations of § 21.

§ 22 deals with the formation of the functional equations corresponding to a given set of integral equations of the kind in question, and § 23 with the transformation of other integral equations by means of the results of § 20.

1, 2. *Preliminary.*

1. The result which it is the chief object of the present paper to extend is given on pages 266–7 of the *Proc. London Math. Soc.* (2), 27 (1926), and is capable of a remarkable transformation as follows :—

Equation (13 . 4), as there given, may be written

$$\phi(z) = g(z) + \frac{1}{2i\pi}\int_0^\infty \frac{\psi(t)}{z+t}\,dt, \tag{1.1}$$

where
$$\psi(z) = -C_2\phi(ze^{c_2}) - \ldots \text{ when } z \text{ runs from 0 to 1}, \tag{1.21}$$
$$= C_1\phi(ze^{c_1}) + \ldots \text{ when } z \text{ runs from 1 to } \infty. \tag{1.22}$$

Thus the problem of solution is equivalent to that of finding ψ when

$$\psi(z) = f(z) + \frac{1}{2i\pi}\int_0^\infty K(z,t)\,\psi(t)\,dt, \tag{1.3}$$

where
$$K(z,t) = -\frac{C_2}{ze^{c_2}+t} - \ldots \text{ when } z \text{ runs from 0 to 1} \tag{1.41}$$
$$= \frac{C_1}{ze^{c_1}+t} + \ldots \text{ when } z \text{ runs from 1 to } \infty, \tag{1.42}$$

and
$$f(z) = -C_2 g(ze^{c_2}) - \ldots \text{ when } z \text{ runs from 0 to 1}, \tag{1.43}$$
$$= C_1 g(ze^{c_1}) + \ldots \text{ when } z \text{ runs from 1 to } \infty. \tag{1.44}$$

The two forms of $K(z, t)$ are algebraic fractions, homogeneous in z, t of degree -1. Their denominators do not contain the factor z or t but otherwise they would be quite arbitrary if C_1, C_2, ... might be quantics in the operator $z(d/dz)$. Since C_1, C_2, ... were taken constant the denominators must not contain repeated factors. (It is also assumed in the investigation that c_1, c_2, ... are not the logarithms of negative real quantities, but probably this restriction could be removed).

The solution of (1.3) given by equation (13.5) of S.N. is

$$\psi(z) = f(z) + \frac{1}{2i\pi}\int_0^\infty \kappa(z, t)\, f(t)\, dt, \qquad (1.5)$$

where $\kappa(z, t) = -C_2\chi(ze^{c_2}, -t) - \ldots$ when z runs from 0 to 1, (1.61)

$= C_1\chi(ze^{c_1}, -t) + \ldots$ when z runs from 1 to ∞. (1.62)

In (1.3) there is a discontinuity in the form of the nucleus $K(z, t)$ as a function of z, but not of t, and accordingly there is a discontinuity of form in f and ψ, but this need not happen in the case of the adjoint equation

$$\psi(z) = f(z) + \frac{1}{2i\pi}\int_0^\infty K(t, z)\, \psi(t)\, dt, \qquad (1.7)$$

for which $\kappa(t, z)$ is the solving function.

I have not met with the remark that the form of the equation adjoint to a given one depends partly on the range of the variables. For instance

$$\psi(z) = f(z) + \frac{\theta}{2i\pi}\int_0^1 \frac{\psi(t)}{z+t}\, dt \qquad (1.81)$$

is adjoint to itself when the range is from 0 to 1, but if the same equation is written

$$\psi(z) = f(z) + \frac{1}{2i\pi}\int_0^2 K(z, t)\, \psi(t)\, dt, \qquad (1.82)$$

where $K(z, t) = \theta/(z+t)$ when $t < 1$, but $K(z, t) = 0$ when $t > 1$, the adjoint equation

$$\psi(z) = f(z) + \frac{1}{2i\pi}\int_0^2 K(t, z)\, \psi(t)\, dt, \qquad (1.91)$$

gives $\psi(z) = f(z)$ when $1 < z < 2$, but when $0 < z < 1$,

$$\psi(z) = f(z) + \frac{\theta}{2i\pi}\int_0^2 \frac{\psi(t)}{t+z}\, dt,$$

so that

$$\psi(z) = f(z) + \frac{\theta}{2i\pi}\int_1^2 \frac{f(t)}{t+z}\, dt + \frac{\theta}{2i\pi}\int_0^1 \frac{\psi(t)}{t+z}\, dt. \qquad (1.92)$$

2. For the extension of (1.3), (1.5) suppose that it is required to construct a row of n functions $f^{(1)}(z)$, ..., $f^{(n)}(z)$ or collectively $f(z)$, having the following properties:—

(*a*) When $R(z) > 0$, that is, in the half-plane on the right of the lateral axis,

$$f(z+i\pi)-f(z-i\pi) = S_1(D)f(z). \tag{2.11}$$

(*b*) When $R(z) < 0$, that is, in the left half-plane,

$$f(z+i\pi)-f(z-i\pi) = S_2(D)f(z). \tag{2.12}$$

(*c*) In the strip $|I(z)| \leqslant \pi$ the functions are regular except at $\pm i\pi$, at infinity, and at u where they have a simple pole with residues A (a row).

In (*a*) and (*b*) D is d/dz; $S_r(D)$ ($r = 1, 2$) stands for

$$C_r \exp cD + C_r' \exp c'D + C_r'' \exp c''D \ldots, \quad \text{or} \quad \Sigma C_r \exp cD,$$

where c, c', c'', ..., are different simple constants each having a lateral part between $\pm i\pi$, while C_r, C_r', ..., are matrices of order n, whose constituents are quantics in D: $\exp cD f(z)$ is interpreted as $f(z+c)$.

It is convenient to suppose that the series c, c', c'', ... contains ..., $-c''$, $-c'$, $-c$ and that π, $I(c)$, $I(c')$, $I(c'')$, ... are in descending order and also to use $c`$, $c``$ for any two consecutive members of the series $i\pi$, c, c', ..., $-i\pi$.

Suppose first that $S_1 = S_2 = S$; let $\Delta(\lambda)$ be the determinant of

$$e^{i\pi\lambda} - e^{-i\pi\lambda} - S(\lambda)$$

and $Q(\lambda)$ a matrix such that

$$\{e^{i\pi\lambda} - e^{-i\pi\lambda} - S(\lambda)\} Q(\lambda) = \Delta(\lambda);$$

thus

$$\{e^{i\pi\lambda} - e^{-i\pi\lambda} - S(\lambda)\}^{-1} = Q(\lambda) \div \Delta(\lambda). \tag{2.2}$$

Towards infinity on the lateral axis

$$\Delta(\lambda) \sim (e^{i\pi\lambda} - e^{-i\pi\lambda})^n, \tag{2.31}$$

$$Q(\lambda) \sim (e^{i\pi\lambda} - e^{-i\pi\lambda})^{n-1}, \tag{2.32}$$

and, so long as $|I(z)| < \pi$,

$$\int_{-i\infty}^{i\infty} \frac{Q(\lambda) A}{\Delta(\lambda)} e^{\lambda z} d\lambda \tag{2.4}$$

represents a row of regular functions which may be written

$$Q(D) A\tau(z) \quad \text{or} \quad \sigma(z) A, \tag{2.41}$$

$\tau(z)$ being the single function denoted by $\int_{-i\infty}^{i\infty} \frac{e^{\lambda z}}{\Delta(\lambda)} d\lambda$, when $|I(z)| < n\pi$, and $\sigma(z)$ denoting $Q(D)\tau(z)$. (2.42)

If $I(z)$ approaches π, we may put

$$\sigma(z)A = \int_0^{i\infty} \frac{Q(\lambda)Ae^{\lambda z}}{\Delta(\lambda)} d\lambda + \int_{-i\infty}^{0} \frac{\{e^{-2i\pi\lambda} + e^{-i\pi\lambda}S(\lambda)\}Q(\lambda)Ae^{\lambda z}}{\Delta(\lambda)} d\lambda + \int_{-i\infty}^{0} Ae^{\lambda(z-i\pi)} d\lambda, \quad (2.43)$$

where the last term is $A/(z-i\pi)$ and the rest represents functions regular so long as $-\pi < I(z) < 2\pi - I(c)$. Thus in this wider strip $\sigma(z)A$ is regular except for a pole at $i\pi$ with residue A. When $I(z) > \pi$, $A/(z-i\pi)$ may be written

$$-\int_0^{i\infty} Ae^{\lambda(z-i\pi)} d\lambda,$$

and thus, if $0 < I(z) < \pi - I(c)$,

$$\sigma(z+i\pi)A = \int_{-i\infty}^{i\infty} \frac{\{e^{-2i\pi\lambda} + e^{-i\pi\lambda}S(\lambda)\}Q(\lambda)Ae^{\lambda(z+i\pi)}}{\Delta(\lambda)} d\lambda$$
$$= \{e^{-i\pi D} + S(D)\}\sigma(z)A. \quad (2.5)$$

Hence $\sigma(z-u+i\pi)A$ satisfies the prescribed conditions (a), (b), (c) so long as $I(c) < I(u) < \pi$.

The matrix $\sigma(z)$ satisfies the conditions (a), (b) when $S_1 = S_2 = S$, and its singularity in the strip $|I(z)| \leqslant \pi$ is a simple pole at $\pm i\pi$ with the unit matrix as residue. We also have

$$\{e^{i\pi D} - e^{-i\pi D} - jS(D)\}\, j\sigma(z) = \Delta(D)\tau(z) = 0, \quad (2.51)$$

where j denotes the operation of turning the rows of a matrix into columns. (2.51) expresses the fact that $\sigma(z)$ is reduced to zero by the operator $e^{i\pi D} - e^{-i\pi D} - S(D)$ in the position of a post-factor.

For a position of u in which $I(c') > I(u) > I(c'')$ let S^+ denote the sum of the terms in S which involve c, c', ... as far as c' and S^- the rest, involving c'' ..., $-c'$, $-c$: then a row of functions satisfying the conditions (a), (b), (c) is $F(z,u)A$, where

$$F(z,u) = Q(D)\{e^{i\pi D} - S^+(D)\}\tau(z-u) = Q(D)\{e^{-i\pi D} + S^-(D)\}\tau(z-u). \quad (2.6)$$

The path of integration in (2.4) may be any parallel to the lateral axis which does not pass through a zero of $\Delta(\lambda)$. A change in the path may add

or subtract exponential terms in $\tau(z)$ and may thus affect the behaviour of the functions at infinity. Near $\pm i\pi$, $F(z, u)$ has no singularity unless $u = c'$ or c''. By shifting the path to the right we find, as in other such cases*, suggested expansions for $\tau(z)$ and its derivatives, and therefore for $\sigma(z)$ and $F(z, u)$, in the form

$$A_1 e^{\lambda_1 z} + A_2 e^{\lambda_2 z} + \ldots, \tag{2.7}$$

where $\lambda_1, \lambda_2, \ldots$ are the zeros of $\Delta(\lambda)$ in the right half-plane in ascending order of real parts. The error in stopping at any stage is of the order of the first term omitted when $R(z)$ is negative and great: there is a similar expansion (2.71) when $R(z)$ is positive and great.

3. *A lemma on integration.*

If U, V are matrices of functions and $S(D)$ is a matrix involving only quantics in D, then

$$US(D)V - j[(jV)\{jS(-D)\}(jU)] \tag{3.1}$$

is immediately integrable.

This follows from its simplest particular case, namely,

$$D(UV) = (DU)V + U(DV), \tag{3.2}$$

where the first term on the right is $j\{(jV)D(jU)\}$.

If the coefficient of D^r in $S(D)$ is S_r, a constant numerical matrix, then the corresponding terms in (3.1) are

$$US_r(D^r V) - \{(-D)^r U\} S_r V,$$

which may be written as

$$D \sum_{s=1}^{r} \{(-D)^{s-1} U\} S_r (D^{r-s} V),$$

whence the lemma follows.

4–7. *Canonical form of a pair of matrices of regular analytical functions.*

4. If M, N are two matrices of order n and θ is a single parameter, it is known† that generally $M - \theta N$ can be expressed in the form

* Compare *Proc. London Math. Soc.* (2), 25 (1925), 433–4. The validity of the expressions in *infinite* series such as (2.7) need not be discussed here.

† Compare *Proc. London Math. Soc.* (1), 31 (1899), 170-6.

$K(U-\theta V)L$, where the matrices K, L, U, V do not involve θ and U, V are "diagonals", that is, matrices in which all constituents vanish except those of the dexter diagonals*. UV^{-1}, which is the same as $V^{-1}U$, is a diagonal formed with the multipliers of MN^{-1}, which are also those of $N^{-1}M$, and are the n values of θ for which the determinant of $M-\theta N$ vanishes. The reduction is possible if all the multipliers are unequal and it is then unique except for permutations among the multipliers and for the change of K, U, V, L into $KE^{-1}, UEG, VEG, G^{-1}L$, where E, G are any two diagonals: they may be so chosen that $VEG = 1$, so that the form of $M-\theta N$ becomes $K(U-\theta)L$.

Suppose now that the constituents of M, N are regular functions of a variable z in a domain, say that of the origin.

Then the multipliers are the branches of what may be called the multiplier-function, an n-valued function of z, uniform on a certain Riemann surface of n sheets. We shall ignore special cases and therefore assume that there are no finite values of z for which more than two of the multipliers become equal. Take $\theta_n(z)$ to be a branch which at the origin is finite† and not equal to any other: it is a regular function of z near the origin. The determinant of $M-\theta_n(z)N$ is zero; but not all of its first minors vanish, and in fact we may permute the rows and columns of $M-\theta_n(z)N$ so that the minor of the last constituent in the last row does not vanish with z: then, adding to the last row the other rows multiplied by suitable coefficients which are all regular functions of z, we reduce the last row to zero and similarly the last column. These operations applied to $M-\theta N$ reduce it to a form in which $\theta-\theta_n(z)$ is a factor throughout the last row and column, the other factor being in each case a regular function of z. The last constituent of the last row does not vanish with z since $\theta_n(z)$ is not equal to any other branch at the origin, and hence all other constituents in the last row and column may be reduced to zero by subtracting the last one multiplied by regular functions of z.

The operations by which the matrix $M-\theta N$ is thus reduced are equivalent to divisions by suitable matrices, not involving θ, taken some as pre-factors, some as post-factors, their constituents being regular functions of z near the origin. They may be combined into one pre-factor and one post-factor, and thus

$$M-\theta N = K'(M'-\theta N')L',$$

where $M'-\theta N'$ has nothing in the last row and column but the common

* Any two "diagonals" are commutative.

† If $\theta_n(0)$ is infinite, interchange M and N.

constituent which is a multiple of $\theta_n(z)-\theta$. The same process may be applied to the last row and column but one and so on until no unrepeated values of the multiplier function are left: thus, if all the multipliers are unequal at the origin, the whole takes the desired form $K(U-\theta V)L$, in in which all the constituents of K, L, U, V are regular.

If the reduction fails it is at the last stage and because $\theta_1(0)=\theta_2(0)$: the problem is then that of reducing two matrices M, N of the second order, for which the two multipliers become equal at the origin.

There are three ordinary cases.

I. $M-\theta_1(0)N$ may vanish at the origin, so that

$$M=\theta_1(0)N+zQ,$$

where Q is a matrix of regular functions. Then the above method may be applied to N and Q, and if

$$N=KVL, \quad Q=KWL,$$

where V, W are diagonals, then

$$M=K\{\theta_1(0)V+zW\}L,$$

which is as desired, since $\theta_1(0)V+zW$ is a diagonal.

II. $M-\theta_1(0)N$ does not vanish at the origin, which is a branch-point.

III. $M-\theta_1(0)N$ does not vanish at the origin, which is not a branch-point though $\theta_1(0)=\theta_2(0)$.

In each of these cases let $M=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$, $N=\begin{pmatrix}a' & b'\\ c' & d'\end{pmatrix}$. The ratios a/a', b/b', c/c', d/d' are not all equal at the origin; suppose that $ab'-a'b$ does not vanish with z. Then the result of post-multiplying $M-\theta N$ by $\begin{pmatrix}b-\theta_1 b' & b-\theta_2 b'\\ \theta_1 a'-a & \theta_2 a'-a\end{pmatrix}$ is

$$\begin{pmatrix}a'b-ab' & a'b-ab'\\ bc'-ad'+\theta_1(a'd'-b'c') & bc'-ad'+\theta_2(a'd'-b'c')\end{pmatrix}\begin{pmatrix}\theta_1-\theta & 0\\ 0 & \theta_2-\theta\end{pmatrix}.$$

The reciprocal of $\begin{pmatrix}b-\theta_1 b' & b-\theta_2 b'\\ \theta_1 a'-a & \theta_2 a'-a\end{pmatrix}$ is

$$\begin{pmatrix}\theta_2 a'-a & \theta_2 b'-b\\ a-\theta_1 a' & b-\theta_1 b'\end{pmatrix}\div\{(ab'-a'b)(\theta_1-\theta_2)\},$$

so that we may put

$$K = \begin{pmatrix} a'b-ab' & a'b-ab' \\ bc'-ad'+\theta_1(a'd'-b'c') & bc'-ad'+\theta_2(a'd'-b'c') \end{pmatrix},$$

$$L = \begin{pmatrix} \theta_2 a'-a & \theta_2 b'-b \\ a-\theta_1 a' & b-\theta_1 b' \end{pmatrix},$$

$$U = \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{pmatrix} \div \{(ab'-a'b)(\theta_1-\theta_2)\}, \quad V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \div \{(ab'-a'b)(\theta_1-\theta_2)\}.$$

The same method can be applied to M, N if $cd'-c'd$ does not vanish, or to jM, jN if $ac'-a'c$ and $bd'-b'd$ do not both vanish, and it is therefore always effective in Cases II, III. In each case then K, L, $U(\theta_1-\theta_2)$ and $V(\theta_1-\theta_2)$ are regular on the Riemann surface: in Case II $\theta_1-\theta_2$ is of the order $\sqrt{z}$ and is a regular function of $\sqrt{z}$; in Case III $\theta_1-\theta_2$ is of order z and is a regular function of z.

6. The conclusion is then that M, N can be expressed in the forms KUL, KVL, where U, V are diagonals, and that K, L, U, V are regular analytical functions of z in the domain of any point where the constituents of M, N are so, except where two branches of the multiplier function become equal. On the n-sheeted surface K, L, U, V are in fact uniform, and they are regular except that if z_0 is a branch-point the constituents of U, V affected by that branch-point contain terms in $(z-z_0)^{-\frac{1}{2}}$, and if z_0 is not a branch-point but a point at which two values of the multiplier function happen to be equal, then the corresponding constituents of U, V sometimes have a simple pole at z_0. Except in special cases, there are no higher singularities in the finite part of the plane.

7. Again, let there be two matrices $M+\mu$, $N+\nu$, where M, N are constant and μ, ν variable but small: suppose that

$$M = KUL, \quad N = KL, \tag{7.1}$$

where U is a diagonal all of whose constituents $\theta_1, \theta_2, \ldots, \theta_n$ are unequal, and the determinants of K, L do not vanish. It is to be proved that there are variable matrices κ, λ, and a diagonal υ such that

$$M+\mu = (K+\kappa)(U+\upsilon)(L+\lambda), \quad N+\nu = (K+\kappa)(L+\lambda) \tag{7.2}$$

and that these will be of the order of μ, ν at most. Now κ, υ, λ can be found by successive approximation in ascending series of powers and products of the constituents of μ, ν, and by the ordinary theory of reversion these series will converge so long as the absolute values of those constituents do not exceed a certain quantity, provided that the linear equations that give the first approximation are generally soluble, that is,

have a determinant that does not vanish. The sums of these series will then moreover be of the order of μ and ν at most.

Now the linear equations for the first approximation may be written

$$K\lambda+\kappa L = \nu, \tag{7.31}$$

$$KU\lambda+K\nu L+\kappa UL = \mu, \tag{7.32}$$

or

$$\lambda L^{-1}+K^{-1}\kappa = K^{-1}\nu L^{-1}, \tag{7.33}$$

$$U\lambda L^{-1}+\nu+K^{-1}\kappa U = K^{-1}\mu L^{-1}, \tag{7.34}$$

whence

$$\nu-U(K^{-1}\kappa)+(K^{-1}\kappa)\,U = K^{-1}\mu L^{-1}-UK^{-1}\nu L^{-1}. \tag{7.4}$$

The diagonal on the left side in (7.4) is ν, and therefore

$$\nu = \text{diagonal of } K^{-1}\mu L^{-1}-UK^{-1}\nu L^{-1}.$$

When r, s are unequal the s-th constituent of the r-th row on the left in (7.4) is the corresponding constituent in $K^{-1}\kappa$ multiplied by $\theta_s-\theta_r$, and hence this constituent in $K^{-1}\kappa$ is the corresponding one in $K^{-1}\mu L^{-1}-UK^{-1}\nu L^{-1}$ divided by $\theta_s-\theta_r$.

The diagonal of $K^{-1}\kappa$ is quite arbitrary and may be taken to be 0, and then λ is given by (7.33).

Hence the linear equations giving the first approximation are soluble even with the added condition that the diagonal of $K^{-1}\kappa$ shall vanish, and thus it follows that κ, ν, λ are of the order μ, ν at most.

8. Thus the matrices $e^{i\pi z}-e^{-i\pi z}-S_r(z)$ $(r=1, 2)$ of § 2, in any finite domain on the appropriate Riemann surface, may be reduced to the forms KU_1L, KU_2L, where K, L are matrices and U_1, U_2 are diagonals, all being regular except for possible simple poles of U_1, U_2, where two values of the multiplier-function are equal.

To discuss the state of affairs at infinity, say when $I(z)\to-\infty$, let us ignore special cases and suppose the terms containing the highest power of z in C_1-C_2 (§ 2) to form the matrix Cz^q, the multipliers of C being all different, so that

$$C = K_0U_0K_0^{-1},$$

where K_0, U_0 are numerical and U_0 is a diagonal.

Then the matrices to be reduced are M, N, where

$$N = e^{i\pi z}(1+\nu),$$

$$M = N-e^{cz}z^q(C+\rho)$$

and ν, ρ are small.

As in § 7, $C+\rho$ and $1+\nu$ can be expressed as

$$(K_0+\kappa)(U_0+\upsilon)(K_0^{-1}+\lambda) \quad \text{and} \quad (K_0+\kappa)(K_0^{-1}+\lambda),$$

where κ, υ, λ are of the order of ν and ρ. Thus

$$MN^{-1} = (K_0+\kappa)\{1-e^{(c-i\pi)z}z^q(U_0+\upsilon)\}(K_0+\kappa)^{-1},$$

so that all the values of the multiplier-function tend to unity when $z\to -i\infty$, the error being of the order of

$$e^{(c-i\pi)z}z^q.$$

We have then $\quad e^{i\pi z}-e^{-i\pi z}-S_r(z) = KU_rL \quad (r=1, 2),$

where $K = K_0+\kappa$, $L = K_0^{-1}+\lambda$, these being bounded,

$$U_1 = e^{i\pi z}+O(e^{cz}z^q), \quad U_2 = e^{i\pi z}.$$

Similarly when $I(z)\to+\infty$.

9. *The logarithm of a matrix.*

Let M be any matrix, then

$$1+M+\frac{M^2}{2!}+\frac{M^3}{3!}+\ldots$$

is also a matrix of the same order, which may be denoted by $\exp M$, and if M, M' are commutable matrices, or in particular diagonals, then

$$\exp M\times\exp M' = \exp(M+M').$$

If h, k are scalar, then

$$\exp hM\times\exp kM = \exp(hM+kM),$$

and, if q is an integer, $\quad (\exp M/q)^q = \exp M.$

Thus, if $N = \exp M$, a value of

$$q(N^{1/q}-1) \quad \text{is} \quad q\{\exp(M/q)-1\},$$

which tends to M when $q\to\infty$, and we may say that

$$M = \log N = \lim_{q\to\infty}\{q(N^{1/q}-1)\}.$$

The exponential function of M has one value only, but the logarithm of N has many, which may be compared by reducing M to the form* KUK^{-1} in which U is a diagonal. Then $\exp U$ is also a diagonal whose constituents are the exponential functions of the constituents of U and,

* This assumes that all the multipliers of M are unequal. Otherwise there may be complications.

term by term,

$$\exp M = K(\exp U)\,K^{-1}.$$

Thus, to find the logarithm of N, we are to express it in the form KVK^{-1}, where V is a diagonal, and

$$\log N = K(\log V)\,K^{-1},$$

$\log V$ being a diagonal whose constituents are the logarithms of those of V: each of them may be increased by any multiple of $2i\pi$.

The binomial and logarithmic series are also valid under restrictions, for, if M, N are of the order n and the absolute values of the constituents are less than a in M, b in N, then each constituent in MN is less than nab in absolute value, and thus each in M^q is less than $n^{q-1}a^q$, q being a positive integer.

The series $$M+\frac{h-1}{2!}M^2+\frac{(h-1)(h-2)}{3!}M^3+\ldots$$

converges absolutely and uniformly if $|h|\leqslant 1$ and $na<1-0$. Thus the ordinary proofs of the binomial and logarithmic series may be applied if $a<1/n$, and in particular

$$\log(1+M) = M-\tfrac{1}{2}M^2+\tfrac{1}{3}M^3-\ldots,$$

the error at the q-th term being

$$O\left(\frac{1}{q+1}n^q a^{q+1}\right).$$

10. *The diagonal* U_{12}.

It follows from the results of §§ 4–9 that

$$\log\left[\left\{e^{i\pi z}-e^{-i\pi z}-S_1(z)\right\}\left\{e^{i\pi z}-e^{-i\pi z}-S_2(z)\right\}^{-1}\right]$$

may be written $$K\log(U_1/U_2)\,K^{-1}, \qquad (10.1)$$

where in each sheet one value of $\log(U_1/U_2)$ is $O\left\{e^{(c-i\pi)z+0}\right\}$ when $I(z)\to-\infty$ and one value is $O\left\{e^{(i\pi-c)z+0}\right\}$ when $Iz\to+\infty$; also K, K^{-1} are bounded in either case.

Further $\log(U_1/U_2)$ is the diagonal formed with the logarithms of $\theta_1, \theta_2, \ldots, \theta_n$, the different values or branches of the multiplier-function $\theta(z)$, which is such that the determinant of the matrix

$$e^{i\pi z}-e^{-i\pi z}-S_1(z)-\left\{e^{i\pi z}-e^{-i\pi z}-S_2(z)\right\}\theta(z) \qquad (10.2)$$

is zero.

If z travels along a line $x=a$ from $a-i\infty$ to $a+i\infty$, the increase in value of $\log\theta(z)$ may be zero or may be a multiple of $2i\pi$. It will be

assumed that there is a value of a for which this increase is zero in each sheet, so that the same value of $\log\theta(z)$ is exponentially small at each end. It will also be assumed that this path avoids branch-points and places where $\theta(z)$ becomes zero or infinite in any sheet.

A further assumption, which simplifies the work but seriously limits its generality, will be made, namely, that the n-sheeted surface is simply-connected and that K, L contain only algebraic functions*. The $2n-2$ branch-points will be supposed to lie at finite distances from the origin. There is then a variable ξ, such that on the n-sheeted surface z and the multiplier-function $\theta(z)$ are both regular functions of ξ, except for the n poles of z which are essential singularities of θ, and for the poles of θ. Then take

$$\int \log\theta(z)\,\frac{d\xi}{\tau-\xi} \tag{10.3}$$

along the ξ-path composed of n pieces corresponding to the line $x=a$, from $a-i\infty$ to $a+i\infty$, in the n sheets. It may be written

$$\int_{a-i\infty}^{a+i\infty} \Sigma\left\{\frac{\log\theta(z)}{\tau-\xi}\,\frac{d\xi}{dz}\right\}dz, \tag{10.31}$$

the summation referring to the different sheets. The value of $\log\theta(z)$ to be taken is that which tends to zero as above mentioned when $z\to a\pm i\infty$, and then the integral represents a regular function of τ so long as τ does not approach the ξ-path. If t is the same function of τ that z is of ξ, the integral is a function of t uniform on the n-sheeted surface so long as t does not approach the line $x=a$. If we denote it by $2i\pi\,\omega(t)$, when t is in the right half-plane in any sheet, then it becomes

$$2i\pi\,\omega(t)-2i\pi\log\theta(t),$$

when t crosses into the left half-plane.

When $t\to\pm i\infty$, suppose that γ is the value to which τ tends. The path of integration in (10.31) may be bent to the right or left so that z, t tend to infinity at different amplitudes and $t/(z-t)$ is bounded: then $(\xi-\gamma)/(\tau-\xi)$ is also bounded.

---

* As an example, take

$$M=\begin{pmatrix} e^{i\pi z}-e^{-i\pi z}+z, & -z-z^2 \\ -z+z^2, & e^{i\pi z}-e^{-i\pi z}+z\end{pmatrix},\quad N=\begin{pmatrix} e^{i\pi z}-e^{-i\pi z}+z, & z+z^2 \\ z-z^2, & e^{i\pi z}-e^{-i\pi z}+z\end{pmatrix},$$

where

$$2(M-\theta N)(1-z^2)^{\frac{1}{2}}$$
$$=\begin{pmatrix}(1+z)^{\frac{1}{2}}, & -(1+z)^{\frac{1}{2}}\\ (1-z)^{\frac{1}{2}}, & (1-z)^{\frac{1}{2}}\end{pmatrix}\begin{pmatrix}(1-\theta)(e^{i\pi z}-e^{-i\pi z}+z)-(1+\theta)\,z\,(1-z^2)^{\frac{1}{2}}, & 0\\ 0, & (1-\theta)(e^{i\pi z}-e^{i\pi z}+z)+(1+\theta)\,z\,(1-z^2)^{\frac{1}{2}}\end{pmatrix}\begin{pmatrix}(1-z)^{\frac{1}{2}}, & (1+z)^{\frac{1}{2}}\\ -(1-z)^{\frac{1}{2}}, & (1+z)^{\frac{1}{2}}\end{pmatrix}.$$

There are simple branch-points at ±1 only, and the variable ζ, of which z and the multiplier $\theta(z)$ are uniform functions, is

$$\{(1-z)/(1+z)\}^{\frac{1}{2}}.$$

Since $\log\theta(z)$ is exponentially small and $1/(\zeta-\gamma)$ is $O(z)$, it follows that

$$\int |\log\theta(z)/(\zeta-\gamma)|\,d|\zeta|$$

is convergent, and thus

$$\int \{\log\theta(z)/(\tau-\zeta)\}\,d\zeta$$

is bounded when $\tau\to\gamma$. Therefore $\omega(t)$ is bounded when $t\to\pm i\infty$.

Let $\omega_1, \omega_2, \ldots, \omega_n$ be the values of ω in the different sheets corresponding to $\theta_1, \theta_2, \ldots, \theta_n$, and let $U_{12}(z)$ be the product of the diagonal $U_2(z)$ by the diagonal formed with

$$e^{\omega_1(z)},\ e^{\omega_2(z)},\ \ldots,\ e^{\omega_n(z)}.$$

Then on the right of the path the zeros and poles of $U_{12}(z)$ in any sheet are those of $U_2(z)$ in that sheet, but on the left those of $U_1(z)$.

Again, the constituents of KU_1L and KU_2L are uniform functions of z and the same is true of $KU_{12}L$, or indeed KVL if V is any diagonal with the same branching as U_1, U_2. For, if γ denotes for the moment the unit matrix with its r-th and s-th rows interchanged,

$$\gamma^2=1 \quad\text{and}\quad KVL=K\gamma.\gamma V\gamma.\gamma L.$$

Now $\gamma V\gamma$ is V with its r-th and s-th elements interchanged,
$K\gamma$ is K with its r-th and s-th columns interchanged,
γL is L with its r-th and s-th rows interchanged,

that is, KVL is unchanged by the interchange of the r-th and s-th sheets and is uniform in the z-plane. Similarly KL is uniform. Near a branch-point z_0, if K, L are bounded, U_1, U_2, and U_{12} will have infinite parts of the order of $(z-z_0)^{-\frac{1}{2}}$, but by the symmetry these must disappear in $KU_{12}L$, which is therefore bounded near z_0. If two of the multipliers become equal at z_0, but it is not a branch-point, $KU_{12}L$ may have a simple pole.

Towards $\pm i\infty$, $\omega(z)$ is bounded, and therefore $U_{12}(z)$, like $U_2(z)$, is of the order of $e^{\mp imz}$. There are asymptotic expressions for $\omega(z)$ of the form

$$A+\frac{B}{z}+\frac{C}{z^2}+\ldots,$$

different in different sheets. (Compare S.N., 254.)

11. *Formation of a matrix having the properties* (2.11), (2.12).

In
$$\int_{a-i\infty}^{a+i\infty}\{L(\lambda)\}^{-1}\{U_{12}(\lambda)\}^{-1}\{K(\lambda)\}^{-1}e^{\lambda z}\,d\lambda \qquad (11.1)$$

the outside matrix factors* have been assumed algebraic, and the middle

* When $n=1$ these outside factors are absent.

one is a diagonal of the order $e^{\pm i\pi\lambda}$ when $\lambda \to \pm i\infty$. Hence the integral converges absolutely and represents a matrix of regular functions of z so long as $|I(z)| < \pi$. Call it $\sigma_{12}(z)$.

When S_1 is put for S_2 it may be called $\sigma_{11}(z)$, and if $S_1 = S$, it is the function $\sigma(z)$ of § 2.

The path may be bent to the right so as to tend to infinity in the first and fourth quadrants. The region of validity of the expression so modified is a wedge opening out to the left, bounded by lines through $\pm i\pi$, and when all the arguments involved are in this wedge, that is, when z lies in a similar wedge with its apex at the origin,

$$\{e^{i\pi D} - e^{-i\pi D} - S_2(D)\}\,\sigma_{12}(z)$$
$$= \int K(\lambda)\, U_2(\lambda)\, L(\lambda) \{L(\lambda)\}^{-1} \{U_{12}(\lambda)\}^{-1} \{K(\lambda)^{-1} e^{\lambda z}\, d\lambda$$
$$= \int K(\lambda) \{U_2(\lambda)/U_{12}(\lambda)\} \{K(\lambda)\}^{-1} e^{\lambda z}\, d\lambda. \qquad (11.2)$$

Here $U_2(\lambda)/U_{12}(\lambda)$ is bounded on the right of the path and $K(\lambda)$ contains only algebraic functions: hence the subject of integration is exponentially small towards infinity on the right, z being in the wedge, and by moving the path away to infinity we have, in the left half-plane,

$$\sigma_{12}(z + i\pi) - \sigma_{12}(z - i\pi) = S_2(D)\,\sigma_{12}(z). \qquad (11.31)$$

Similarly in the right half-plane

$$\sigma_{12}(z + i\pi) - \sigma_{12}(z - i\pi) = S_1(D)\,\sigma_{12}(z), \qquad (11.32)$$

since on the left of the path $U_1(\lambda)/U_{12}(\lambda)$ is bounded. Also in the left half-plane

$$j[\{e^{i\pi D} - e^{-i\pi D} - jS_2(D)\}\, j\sigma_{12}(z)]$$
$$= \int \{L(\lambda)\}^{-1} \{U_{12}(\lambda)\}^{-1} \{K(\lambda)\}^{-1} K(\lambda)\, U_2(\lambda)\, L(\lambda)\, e^{\lambda z}\, d\lambda = 0 \qquad (11.33)$$

similarly, and the like holds in the right half-plane with S_1 for S_2.

The only singularities of $\sigma_{12}(z)$ in the strip $|I(z)| \leqslant \pi$ are at $\pm i\pi$. To discuss them divide the path at any convenient point, say a.

The part

$$\int_a^{a+i\infty} \{L(\lambda)\}^{-1} \{U_{12}(\lambda)\}^{-1} \{K(\lambda)\}^{-1} e^{\lambda z}\, d\lambda \qquad (11.41)$$

has no singularity at $i\pi$, nor has

$$\int_{a-i\infty}^{a} \{e^{-2i\pi\lambda} + e^{-i\pi\lambda} S_2(\lambda)\} \{L(\lambda)\}^{-1} \{U_{12}(\lambda)\}^{-1} \{K(\lambda)\}^{-1} e^{\lambda z}\, d\lambda, \qquad (11.42)$$

and hence the singular part of $\sigma_{12}(z)$ is that of

$$\int_{a-i\infty}^{a} \{1-e^{-2i\pi\lambda}-e^{-i\pi\lambda}S_2(\lambda)\}\{L(\lambda)\}^{-1}\{U_{12}(\lambda)\}^{-1}\{K(\lambda)\}^{-1}e^{\lambda z}\,d\lambda, \quad (11.43)$$

which is

$$\int_{a-i\infty}^{a} K(\lambda)\{U_2(\lambda)/U_{12}(\lambda)\}\{K(\lambda)\}^{-1}e^{\lambda(z-i\pi)}d\lambda. \quad (11.44)$$

Now it has been assumed that $K(\lambda)$ is algebraic and that infinity is not a branch-point. Hence $K(\lambda)$ and its reciprocal have regular expansions of the form

$$K_0+K_1/\lambda+K_2/\lambda^2+\ldots, \quad (11.5)$$

while $U_2(\lambda)/U_{12}(\lambda)$, which is the diagonal formed with the exponential functions of

$$-\omega_1(\lambda), \quad -\omega_2(\lambda), \quad \ldots,$$

has an asymptotic expansion of the same form (11.51). Hence the matrix part of the integrand in (11.44) is asymptotically represented by a series

$$M_0+M_1/\lambda+M_2/\lambda^2+\ldots$$

and the result of integration is (as at S.N., § 10)

$$M_0(z-i\pi)^{-1}+\Phi(z-i\pi)\log(z-i\pi)+\Psi(z-i\pi), \quad (11.6)$$

where Φ, Ψ are matrices of functions bounded near the origin.

On account of (11.31) or (11.32) the singularity at $-i\pi$ is that of $M_0(z+i\pi)^{-1}+\Phi(z+i\pi)\log(z+i\pi)$ and the value of $\Phi(z)$ in (11.6) is

$$\frac{1}{2i\pi}\{S_1(D)-S_2(D)\}\,\sigma_{12}(z).$$

The determinant of M_0 does not vanish since $\omega(\lambda)$ is bounded. Hence

$$\sigma_{12}(z)\,M_0^{-1}$$

is a matrix with properties similar to those of $\sigma_{12}(z)$, but the coefficient of $(z-i\pi)^{-1}$ in the expansion near $i\pi$ is the unit matrix.

By shifting the path of integration in (11.1) to the right, we find, as at (2.7), the suggestion that $\sigma_{12}(z)$ can be expanded in the form

$$A_1e^{\mu_1 z}+A_2e^{\mu_2 z}+\ldots \quad (11.7)$$

in the left half-plane. Here $\ldots, \mu_{-1}, \mu_0, \mu_1, \mu_2, \ldots$ are the zeros of the determinant of $e^{i\pi\mu}-e^{-i\pi\mu}-S_2(\mu)$ in ascending order of real parts and the original path in (11.1) separates μ_0, μ_1.

Similarly, if $\ldots, \lambda_{-1}, \lambda_0, \lambda_1, \lambda_2, \ldots$ are the zeros of the determinant of $e^{i\pi\lambda}-e^{-i\pi\lambda}-S_1(\lambda)$ and the path separates λ_0, λ_1, the suggested expansion of $\sigma_{12}(z)$ in the right half-plane is of the form

$$B_0 e^{\lambda_0 z}+B_{-1}e^{\lambda_{-1}z}+\ldots. \qquad (11.71)$$

Without discussing the validity of the infinite series (11.7), (11.71) we are entitled again to say that when $|R(z)|$ is great the error in stopping at any stage is of the order of the first term omitted.

12–15. *The matrix $F_{12}(z, u)$: its construction and properties.*

12. It would be possible to carry out the above work with S_1 and S_2 interchanged. If we used the same path of integration, the line $x = a$, we should have in each sheet the same integral $\omega(\zeta)$, but with the opposite sign. Hence

$$\log U_{21}(z)-\log U_1(z) = -\log U_{12}(z)+\log U_2(z),$$

that is
$$U_{12}(z)\times U_{21}(z) = U_1(z)\times U_2(z); \qquad (12.21)$$

K, L are not changed and therefore

$$\sigma_{21}(z) = \int_{a-i\infty}^{a+i\infty} \{L(\lambda)\}^{-1}\{U_{21}(\lambda)\}^{-1}\{K(\lambda)\}^{-1}e^{\lambda z}\,dz. \qquad (12.2)$$

Now $K(\lambda)L(\lambda)$ is a matrix of rational integral algebraic functions of λ and it is therefore possible to operate with $K(D)L(D)$. Consider then

$$\int_{-\infty}^{\infty} \sigma_{12}(t+\alpha)K\left(-\frac{\partial}{\partial u}\right)L\left(-\frac{\partial}{\partial u}\right)\sigma_{21}(-u-\alpha)\,d\alpha, \qquad (12.3)$$

where the path is the real axis and $I(t)$, $I(u)$ are both between $\pm\pi$.

Towards the upper limit the integrand is $O[\exp(\lambda_0-\lambda_1)\alpha]$ and towards the lower limit it is $O[\exp(\mu_1-\mu_0)\alpha]$, so that the integral is absolutely convergent. It may be written

$$\int_{-\infty}^{\infty}\int_{a-i\infty}^{a+i\infty}\int_{a-i\infty}^{a+i\infty} \{L(\lambda)\}^{-1}\{U_{12}(\lambda)\}^{-1}\{K(\lambda)\}^{-1}K(\mu)L(\mu)\{L(\mu)\}^{-1} \times\{U_{21}(\mu)\}^{-1}\{K(\mu)\}^{-1}e^{\lambda t-\mu u+(\lambda-\mu)\alpha}\,d\lambda\,d\mu\,d\alpha, \qquad (12.4)$$

where the α-integration is last. This reduces, as in the simpler case* $n = 1$, to

$$2i\pi\int_{a-i\infty}^{a+i\infty}\{L(\lambda)\}^{-1}\{U_{12}(\lambda)\}^{-1}\{U_{21}(\lambda)\}^{-1}\{K(\lambda)\}^{-1}e^{\lambda(t-u)}d\lambda$$

or
$$2i\pi\int_{a-i\infty}^{a+i\infty}\{L(\lambda)\}^{-1}\{U_1(\lambda)U_2(\lambda)\}^{-1}\{K(\lambda)\}^{-1}e^{\lambda(t-u)}d\lambda. \qquad (12.5)$$

* S. N. (11.31).

13. An operation $S_5(D)$ is next constructed as follows. We have

$$\{e^{i\pi\lambda}-e^{-i\pi\lambda}-S_2(\lambda)\}\{L(\lambda)\}^{-1}\{K(\lambda)\}^{-1}\{e^{i\pi\lambda}-e^{-i\pi\lambda}-S_1(\lambda)\}$$
$$= KU_2U_1L = KU_1U_2L$$
$$= \{e^{i\pi\lambda}-e^{-i\pi\lambda}-S_1(\lambda)\}\{L(\lambda)\}^{-1}\{K(\lambda)\}^{-1}\{e^{i\pi\lambda}-e^{-i\pi\lambda}-S_2(\lambda)\}. \quad (13.1)$$

Hence, with a notation as at (2.6), we may write $S_5(\lambda)$ for

$$\{e^{i\pi\lambda}-e^{-i\pi\lambda}-S_2(\lambda)\}\{L(\lambda)\}^{-1}\{K(\lambda)\}^{-1}\{e^{i\pi\lambda}-S_1^+(\lambda)\}$$
$$-\{e^{i\pi\lambda}-e^{-i\pi\lambda}-S_1(\lambda)\}\{L(\lambda)\}^{-1}\{K(\lambda)\}^{-1}\{e^{i\pi\lambda}-S_2^+(\lambda)\}, \quad (13.2)$$

or

$$\{e^{i\pi\lambda}-e^{-i\pi\lambda}-S_2(\lambda)\}\{L(\lambda)\}^{-1}\{K(\lambda)\}^{-1}\{e^{-i\pi\lambda}+S_1^-(\lambda)\}$$
$$-\{e^{i\pi\lambda}-e^{-i\pi\lambda}-S_1(\lambda)\}\{L(\lambda)\}^{-1}\{K(\lambda)\}^{-1}\{e^{-i\pi\lambda}+S_2^-(\lambda)\}, \quad (13.3)$$

since these two expressions are equal.

The first form shows that the coefficients of λ in the exponents are all in the horizontal strip between $2i\pi$ and $c'-i\pi$, the second that they are all in the strip between $c''+i\pi$ and $-2i\pi$, and hence actually they are all in the horizontal strip between $c'-i\pi$ and $c''+i\pi$. So then the operation $S_5\{-\partial/(\partial u)\}$, in the position of a post-factor, acting upon the function of u represented by (12.3) will introduce arguments of σ_{21} lying in the horizontal strip between $-u+c'-i\pi$ and $-u+c''+i\pi$, and, if u lies in the strip between c' and c'', all such arguments are in the strip between $\pm i\pi$, so that for all of them the expressions (12.3), (12.5) are valid. We have then

$$j\Big[jS_5\Big(-\frac{\partial}{\partial u}\Big)\,j\int_{-\infty}^{\infty}\sigma_{12}(t+\alpha)\,K\Big(-\frac{\partial}{\partial u}\Big)\,L\Big(-\frac{\partial}{\partial u}\Big)\,\sigma_{21}(-u-\alpha)\,d\alpha\Big]$$
$$= 2i\pi\int_{a-i\infty}^{a+i\infty}\{L(\lambda)\}^{-1}\{U_1(\lambda)\,U_2(\lambda)\}^{-1}\{K(\lambda)\}^{-1}S_5(\lambda)\,e^{\lambda(t-u)}\,d\lambda$$
$$= 2i\pi\int_{a-i\infty}^{a+i\infty}[\{e^{i\pi\lambda}-e^{-i\pi\lambda}-S_1(\lambda)\}^{-1}\{e^{i\pi\lambda}-S_1^+(\lambda)\}$$
$$-\{e^{i\pi\lambda}-e^{-i\pi\lambda}-S_2(\lambda)\}^{-1}\{e^{i\pi\lambda}-S_2^+(\lambda)\}]\,e^{\lambda(t-u)}\,d\lambda$$
$$= 2i\pi F_{11}(t,u)-2i\pi F_{22}(t,u), \quad (13.4)$$

where F_{11}, F_{22} denote the matrix F of § 2 with S_1, S_2 in the place of S. And we may write $F_{12}(z,u)$ for

$$F_{11}(z,u)-\frac{1}{2i\pi}j\Big[jS_5\Big(-\frac{\partial}{\partial u}\Big)\,j\int_0^{\infty}\sigma_{12}(z+\alpha)\,K\Big(-\frac{\partial}{\partial u}\Big)$$
$$\times L\Big(-\frac{\partial}{\partial u}\Big)\,\sigma_{21}(-u-\alpha)\,d\alpha\Big], \quad (13.51)$$

or the equivalent expression (13.52), in which the first term is $F_{22}(z, u)$ and the upper limit in the integral is $-\infty$.

The path of integration in (13.51) may be deformed so as to avoid singularities of the subject unless they occur at the lower limit. Hence in the right half-plane, except at the singularities of $F_{11}(z, u)$ and $\sigma_{12}(z)$, (13.51) is a valid representation of a matrix of regular functions of z, the matrix $F_{12}(z, u)$. This matrix has no singularity in the strip $|I(z)| \leqslant \pi$, except a pole at u with unit residue, and logarithmic terms at $\pm\pi$ (compare S.N., page 265); it satisfies (2.11) in the half-plane. The functions represented by (13.52) are the continuation of $F_{12}(z, u)$ in the left half-plane, where, again, they are generally regular and satisfy (2.12). Their properties as functions of u must also be considered.

The expression (13.51) may be transformed, since the matrix $j\sigma_{21}(-u-a)$ is reduced to zero by the operation

$$e^{-i\pi(\partial/\partial u)} - e^{i\pi(\partial/\partial u)} - jS_r\left(-\frac{\partial}{\partial u}\right),$$

where $r = 1$ in the right half-plane, $r = 2$ in the left. Hence

$$\begin{aligned} F_{12}(z, u) &= j\left[\left\{e^{-i\pi(\partial/\partial u)} - jS_r^+\left(-\frac{\partial}{\partial u}\right)\right\} jG(z, u)\right] \\ &= j\left[\left\{e^{i\pi(\partial/\partial u)} + jS_r^-\left(-\frac{\partial}{\partial u}\right)\right\} jG(z, u)\right], \end{aligned} \tag{13.6}$$

where $G(z, u)$ or $G_{12}(z, u)$ stands for

$$\sigma_{11}(z-u) - \frac{1}{2i\pi} j\left[j\left\{K\left(-\frac{\partial}{\partial u}\right) L\left(-\frac{\partial}{\partial u}\right)\right\}^{-1} j\left\{S_1\left(-\frac{\partial}{\partial u}\right) - S_2\left(-\frac{\partial}{\partial u}\right)\right\}$$

$$\times j\int_0^\infty \sigma_{12}(z+a)\, K\left(-\frac{\partial}{\partial u}\right) L\left(-\frac{\partial}{\partial u}\right) \sigma_{21}(-u-a)\, da\Big], \tag{13.61}$$

which, as a function of u, has no singularity in the horizontal strip between $\pm(i\pi - c)$, unless at $z \pm i\pi$. If $F_{12}(z, u)$ is distinguished as $F_{12}(z, u, c', c'')$ when $I(c') > I(u) > I(c'')$, then

$$F_{12}(z, u+i\pi, i\pi, c) = F_{12}(z, u-i\pi, -c, -i\pi) = G(z, u). \tag{13.7}$$

14. As a matrix of functions of u, $F_{12}(t, u)$ has singularities at $c, c', \ldots$, and, moreover, it is equal to different sets of analytical functions in the horizontal strips whose bounding lines pass through these points. It is remarkable that the difference between the two sets of analytical functions represented by $F_{12}(t, u)$ in two adjoining strips has no singularity on the bounding line, or, in other words, the singularities of

the two sets of functions on the bounding line are the same. For instance, in $F_{12}(t, u, i\pi, c)$, the singular part when u approaches c is contributed by the terms in $e^{(c+i\pi)\lambda}$ in $S_5(\lambda)$ which are

$$e^{i\pi\lambda}\{L(\lambda)\}^{-1}\{K(\lambda)\}^{-1}\{C_1(\lambda)-C_2(\lambda)\}e^{c\lambda}, \tag{14.1}$$

and the contribution of these terms in (13.51) is

$$-\frac{1}{2i\pi}j\Big[j\Big\{C_1\Big(-\frac{\partial}{\partial u}\Big)-C_2\Big(-\frac{\partial}{\partial u}\Big)\Big\}jH(u-c)\Big], \tag{14.2}$$

where

$$H(u)=j\Big\{K\Big(-\frac{\partial}{\partial u}\Big)L\Big(-\frac{\partial}{\partial u}\Big)\Big\}^{-1}$$
$$\times j\int_0^\infty \sigma_{12}(t+a)K\Big(-\frac{\partial}{\partial u}\Big)L\Big(-\frac{\partial}{\partial u}\Big)\sigma_{21}(-u+i\pi-a)\,da. \tag{14.3}$$

In $F_{12}(t, u, c, c')$ the singular part when u approaches c is contributed by the terms in $e^{(c-i\pi)\lambda}$ in $S_5(\lambda)$ which are the same as (14.1) except that the first factor is not $e^{i\pi\lambda}$ but $e^{-i\pi\lambda}$. This does not affect the singularity, so that the values of $F_{12}(t, u, c, c')-F_{12}(t, u, i\pi, c)$ are regular at c. In fact, this difference by (13.6) is

$$-j\Big[jC_r\Big(-\frac{\partial}{\partial u}\Big)jG(t, u-c)\Big], \tag{14.4}$$

where $r=1$ or 2 according as $R(u-c)$ is positive or negative. The two expressions are not the same, because the singular part, though common to the two functions, is not uniform in the neighbourhood of c.

Similarly the singular parts of $F_{12}(t, u)$ when u approaches $c', c'', \ldots$ all depend on those of $H(u)$ when u approaches 0. Now

$$H(u)=\int_0^\infty\int_{a-i\infty}^{a+i\infty}\int_{b-i\infty}^{b+i\infty}\{L(\lambda)\}^{-1}\{U_{12}(\lambda)\}^{-1}\{K(\lambda)\}^{-1}K(\mu)\{U_{21}(\mu)\}^{-1}\{K(\mu)\}^{-1}$$
$$\times e^{i\pi\mu}\{L(\mu)\}^{-1}\{K(\mu)\}^{-1}e^{\lambda t-\mu u+(\lambda-\mu)a}\,d\lambda\,d\mu\,da, \tag{14.5}$$

where the convergence is absolute and the order of integration may be changed, if t and $u-i\pi$ lie in the horizontal strip between $\pm i\pi$ and if $R(\lambda-\mu)$, that is $R(b-a)$ is negative, the λ path having been shifted slightly to the left. The failure of convergence, when $I(u)=0$, and therefore the singularity when u approaches zero, is due to the lower part of the μ integration, which may be taken as that from $a-i\infty$ to a, in which the effect of changing the factor $e^{i\pi\mu}$ into

$$e^{i\pi\mu}-e^{-i\pi\mu}-S_r(\mu) \quad \text{or} \quad K(\mu)U_r(\mu)L(\mu) \quad (r=1, 2)$$

is to add an expression without singularity at the origin. Thus the singularity of $H(u)$ at the origin is the same as that of

$$\int_{b-i\infty}^{b+i\infty}\int_{a-i\infty}^{a}\{L(\lambda)\}^{-1}\{U_{12}(\lambda)\}^{-1}\{K(\lambda)\}^{-1}K(\mu)\{U_{21}(\mu)\}^{-1} \times U_r(\mu)\{K(\mu)\}^{-1}(\mu-\lambda)^{-1}e^{\lambda t-\mu u}\,d\mu\,d\lambda, \quad (14.6)$$

the factor $(\mu-\lambda)^{-1}$ being $\int_0^\infty e^{(\lambda-\mu)a}\,da.$

The restriction upon u necessary for the validity of this expression is that μu must have a positive real part at the lower limit of the μ integration, that is to say, when the μ path is parallel to the lateral axis, u must be in the first or second quadrant. If the μ path is rotated, the boundary of the region in which u must lie rotates in the opposite sense. Thus u can be brought into the first negative quadrant if the μ path rotates counterclockwise, and this rotation may go on so long as the path does not pass over a pole of the subject. Similarly u can be brought into the third quadrant if the μ path rotates clockwise.

Now, if $r=1$, the subject of integration with respect to μ in (14.6) has no poles on the right of the λ path, and the μ path may be rotated through $+\pi$ so as to run from $a+i\infty$ to a, u being brought round into the first two negative quadrants.

Thus, when u approaches the origin from the first negative quadrant, the expression for $H(u)$ is (14.5), with the substitution of

$$e^{i\pi\mu}-K(\mu)\,U_1(\mu)\,L(\mu)$$

for the factor $e^{i\pi\mu}$.

If $r=2$ the subject in the μ integration has no pole on the left of the λ path, but it has a pole at λ; thus the μ path may be rotated clockwise through the left half-plane if we make the proper allowance for the pole at λ and also rotate the boundary of the region in which u may lie counterclockwise, so that u is admitted to the third and fourth quadrants.

The quantity to be added when the μ path passes over λ is

$$2i\pi\int_{b-i\infty}^{b+i\infty}\{L(\lambda)\}^{-1}\{U_1(\lambda)\}^{-1}\{K(\lambda)\}^{-1}e^{\lambda(t-u)}\,d\lambda$$

or $2i\pi\sigma_{11}(t-u)$, and thus, when u approaches the origin from the third or fourth quadrant, the expression (14.5), with

$$e^{i\pi\mu}-K(\mu)\,U_2(\mu)\,L(\mu)$$

written for the factor $e^{i\pi\mu}$, represents

$$H(u)-2i\pi\sigma_{11}(t-u).$$

By comparing the expression in the fourth quadrant with that in the first negative quadrant we have, near the origin,

$$H(ue^{2i\pi})-H(u)$$
$$= 2i\pi\sigma_{11}(t-u)+\int_0^\infty\int_{a-i\infty}^{a+i\infty}\int_{b-i\infty}^{b+i\infty}\{L(\lambda)\}^{-1}\{U_{12}(\lambda)\}^{-1}\{K(\lambda)\}^{-1}K(\mu)$$
$$\times\{U_{21}(\mu)\}^{-1}\{U_1(\mu)-U_2(\mu)\}\{K(\mu)\}^{-1}e^{\lambda t-\mu u+(\lambda-\mu)a}\,d\lambda\,d\mu\,da$$
$$= 2i\pi G(t,u), \text{ by (13.61).}$$

Hence $H(u)-G(t,u)\log u$ is uniform when u is near the origin.

It is also bounded and therefore regular. For, in (14.6), take $r=1$, and let the μ path be rotated counterclockwise until it is parallel to the real axis. Then we may put

$$\frac{1}{\mu-\lambda}=\frac{1}{\mu}+\frac{\lambda}{\mu(\mu-\lambda)},$$

where the second term is of the order of $\lambda\mu^{-2}$ when μ is great, wherever λ may be on its path.

The factor $K(\mu)\{U_{21}(\mu)\}^{-1}U_1(\mu)\{K(\mu)\}^{-1}$ can also be expanded in the form

$$A+\frac{B}{\mu}+\frac{C}{\mu^2}+\ldots,$$

by (11.5) and (11.51). Thus, in (14.6), the principal part, when u is small, is a multiple of

$$\int^\infty\frac{e^{-\mu u}}{\mu}\,d\mu,$$

that is, of $\log u$.

Hence* $H(u)-G(t,u)\log u$ is at most of the order of $\log u$, and must therefore be bounded, since it is uniform.

15. From the form (13.51) by substitution from (11.71) it follows that there is an expansion of the form (11.71) in which, when $R(z)$ is great and positive, the sum of any number of terms at the beginning differs from $F_{12}(z,u)$ by a quantity of the order of the first term omitted, and from (13.52) there is an expansion of the form (11.7) for which the same holds when $R(z)$ is great and negative.

* At (14.5) the domain of t was restricted to the horizontal strip between $\pm i\pi$, but if the λ-path is bent to the left the domain of t becomes a wedge opening out to the right, so that $|I(t)|$ may exceed π if $R(t)$ is positive; by the equation (2.11) the domain may then be extended over the right half-plane. If the form (13.52) is used instead of (13.51) and the λ-path is bent to the right, the domain of t may similarly be extended over the left half-plane.

The expansions having similar properties when $R(u)$ is great are of the forms

$$A_1 e^{-\lambda_1 u} + A_2 e^{-\lambda_2 u} + \dots \quad \text{when } R(u) \text{ is positive,}$$

$$B_0 e^{-\mu_0 u} + B_1 e^{-\mu_1 u} + \dots \quad \text{when } R(u) \text{ is negative.}$$

16. *An important theorem deduced by contour integration.*

Let $f(u)$ denote a matrix of functions of u, regular in the whole strip $|I(u)| \leqslant \pi$, except that they may have poles at points where Iu is not equal to π, Ic, Ic', ..., $-\pi$, and that near $\pm i\pi$ they may contain products of $\log(z \mp i\pi)$ by regular functions. Consider the integral

$$\int F(t, u, c, c') f(u)\, du$$

taken round the parallelogram whose vertices are

$$c'-\beta, \quad c'+\alpha, \quad c+\alpha, \quad c-\beta,$$

where α, β are real and generally positive; when both α and β tend to $+\infty$ this parallelogram becomes the strip $I(c) > I(u) > I(c')$. The points c, c' themselves are to be excluded by means of small arcs.

Within the contour the functions f are regular except for poles, say at k, ..., with residues (matrices) A, ..., and thus, if t lies within the contour,

$$\left[\int_{c-\beta}^{c+\alpha} - \int_{c'-\beta}^{c'+\alpha} - \int_{c'+\alpha}^{c+\alpha} + \int_{c'-\beta}^{c-\beta}\right] F(t, u, c, c') f(u)\, du = 2i\pi f(t) - 2i\pi \Sigma F(t, k, c, c') A. \quad (16.1)$$

If $|I(t)| < \pi$, but t does not lie within the contour, the first term on the right is absent. Let (16.1) be summed over the different strips bounded by parallels to the real axis through $i\pi$, c, c', ..., $-c$, $-i\pi$, it being supposed that t lies inside one of these strips. Thus

$$\begin{aligned} 2i\pi f(t) - 2i\pi \Sigma\Sigma F(t, k) A + \int_{-i\pi+\alpha}^{i\pi+\alpha} F(t, u) f(u)\, du - \int_{-i\pi-\beta}^{i\pi-\beta} F(t, u) f(u)\, du \\ = \int_{i\pi-\beta}^{i\pi+\alpha} F(t, u, i\pi, c) f(u)\, du - \int_{c-\beta}^{c+\alpha} F(t, u, i\pi, c) f(u)\, du \\ + \int_{c-\beta}^{c+\alpha} F(t, u, c, c') f(u)\, du - \dots \\ - \int_{-i\pi-\beta}^{-i\pi+\alpha} F(t, u, -c, -i\pi) f(u)\, du. \quad (16.2) \end{aligned}$$

The second term on the left in (16.2) involves two summations, one over the strips, the other over the different poles of f in any strip, that is a single summation over all the poles of f in the whole strip $|I(u)| < \pi$.

In the third term on the left the path from $-i\pi+a$ to $i\pi+a$ goes by way of $-c+a$, $-c'+a$, ..., $c+a$, and F has different meanings on the different segments; similarly in the fourth term.

In the second term on the right

$$F(t, u, i\pi, c) = Z(u)+\Phi(u), \tag{16.31}$$

and in the third term on the right

$$F(t, u, c, c') = Z(u)+\Psi(u), \tag{16.32}$$

where

$$Z(u) = -\frac{1}{2i\pi} j\left[j\left\{C_1\left(-\frac{\partial}{\partial u}\right)-C_2\left(-\frac{\partial}{\partial u}\right)\right\}\{jG(t, u-c)\log(u-c)\}\right],$$

and $\Phi(u)$, $\Psi(u)$ are regular when u is near c.

For definiteness we shall take $\log(u-c)$ and therefore $Z(u)$ to have the same meanings in (16.31) and (16.32) when $u-c$ is real and negative. Then the second and third terms on the right in (16.2) are together equal to

$$\int_{-\beta}^{a}[\Psi(u+c)-\Phi(u+c)]f(u+c)\,du+\int_{a}^{(0+)} Z(u+c)f(u+c)\,du. \tag{16.4}$$

In the first part of (16.4) the path need not avoid the origin, since the integrand is regular; also, since, when u is negative, by (14.4),

$$\begin{aligned}\Psi(u+c)-\Phi(u+c) &= F(t, u+c, c, c')-F(t, u+c, i\pi, c)\\ &= -j\left\{jC_2\left(-\frac{\partial}{\partial u}\right).jG(t, u)\right\},\end{aligned} \tag{16.41}$$

(16.4) is

$$\begin{aligned}&-j\int_{-\beta}^{a} jf(u+c).jC_2\left(-\frac{\partial}{\partial u}\right).jG(t, u)\,du\\ &+\frac{1}{2i\pi}j\int_{a}^{(0+)} jf(u+c).j\left\{C_2\left(-\frac{\partial}{\partial u}\right)-C_1\left(-\frac{\partial}{\partial u}\right)\right\}\{jG(t, u)\log u\}\,du.\end{aligned} \tag{16.42}$$

By the lemma of § 3 this becomes

$$\begin{aligned}&-\int_{-\beta}^{a} G(t, u)\,C_2\left(\frac{d}{du}\right)f(u+c)\,du\\ &\qquad+\frac{1}{2i\pi}\int_{a}^{(0+)} G(t, u)\log u\left\{C_2\left(\frac{d}{du}\right)-C_1\left(\frac{d}{du}\right)\right\}f(u+c)\,du\\ &+\Omega_2(t, a, c)-\Omega_2(t, -\beta, c)-\frac{1}{2i\pi}\{W_2(ae^{2i\pi}, c)-W_1(ae^{2i\pi}, c)\\ &\qquad-W_2(a, c)+W_1(a, c)\},\end{aligned} \tag{16.43}$$

where

$$\frac{\partial}{\partial u} W_r(u, c) = G(t, u) \log u C_r\left(\frac{d}{du}\right) f(u+c)$$
$$-j\left[jf(u+c).jC_r\left(-\frac{\partial}{\partial u}\right) j\{G(t, u)\log u\}\right], \quad (16.5)$$

and
$$2i\pi\Omega_r(t, u, c) = W_r(ue^{2i\pi}, c) - W_r(u, c), \quad (16.51)$$

so that

$$\frac{\partial}{\partial u}\Omega_r(t, u, c) = G(t, u)\, C_r\left(\frac{d}{du}\right) f(u+c)$$
$$-j\left[jf(u+c).jC_r\left(-\frac{\partial}{\partial u}\right).jG(t, u)\right]. \quad (16.52)$$

In the second part of (16.43) the small circuit about the origin may be reduced to a point, since the integrand is only of the order of $\log u$, and the loop integral thus becomes

$$2i\pi\int_0^\alpha G(t, u)\left\{C_2\left(\frac{d}{du}\right) - C_1\left(\frac{d}{du}\right)\right\} f(u+c)\, du. \quad (16.6)$$

Thus, in (16.2), the second and third terms on the right are together equal to

$$-\int_{-\beta}^0 G(t, u)\, C_2\left(\frac{d}{du}\right) f(u+c)\, du$$
$$-\int_0^\alpha G(t, u)\, C_1\left(\frac{d}{du}\right) f(u+c)\, du + \Omega_1(t, \alpha, c) - \Omega_2(t, -\beta, c). \quad (16.7)$$

If there are other such pairs they may be treated similarly. In the first term on the right in (16.2) put $u+i\pi$ for u, and in the last put $u-i\pi$ for u. Thus, on the whole,

$$f(t) - \Sigma F(t, k)\, A$$
$$= \frac{1}{2i\pi}\int_{-\beta}^0 G(t, u)\left\{f(u+i\pi) - f(u-i\pi) - S_2\left(\frac{d}{du}\right) f(u)\right\} du$$
$$+ \frac{1}{2i\pi}\int_0^\alpha G(t, u)\left\{f(u+i\pi) - f(u-i\pi) - S_1\left(\frac{d}{du}\right) f(u)\right\} du$$
$$+ \frac{1}{2i\pi}\Sigma\Omega_1(t, \alpha, c) - \frac{1}{2i\pi}\Sigma\Omega_2(t, -\beta, c)$$
$$- \frac{1}{2i\pi}\int_{-i\pi+\alpha}^{i\pi+\alpha} F(t, u) f(u)\, du + \frac{1}{2i\pi}\int_{-i\pi-\beta}^{i\pi-\beta} F(t, u) f(u)\, du. \quad (16.8)$$

Here $\Omega_1(t, u, c)$ and $\Omega_2(t, u, c)$ denote homogeneous bilinear expressions in $f(u+c)$ and its derivatives on the one hand, $G(t, u)$ and its derivatives with respect to u on the other.

In the foregoing work the values $i\pi$, c, c', ..., $-i\pi$ for t have been excluded. The positions $\pm i\pi$ cannot well be allowed, since at those points $F(t, u)$ is infinite; but c, c', ... need not be excluded. If, for instance, $t = c$, the common singular part $Z(u)$ in (16 . 31) and (16 . 32) is increased by the addition of $(c-u)^{-1}$ from the term $F_{11}(z, u)$ in (13 . 51). Thus the loop integral in (16 . 4) is increased by $-2i\pi f(c)$, which compensates for the term $2i\pi f(t)$ on the left, lost since t is not within the contour of integration.

Other cases when t lies on the paths of integration may be met by deformation of one of the paths in (16 . 8). If this affects the first or second integral, it involves corresponding deformations in the first and second terms on the left of (16 . 1), and in each term on the right of (16 . 2).

The values excluded for t were also excluded for k, and it was further supposed that α, β were positive. We may, however, take α negative, calling it $-\beta'$, and then the work is simplified by the absence of the loop integral in (16 . 4) and the W terms in (16 . 43). The right-hand side of the result is

$$\frac{1}{2i\pi}\int_{-\beta}^{-\beta'} G(t, u)\left\{f(u+i\pi)-f(u-i\pi)-S_2\left(\frac{d}{du}\right)f(u)\right\}du+\frac{1}{2i\pi}\Sigma\Omega_2(t, -\beta', c)$$

$$-\frac{1}{2i\pi}\Sigma\Omega_2(t, -\beta, c)-\frac{1}{2i\pi}\int_{-i\pi-\beta'}^{i\pi-\beta'} F(t, u)f(u)\,du$$

$$+\frac{1}{2i\pi}\int_{-i\pi-\beta}^{i\pi-\beta} F(t, u)f(u)\,du, \quad (16 . 81)$$

and on the left side only terms are to be included which arise from singularities of $F(t, u)f(u)$ within the contours, so that the left side is zero if t and all the points k lie outside, even on the lines

$$I(z) = \pi, \ I(c), \ \ldots, \ -\pi,$$

unless $t = \pm i\pi$.

Hence, if on the negative real axis on the left of $-\beta_1$,

$$f(u+i\pi)-f(u-i\pi)-S_2\left(\frac{d}{du}\right)f(u) = 0,$$

and if there are no singularities of f on the left of the path from

$-i\pi-\beta_1$ to $i\pi-\beta_1$, and t is not on the left of the path, the expression

$$\int_{-i\pi-\beta}^{i\pi-\beta} F(t, u) f(u)\, du - \Sigma\Omega_2(t, -\beta, c) \qquad (16.82)$$

is independent of β so long as $\beta > \beta_1$.

Here $f(u)$ may be $F(u, k)A$ if k is not on the left of the path from $-i\pi-\beta_1$ to $i\pi-\beta_1$, and since $F(t, u)F(u, k)$ and its derivatives are $O(e^{(\mu_1-\mu_0)u})$ when $R(u)$ is great and negative, β may be increased without limit and the value of (16.82) is zero.

Thus, in (16.8), when β exceeds β_1, the terms depending on β, namely

$$\frac{1}{2i\pi}\int_{-\beta}^{-\beta_1} G(t, u)\left\{f(u+i\pi)-f(u-i\pi)-S_2\left(\frac{d}{du}\right)f(u)\right\}du$$

$$-\frac{1}{2i\pi}\Sigma\Omega_2(t, -\beta, c)+\frac{1}{2i\pi}\int_{-i\pi-\beta}^{i\pi-\beta} F(t, u) f(u)\, du, \quad (16.83)$$

are unaltered by the substitution of $f(t)-\Sigma F(t, k)A$ for $f(t)$, even when $I(k)=\pi$, $I(c)$, ..., $-\pi$.

The same holds for the terms in (16.8) depending on α, namely

$$\frac{1}{2i\pi}\int_{\alpha_1}^{\alpha} G(t, u)\left\{f(u+i\pi)-f(u-i\pi)-S_1\left(\frac{d}{du}\right)f(u)\right\}du$$

$$+\frac{1}{2i\pi}\Sigma\Omega_1(t, \alpha, c)-\frac{1}{2i\pi}\int_{-i\pi+\alpha}^{i\pi+\alpha} F(t, u) f(u)\, du,$$

when α exceeds a certain positive value α_1.

Hence, even when $I(k)=\pi$, $I(c)$, ..., $-\pi$, the result (16.8) still holds if, in the integrals along the real axis, $f(u)$ is replaced by

$$f(u)-\Sigma F(u, k)A.$$

This does not alter the integrand in general; it only removes certain terms which are generally zero but are infinite at isolated points. The result therefore holds in this slightly modified form for all positions of the poles k in the strip $\pi > I(z) \geqq -\pi$ or $\pi \geqq I(z) > -\pi$, but it should be noted that the singularities of f on one of the bounding lines $I(z) = \pm\pi$ must be consistent with the conditions (a), (b) of § 2 if those on the other bounding line and those within the strip are assigned.

17. *The case* $S_1 = S_2$.

The general result (16.8) leads to many important particular ones. For instance, take $S_1 = S_2 = S$, so that $F(z, u)$ is the simpler

matrix of § 2 and $G(z, u) = \sigma(z-u)$. Then

$$f(t) = \Sigma F(t, k) A + \frac{1}{2i\pi} \int_{-\beta}^{\alpha} \sigma(t-u) \left\{ f(u+i\pi) - f(u-i\pi) - S\left(\frac{d}{du}\right) f(u) \right\} du$$
$$- \frac{1}{2i\pi} \int_{-i\pi+\alpha}^{i\pi+\alpha} F(t, u) f(u)\, du + \frac{1}{2i\pi} \Sigma\Omega(t, \alpha, c)$$
$$+ \frac{1}{2i\pi} \int_{-i\pi-\beta}^{i\pi-\beta} F(t, u) f(u)\, du - \frac{1}{2i\pi} \Sigma\Omega(t, -\beta, c). \quad (17.1)$$

A particular case of (17 . 1), in which the matrices are of the first order and S is linear, is included in equation (6 . 3) on page 279 of *Proc. London Math. Soc.* (2), 21 (1921), and here, as there, conclusions may be drawn concerning a matrix or row f such that

$$f(z+i\pi) - f(z-i\pi) = S(D) f(z). \quad (17.2)$$

Let us regard $f(z)$ as having a singularity at infinity if, when $R(z) \to +\infty$, $f(z) e^{-\lambda_1 z}$ with all its derivatives to the finite orders involved does not tend to zero uniformly in the strip $|I(z)| \leqslant \pi$, or if the same failure occurs with $f(z) e^{-\lambda_0 z}$ when $R(z) \to -\infty$.

Then, by increasing α, β without limit in (17 . 1), we prove that a matrix or row must vanish if it satisfies (17 . 2) and has no singularity in a strip of width 2π bounded by parallels to the real axis.

Again, if the functions f are regular towards infinity on the right, so that a value of β can be taken, perhaps a negative value, such that there is in the strip no singularity of f on the right of the path from $-i\pi-\beta$ to $i\pi-\beta$, then the right side of (17 . 1) may, by increasing α, be reduced to its last two terms, so that there is an expansion of the form (2 . 71) or (11 . 71) in which, when $R(z)$ is positive and great, the sum of any number of terms at the beginning differs from $f(z)$ by a quantity of the order of the first term omitted. There is a similar result for functions regular towards infinity on the left, the expansion being of the form (2 . 7).

These results include the statements about $F_{12}(z, u)$, as a function of z, made in the first paragraph of § 15.

The possible forms of singularity at infinity may also be investigated. The proof of (17 . 1) is included in that of (16 . 8), but is simpler because $S_1 = S_2$. Moreover, it applies if, in (13 . 6), (16 . 41), ..., (16 . 8), we substitute for $G(z, u)$, not simply $\sigma(z-u)$, but

$$\sigma(z-u) - \sum_{r=1}^{m} Q(\lambda_r)\, a_r\, e^{\lambda_r (z-u)}, \quad (17.3)$$

where

$$a_1 e^{\lambda_1 z} + a_2 e^{\lambda_2 z} + \ldots + a_m e^{\lambda_m z} \quad (17.4)$$

are the first m terms in the expansion (2 . 7) for $\tau(z)$, so that now, when

$R(u)$ is great and positive, the functions substituted for $G(z, u)$ are of the order of $e^{-\lambda_{m+1}u}$.

If, then, $f(z)$ has a singularity towards infinity on the right, but is yet such that $f(z)e^{-\lambda_{m+1}z}$ and its derivatives tend to zero uniformly in the strip, the right side of (17.1) again reduces to its last two terms, provided that there are no other singularities to the right of the path from $-i\pi-\beta$ to $i\pi-\beta$. Hence the part of $f(z)$ which is singular towards infinity on the right consists of postmultiples of

$$Q(\lambda_1)e^{\lambda_1 z},\quad Q(\lambda_2)e^{\lambda_2 z},\quad \ldots,\quad Q(\lambda_m)e^{\lambda_m z}. \tag{17.5}$$

Such a row of functions may be considered as having an ordinary singularity at infinity on the right. Similarly, if towards infinity on the left $f(z)e^{-\lambda_{-m}z}$ and its derivatives tend to zero uniformly in the strip, the singular part of $f(z)$ towards infinity on the left consists of postmultiples of

$$Q(\lambda_0)e^{\lambda_0 z},\quad Q(\lambda_{-1})e^{\lambda_{-1}z},\quad \ldots,\quad Q(\lambda_{1-m})e^{\lambda_{1-m}z}, \tag{17.6}$$

and this may also be regarded as an ordinary singularity for a row of functions satisfying (17.2).

18–19. *The case* $S_1 \neq S_2$.

18. The discussion of singularities at infinity on the right, leading up to (17.5), applies to rows satisfying the conditions (a), (b) of § 2, if we suppose S to be S_1 and take β negative, so that anything in the left half-plane is to the left of the path from $-i\pi-\beta$ to $i\pi-\beta$ in (17.1).

Thus a row $f(z)$ satisfying the conditions (a), (b) of § 2 is considered to have a singularity at $+\infty$ or $-\infty$ as the case may be if, when $R(z)\to+\infty$, $f(z)e^{-\lambda_1 z}$ and $f(-z)e^{\mu_0 z}$ with their derivatives do not tend to zero uniformly in the strip $|I(z)|\leqslant\pi$; these singularities are ordinary if $f(z)e^{-\lambda_{m+1}z}$ and $f(-z)e^{\mu_{-p}z}$ with their derivatives do tend to zero uniformly in the strip, and then there are expansions in postmultiples of

$Q_1(\lambda_m)e^{\lambda_m z},\ Q_1(\lambda_{m-1})e^{\lambda_{m-1}z},\ \ldots,\ Q_1(\lambda_1)e^{\lambda_1 z},\ Q_1(\lambda_0)e^{\lambda_0 z},\ \ldots$ on the right,

$Q_2(\mu_{1-p})e^{\mu_{1-p}z},\ Q_2(\mu_{2-p})e^{\mu_{2-p}z},\ \ldots,\ Q_2(\mu_0)e^{\mu_0 z},\ Q_2(\mu_1)e^{\mu_1 z},\ \ldots$ on the left,

such that the sum of any number of terms at the beginning differs from $f(z)$ by quantities of the order of the first term omitted.

19. Then on the right side of (16.8) the first two parts vanish if $f(z)$ satisfies the conditions (a), (b) of § 2; the rest tends to zero as α, β increase if $f(z)$ has no singularity at infinity; the second term on the left

vanishes if there are no singularities at finite points in the strip. Hence, if $f(z)$ satisfies the functional equation (2.11) in the right half-plane, and (2.12) in the left half-plane, and has no singularity in the strip $|I(z)| \leqslant \pi$, it must vanish. The logarithmic terms mentioned at the opening of § 16 are not to be regarded as singular.

In particular

$$\sigma_{12}(z) = G(z, 0)\, M_0, \tag{19.0}$$

since their difference has no singularities in the strip.

Rows of functions having prescribed ordinary singularities at infinity and none at finite points can be constructed as follows.

In (16.8) let

$$f(z) = Be^{\lambda z} \tag{19.10}$$

where

$$\{e^{i\pi\lambda} - e^{-i\pi\lambda} - S_1(\lambda)\}\, B = 0, \tag{19.11}$$

so that $\Delta_1(\lambda) = 0$ and B is a multiple of one of the columns of $Q_1(\lambda)$.

Then the residues A are zero and

$$\begin{aligned} 2i\pi Be^{\lambda t} = \int_{-\beta}^{0} G(t, u)\{S_1(\lambda) - S_2(\lambda)\}\, Be^{\lambda u}\, du \\ - \int_{-i\pi+\alpha}^{i\pi+\alpha} F(t, u)\, Be^{\lambda u}\, du + \Sigma\omega_1(t, \alpha, c) \\ + \int_{-i\pi-\beta}^{i\pi-\beta} F(t, u)\, Be^{\lambda u}\, du - \Sigma\omega_2(t, -\beta, c), \end{aligned} \tag{19.2}$$

where ω_1, ω_2 are what Ω_1, Ω_2 become when $f(z) = Be^{\lambda z}$; $\omega_1(t, u, c)$ and $\omega_2(t, u, c)$ are linear in $G(t, u)$ and its derivatives with respect to u.

If $R(\lambda - \mu_0)$ is positive we may increase β without limit, and thus, when $\lambda = \lambda_1, \lambda_2, \ldots$,

$$\begin{aligned} 2i\pi Be^{\lambda z} - \int_{-\infty}^{0} G(z, u)\{S_1(\lambda) - S_2(\lambda)\}\, Be^{\lambda u}\, du \\ = -\int_{-i\pi+\alpha}^{i\pi+\alpha} F(z, u)\, Be^{\lambda u}\, du + \Sigma\omega_1(z, \alpha, c). \end{aligned} \tag{19.3}$$

The two sides of (19.3) represent the same row of functions, and this row satisfies the conditions (a), (b) of § 2, but has no singularity in the strip except that of $2i\pi Be^{\lambda z}$ at infinity on the right.

A like construction can be given for a singularity at infinity on the left.

Thus a row of functions may be constructed satisfying the conditions (a), (b) of § 2 and having any assigned ordinary singularities, if this phrase includes a finite number of poles of finite orders in the strip.

$-\pi < I(z) \leqslant \pi$ and also what have been called ordinary singularities at infinity : the logarithmic terms at $i\pi$ are not to be counted as singular.

20–22. *The integral equations.*

20. In (16 . 8) increase a, β without limit, and take f to be such that the last four terms on the right tend to zero. Then

$$f(z) = \Sigma F(z, k) A + \frac{1}{2i\pi}\int_0^\infty G(z, u)\left\{f(u+i\pi) - f(u-i\pi) - S_1\left(\frac{d}{du}\right) f(u)\right\} du$$
$$+ \frac{1}{2i\pi}\int_{-\infty}^0 G(z, u)\left\{f(u+i\pi) - f(u-i\pi) - S_2\left(\frac{d}{du}\right) f(u)\right\} du. \qquad (20 . 1)$$

For instance, let S_3, S_4 be two other matrices which could take the places of S_1, S_2, the corresponding matrix of functions $G(z, u)$ being distinguished as $G_{34}(z, u)$; suppose that a (§ 10) is the same for S_3, S_4 as for S_1, S_2, so that, if

$$\ldots, \lambda_0', \lambda_1', \lambda_2', \ldots \text{ are the zeros of } S_3$$

and
$$\ldots, \mu_0', \mu_1', \mu_2', \ldots \text{ are the zeros of } S_4,$$

both in ascending order of real parts, the path $x = a$ separates λ_0' from λ_1' and μ_0' from μ_1' and $R(\lambda_1 - \lambda_0')$, $R(\mu_1' - \mu_0)$ are positive. Then such products as $G_{12}(z, u) G_{34}(u, v)$ are exponentially small towards infinity both on the right and the left (§ 15), and $f(z)$ in (20 . 1) may be taken to be $G_{34}(z, v)$; k is $v \pm i\pi$ and A is 1, so that, if v is near the real axis, the first term on the right in (20 . 1) is $G_{12}(z, v)$. Thus*

$$G_{34}(z, v) - G_{12}(z, v) = \frac{1}{2i\pi}\int_0^\infty G_{12}(z, u)\, \mathfrak{S}_3\left(\frac{\partial}{\partial u}\right) G_{34}(u, v)\, du$$
$$+ \frac{1}{2i\pi}\int_{-\infty}^0 G_{12}(z, u)\, \mathfrak{S}_4\left(\frac{\partial}{\partial u}\right) G_{34}(u, v)\, du, \qquad (20 . 2)$$

where $\mathfrak{S}_3 = -\mathfrak{S}_1 = S_3 - S_1$, $\mathfrak{S}_4 = -\mathfrak{S}_2 = S_4 - S_2$.

Thus if, when z is real and positive, we put

$$K_1(z, v) = \frac{1}{2i\pi}\mathfrak{S}_1(D)\, G_{12}(z, v), \quad K_3(z, v) = \frac{1}{2i\pi}\mathfrak{S}_3(D)\, G_{34}(z, v), \qquad (20 . 31)$$

but when z is real and negative

$$K_1(z, v) = \frac{1}{2i\pi}\mathfrak{S}_2(D)\, G_{12}(z, v), \quad K_3(z, v) = \frac{1}{2i\pi}\mathfrak{S}_4(D)\, G_{34}(z, v), \qquad (20 . 32)$$

* This includes (13.2) of S.N. by taking $n = 1$, $S_1 = S_2 = 0$: the matrices there reduce to single functions.

we have

$$K_1(t, v)+K_3(t, v) = -\int_{-\infty}^{\infty} K_1(t, u)\, K_3(u, v)\, du, \qquad (20.41)$$

and, similarly, by the interchanges of 1, 3, and of 2, 4,

$$K_1(t, v)+K_3(t, v) = -\int_{-\infty}^{\infty} K_3(t, u)\, K_1(u, v)\, du, \qquad (20.42)$$

In (20.41) and (20.42) the path of u is the real axis and t, v are in the neighbourhood of that axis, so that G_{12}, G_{34} are regular.

21. Hence each of the matrices K_1, K_3 is a solving nucleus for the other. There are four sets of integral equations

$$\phi(t)+\int_{-\infty}^{\infty} K_1(t, u)\, \phi(u)\, du = \psi(t), \qquad (21.1)$$

$$\psi(t)+\int_{-\infty}^{\infty} K_3(t, u)\, \psi(u)\, du = \phi(t), \qquad (21.2)$$

$$\phi(t)+\int_{-\infty}^{\infty} \phi(u)\, K_1(u, t)\, du = \psi(t), \qquad (21.3)$$

$$\psi(t)+\int_{-\infty}^{\infty} \psi(u)\, K_3(u, t)\, du = \phi(t), \qquad (21.4)$$

and of these (21.1) follows from (21.2) if the order of integration may be changed in

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_1(t, u)\, K_3(u, v)\, \psi(v)\, dv\, du\,; \qquad (21.11)$$

(21.2) follows from (21.1) if the order of integration may be changed in

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_3(t, u)\, K_1(u, v)\, \phi(v)\, dv\, du\,; \qquad (21.21)$$

(21.3) follows from (21.4) if the order of integration may be changed in

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi(u)\, K_3(u, v)\, K_1(v, t)\, du\, dv\,; \qquad (21.31)$$

(21.4) follows from (21.3) if the order of integration may be changed in

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(u)\, K_1(u, v)\, K_3(v, t)\, du\, dv.* \qquad (21.41)$$

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* If $S_1 = S_2 = 0$, we may have $0 < R(a) < 1$ so that $\lambda_r = r = \mu_r$ in § 11. Then $G_{12}(z, v)$ is $e^z/(e^z + e^v)$ and the two values of $K_1(z, v)$ are matrices of rational algebraic functions of e^{v-z}, in which each numerator is of lower degree than its denominator. Hence if the variables in (21.3) are taken to be e^u and e^t, the nuclei are matrices of rational algebraic fractions, homogeneous of degree -1, and, as in § 1, if $ke^u - le^t$ is a factor in a denominator, then k, l must not vanish nor must k/l be real and positive.

22. As a verification of the results it is possible to deduce the properties of a solving nucleus for such a system as (21.1) directly from the integral equations of that system; this was, in fact, the original argument of my former paper (S.N., § 1, page 234).

In (21.1) take

$$\psi(t) = K_1(t, v), \tag{22.1}$$

and suppose that

$$\int_{-\infty}^{\infty} G_{12}(z, t)\,\phi(t)\,dt = G_{12}(z, v) - 2i\pi\theta(z) \tag{22.2}$$

so that

$$\phi(z) = \mathfrak{S}_1(D)\,\theta(z) \text{ when } z \text{ is real and positive,}$$

$$= \mathfrak{S}_2(D)\,\theta(z) \text{ when } z \text{ is real and negative.}$$

Then

$$\theta(z) = \frac{1}{2i\pi} G_{12}(z, v) - \frac{1}{2i\pi}\int_0^{\infty} G_{12}(z, t)\,\mathfrak{S}_1\left(\frac{d}{dt}\right)\theta(t)\,dt$$

$$-\frac{1}{2i\pi}\int_{-\infty}^{0} G_{12}(z, t)\,\mathfrak{S}_2\left(\frac{d}{dt}\right)\theta(t)\,dt. \tag{22.3}$$

The right side of (22.3) may be expected to represent a row of functions of z, regular except for the pole at $v \pm i\pi$, so long as z does not move far enough from the real axis to bring a pole of $G_{12}(z, t)$ on to the path of integration, that is, so long as $|I(z)| < \pi$.

If z crosses the line $I(z) = \pi$ where $R(z)$ is positive, then the pole $z - i\pi$ crosses the path, in the first integral in (22.3), from below to above, and the right side will no longer be equal to $\theta(z)$, but to

$$\theta(z) + \mathfrak{S}_1(D)\,\theta(z - i\pi).$$

Putting $z + i\pi$ for z, we have then, just above the real axis on the right,

$$\theta(z+i\pi) + \mathfrak{S}_1(D)\,\theta(z) = \frac{1}{2i\pi} G_{12}(z+i\pi, v)$$

$$-\frac{1}{2i\pi}\int_0^{\infty} G_{12}(z+i\pi, t)\,\mathfrak{S}_1\left(\frac{d}{dt}\right)\theta(t)\,dt$$

$$-\frac{1}{2i\pi}\int_{-\infty}^{0} G_{12}(z+i\pi, t)\,\mathfrak{S}_2\left(\frac{d}{dt}\right)\theta(t)\,dt. \tag{22.4}$$

In each part of the right-hand expression we may use the relation

$$G_{12}(z+i\pi, u) = G_{12}(z-i\pi, u) + S_1(D)\,G_{12}(z, u), \tag{22.5}$$

since z is in the right half-plane, and then, since all the first arguments in the expressions on the right are in the strip $|I(z)|<\pi$, we may use (22.3) for each; hence

$$\theta(z+i\pi)+\mathfrak{S}_1(D)\,\theta(z) = \theta(z-i\pi)+S_1(D)\,\theta(z)$$

or
$$\theta(z+i\pi)-\theta(z-i\pi) = S_3(D)\,\theta(z). \qquad (22.61)$$

This is to hold when z is near the real positive axis, and in the right half-plane by continuation. Similarly in the left half-plane

$$\theta(z+i\pi)-\theta(z-i\pi) = S_4(D)\,\theta(z). \qquad (22.62)$$

The relations (22.61) and (22.62) and the pole at $v\pm i\pi$ with residue 1, which $2i\pi\theta(z)$ must have from (22.3), are the characteristic properties of $G_{34}(z, v)$, and we are thus brought back to K_3 as the solving nucleus of (21.1).

If the solution of (21.1) is not unique there is a row ϕ satisfying the system (21.1) when ψ is zero, and to such a row the above reasoning applies, except that the first term on the right is absent in (22.2), (22.3), and (22.4). The row $\theta(z)$ has then no singularity at a finite distance in the strip, and it must be such that the integrals in (22.3) are convergent. Hence its singularity must be an ordinary one at infinity, as in § 18, and it must be one of the rows constructed in § 19, with the substitution of S_3, S_4 for S_1, S_2. The rows so constructed are of the orders of

$$e^{\lambda_1' z},\ e^{\lambda_2' z},\ \ldots \quad \text{when} \quad R(z)\to+\infty,$$

$$e^{\mu_0' z},\ e^{\mu_{-1}' z},\ \ldots \quad \text{when} \quad R(z)\to-\infty,$$

and
$$G_{12}(z, t) = O(e^{-\lambda_1 t}) \quad \text{when} \quad R(t)\to+\infty$$
$$= O(e^{-\mu_0 t}) \quad \text{when} \quad R(t)\to-\infty.$$

Hence such a row as $\theta(z)$ exists if

$$R(\lambda_1'-\lambda_1) \text{ is negative or if } R(\mu_0'-\mu_0) \text{ is positive,}$$

and there is in fact one for each member of the series

$$\ldots,\ R(\mu_{-1}'),\ R(\mu_0'),\ R(\lambda_1'),\ R(\lambda_2'),\ \ldots$$

which lies between $R(\mu_0)$ and $R(\lambda_1)$. There are no more*, and there is no contradiction of the statement at (21.21), since the order of integration in

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_3(t, u)\,K_1(u, v)\,\phi(v)\,dv\,du$$

* That is to say, no more such rows of analytic functions. The subjects of integration in (22.3) were presumed to be analytic.

may not be changed; in fact, when $u-v$ is finite, $K_1(u, v)$ does not tend to zero as u, v increase, and the factors $K_3(t, u)$, $\phi(v)$ neutralize each other, so that the integrand is not small.

23. *Transformation of other equations by means of the foregoing results.*

Let $(K_1)\phi(z)$ be written for

$$\int_{-\infty}^{\infty} K_1(z, t)\,\phi(t)\,dt,$$

and so in other cases. Then the results stated in § 21 may be used to transform a set of integral equations

$$\{1+(K_1)+(L)\}\,\phi = \psi, \tag{23.1}$$

where $L(z, t)$ is another nucleus. Often the transformation leads to a set which can be solved by Fredholm's method.

Write as a trial solution

$$\phi = \chi+(K_3)\chi, \tag{23.2}$$

then, since

$$(K_1)+(K_3)+(K_1)(K_3) = 0, \tag{23.3}$$

$$\chi+(L)\chi+(L)(K_3)\chi = \psi. \tag{23.4}$$

Suppose that, by a transformation such as

$$\zeta = \frac{e^{\gamma z}}{e^{\gamma z}+e^{-\gamma z}}, \quad \tau = \frac{e^{\gamma t}}{e^{\gamma t}+e^{-\gamma t}},$$

(23.4) is reduced to a system in which the limits of the integrals are 0, 1, and the nucleus is bounded and, except at a finite number of points, continuous, then it can be solved by Fredholm's method, unless there is a row ω such that

$$\omega+(L)\,\omega+(L)(K_3)\,\omega = 0. \tag{23.5}$$

Here $\omega+(K_3)\omega$ cannot vanish, since ω does not, and

$$\{1+(K_1)+(L)\}\,\{\omega+(K_3)\,\omega\} = 0, \tag{23.51}$$

so that the row $\omega+(K_3)\omega$ is a solution of

$$\{1+(K_1)+(L)\}\,\phi = 0. \tag{23.6}$$

Conversely, if there is a row ϕ satisfying (23.6), let

$$\omega = \phi+(K_1)\,\phi = -(L)\,\phi.$$

Then
$$\omega+(K_3)\,\omega=\phi \qquad (23.7)$$

and
$$\omega=-(L)\{\omega+(K_3)\,\omega\},$$

so that the row ω satisfies (23.5) and does not vanish. Hence the proposed method of solving (23.1) fails unless ψ satisfies a certain condition, since it depends on Fredholm's solution for (23.4). This is quite analogous to results in Fredholm's theory.

The reasoning of this section is subject to the possibility of changing the order of integration in

$$(K_1)(K_3)\chi \quad \text{at} \quad (23.3),$$
$$(K_1)(K_3)\omega \quad \text{at} \quad (23.51),$$
$$(K_3)(K_1)\phi \quad \text{at} \quad (23.7).$$

Summary.

The most important equations included under (21.1), (21.3) are probably those of the form

$$\phi(t)=g(t)+\int_0^1 k_1\left(\frac{t}{u}\right)\phi(u)\,\frac{du}{u}+\int_1^\infty k_2\left(\frac{t}{u}\right)\phi(u)\,\frac{du}{u},$$

where k_1, k_2 are rational algebraic fractions, or matrices of such fractions, as in § 1 and the foot-note to § 21. The present investigation began with a simple example of these (S.N., § 1), the result of which led to extensions of such generality as to obscure the original idea, though giving symmetry to the results, as in (20.31), (20.32), where G is the matrix defined at (13.61).

When such equations present themselves, the method of solution indicated is to find the functional equations by the method of § 22 (or S.N., § 1), and thence to construct the matrix G_{34} and the solving nucleus K_3.

ON THE COMPLETE SYSTEM OF TWO QUADRATIC FORMS IN n VARIABLES.

By H. W. TURNBULL.

[Received 21 March, 1929.—Read 25 April, 1929.]

1. *Introduction.*

The following pages deal with the complete system of concomitants of two quadratic forms in n variables :

$$F = \Sigma a_{ij} x_i x_j, \qquad F' = \Sigma r_{ij} x_i x_j.$$

According to Gordan's theorem, the irreducible concomitants are finite in number. Such a finite system, inclusive, with possible redundancies, of all irreducibles, I gave twenty years ago*. In the work, which involved an intricate argument, it was pointed out (page 239) that the results could be simplified by resolving each variable x, or cogredient variable y, into $n-1$ variables u, v, w, ..., each contragredient to x, and then treating the problem for coefficients a_{ij}, r_{ij} and variables u_i, v_i, ..., alone. In the present work this has been done, and the system has been established by what seems to be an essentially simplified form of the original method. The system first given was unsymmetrical, but the result here obtained is perfectly symmetrical in the ground forms. This system agrees with the known binary and ternary cases (*loc. cit.*, page 240) ; but a proof that all the forms retained are strictly irreducible has not been given. As before, some of them may still be redundant for cases when $n > 3$.

The actual number of members of the system has been enumerated, and the results are stated at the end. This brings to light a little theorem in combinatory analysis, showing that the number of forms of weight $2v+1$ is the same as the number of weight $2n-2v-1$.

* *Trans. Camb. Phil. Soc.*, 21 (1909), 197–240.

The methods apply directly to the analogous problems of the complete system for a bilinear form

$$\phi = \Sigma a_{ij} x_i u_j,$$

where x, u are contragredient and ϕ thus represents a *collineation*, and also where x, u are cogredient and ϕ represents a *correlation*. These are not given in the present pages.

It is perhaps a fact of some interest that the systems so found turn out to be comparatively simple. But this simplicity is, in one important respect, an illusion, because the full complement of sets of variables imposed by Clebsch's theorem* has not been utilized. It would obviously be desirable to solve the problem in this exhaustive case; but for values of n greater than four (the quaternary case) only the very simplest such systems are known—those of a few linear forms and of a single quadratic form. As those which have been found have usually emerged by the use of symbolic methods, it affords some guidance to the complexity of the higher cases to consider that, as n increases, the number r of requisite symbolic factor types mounts up very rapidly. For $n = 2$, 3, 4 we have $r = 2$, 3, 13 respectively. For instance, such factor types are (ab), a_x for the binary case $(n = 2)$, and (abc), $(\alpha\beta\gamma)$, a_α for the ternary case $(n = 3)$. A very cursory survey of the case, when $n = 5$, shows the value of r to be disconcertingly high.

The usual symbolic methods are used throughout, with constant reference to the Sylvester fundamental identities†, involving determinants of order n. These can be written

$$\sum_{RS} (ABS)(RX) = \sum_{AB} (ARS)(BX), \tag{I}$$

$$(AB)(RX) = \sum_{AB} (AR)(BX), \tag{II}$$

$$0 = \sum_{AB} (AR)(BX). \tag{III}$$

Here (ABS) denotes a determinant of given order n, partitioned vertically into i, j, k columns. Thus A, or more specifically A_i, denotes the first i columns, B, or B_j, the next j, and S, or S_k, the next k. Also R, or R_j, denotes j columns of another n-rowed determinant, and X the requisite complement of $n-j$ columns.

* Clebsch, *Göttinger Abh.*, 17 (1872), 1–62.

† Turnbull, *Theory of determinants, matrices, and invariants* (1928), 48. This is referred to later as *Invariants*.

By $\sum_{RS}$ is meant the sum of $\binom{j+k}{j}$ *distinct* terms obtained by derangement of the j columns of R with the k columns of S, accompanied by change of sign, as in a determinantal derangement. Similarly for Σ with subscript AB, which denotes $\binom{i+}{j}$ terms.

The formulae are valid as follows :

$$\text{(I)}\quad i+j<n, \qquad \text{(II)}\quad i+j=n, \qquad \text{(III)}\quad i+j>n.$$

For example, if $n=3$, $X=uv$, then

$$(abs)(ruv)-(abr)(suv)=(ars)(buv)-(brs)(auv), \tag{I}$$

$$(aa'b)(ruv)=(aa'r)(buv)+(a'br)(auv)+(bar)(a'uv), \tag{II}$$

$$\begin{aligned}0=(aa'r)(bb'u)&-(abr)(a'b'u)-(ab'r)(ba'u)\\&+(a'br)(ab'u)+(a'b'r)(bau)+(bb'r)(aa'u).\end{aligned} \tag{III}$$

2. *Two quadratics in n variables.*

Let

$$\left.\begin{aligned}f=a_x^2=b_x^2=c_x^2=\ldots\\ f'=r_x^2=s_x^2=t_x^2=\ldots\end{aligned}\right\} \tag{1}$$

be the symbolic forms of two different quadratics

$$F=\Sigma a_{ij}x_ix_j, \qquad F'=\Sigma r_{ij}x_ix_j \tag{2}$$

in n variables $$x=\{x_1,\ x_2,\ \ldots,\ x_n\}. \tag{3}$$

The summation for each suffix i, j runs from 1 to n, and the symbols are defined by the following equivalent relations

$$\left.\begin{aligned}a_{ij}=a_{ji}=a_ia_j=b_ib_j=c_ic_j=\ldots\\ r_{ij}=r_{ji}=r_ir_j=s_is_j=t_it_j=\ldots\end{aligned}\right\}. \tag{4}$$

Now let y, z, ... be any number of other sets of variables cogredient with x, and also let u, v, w, ... be sets contragredient to x. Then, for the absolutely complete theory covered by Clebsch's theorem, it would also be necessary to introduce compound variables $\pi_2=xy$, $\pi_3=xyz$, etc., or else $p_2=uv$, $p_3=uvw$, etc., where, for instance, π_2 is a set of $\binom{n}{2}$ determinants $(x_iy_j-x_jy_i)$. It is this case which is excluded, so that only those concomitants will be sought which involve variables u, and all variables cogredient with u, but *not* necessarily expressed explicitly

in terms of p_2, p , etc. A simpler system, where the variables x, y, z, etc., alone are retained, has already been given*.

The relation between these various systems can best be seen as follows. Let $C\{a, r, x\}$ denote a complete system where a, r, x specify the coefficients a_{ij}, r_{ij} of ground forms, and the variables x_i, which occur explicitly in members of the system. Then the most general Clebsch system can be denoted by

$$C\{a, r, x, \pi_2, \pi_3, \ldots, \pi_n\}.$$

Members of this, together with their polars, cover every possibility in the projective invariant theory of two quadrics. This system is virtually the same as

$$C\{a, r, u, p_2, p_3, \ldots, p_n\},$$

because p_i and π_{n-i} denote the same compound variables to a constant factor. Other possible complete systems are

(i) $C\{a, r\}$,

(ii) $C\{a, r, x, y, z, \ldots\}$,

(iii) $C\{a, r, u, v, w, \ldots\}$,

(iv) $C\{a, r, x, u, v, w, \ldots\}$,

(v) $C\{a, r, x, y, \ldots, u, v, w, \ldots\}$,

to mention some of the principal cases. System (i) gives the $n+1$ invariants of two quadrics; the system (ii) has not actually been given. Geometrically it is a complete system of invariants of two quadrics as point loci, and any number of given points x, y, etc. System (iii) is found below: geometrically it gives the invariants of two quadrics and any number of given primes (lines for the ternary case, planes for the quaternary). System (iv)† covers the case of one point x and any number of primes u, v, w. This includes, as its particular case when $n = 3$, the Clebsch ternary system for two quadrics. The present argument is a simplification of the methods which were originally adopted, and could be invoked to derive system (ii) from (iii), the system about to be given. It could also be used for system (v), if that were needed‡.

Geometrically, the still unknown Clebsch system answers to the in-

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* Turnbull and Williamson, *Proc. Royal Soc. Edinburgh*, 45 (1925), 149–165. The simplicity was attained by implicitly allowing the concomitant $(uv \ldots w)$ to enter as a *denominator*.

† *Trans. Camb. Phil. Soc.*, 21 (1909), 197–240.

‡ Since writing the above, I have constructed both systems (ii) and (v) in collaboration with Dr. J. Williamson. They appear in the *Proc. Royal Soc. Edinburgh*, 50 (1930).

variants of two point-quadric loci, together with a prime, a subprime, and one of each type of linear locus down to a plane, a line, and a point, each such linear element lying within each higher dimensional element. For this peculiar geometrical arrangement is implicit in the algebraic statement of the Clebsch theorem*.

3. *Concomitants of two quadrics.*

In the present case (iii), every variable u, or symbol a or r, which enters into a concomitant is cogredient with every other; so, by the fundamental theorem, every polynomial concomitant is expressible as ΣP, where P is a product of ϖ bracket factors of the type

$$(A_i R_j U_k) \equiv (abc \dots rst \dots uvw \dots), \tag{1}$$

consisting of i different equivalent symbols a, b, c, ..., j equivalent symbols r, s, t, ..., and k $(=n-i-j)$ variables u, v, w, ...

Here A_i denotes a matrix of currency i, consisting of i columns of these symbols of the first quadric, convolved together; while R_j refers similarly to the second quadric, and U_k to the variables.

Since a, b, c, ... belong to a quadric, somewhere among the other $\varpi - 1$ factors of P occur duplicate symbols a, b, c, ... By interchanging these letters a, b, c, ... in all their $i!$ ways, and adding the results, we obtain a series which immediately transforms into a sum of terms Q, where A_i appears in two different factors†. This process works exactly as in the case† of invariants, when $k = 0$. In this way we express a typical product as

$$P = (A_{q_1} \dots)(A_{q_1} \dots)(A_{q_2} \dots)(A_{q_2} \dots) \dots (A_{q_\nu} \dots)(A_{q_\nu} \dots)(\dots), \tag{2}$$

where $n \geqslant q_1 \geqslant q_2 \geqslant \dots \geqslant q_\nu \geqslant 0$.

Here all the symbols of the first quadric are accounted for among the A's; and the pairs of factors have, for convenience, been arranged in the order indicated by descending values of the suffixes. This product P is now said to be *prepared* for the first quadric. The symbols not yet expressed consist entirely of r, s, t, ..., u, v, w, ... If now $q_1 = n$, P contains the invariant $A_n^2 = (abc \dots m)^2$ as a factor, and is reducible.

Further, if no symbol of the second quadric enters into the first two factors, P contains the concomitant

$$(A_{q_1} U_{n-q_1})(A_{q_1} V_{n-q_1}), \tag{3}$$

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* Cf. *Invariants*, 252.

† *Invariants*, 193, also 304.

with $n-q_1$ variables $u, v, \ldots$, convolved in each factor, although the variables contained in the two groups U and V need not necessarily be alike. Again, for our present case (iii), P is reducible, although this would not be so if the general Clebsch conditions were imposed.

This leaves the case where symbols of the second quadric appear in one or other of these two factors, along with A_{q_1}. Once more, if exactly the same symbols R_{k_1}, R_{k_1} occur in these factors, P contains the concomitant

$$(A_{q_1} R_{k_1} U_{t_1})(A_{q_1} R_{k_1} V_{t_1}) \quad (q_1+k_1+t_1 = n), \tag{4}$$

and is reducible. The typical case still to be considered is therefore when a pair of duplicates A_{q_i} occur in the setting

$$Q_i = (A_{q_i} R_{k_i} S_{l_i} U_{t_i})(A_{q_i} R_{k_i} T_{m_i} V_{s_i}), \tag{5}$$

with $$q_i+k_i+l_i+t_i = q_i+k_i+m_i+s_i = n \quad (q_i > 0),$$

where the l_i+m_i symbols of F', contained in S and T, are all entirely different.

Similarly, if P contains a factor $(rs \ldots t)$ composed entirely of symbols of F', it is reducible, by convolving the duplicate symbols in a second factor, so that the discriminant $R_n^2 = (rs \ldots t)^2$ emerges. This only happens if $\varpi > 2\nu$, or else if $\nu = 0$.

Again, if P contains a factor $(uv \ldots w)$ composed entirely of variables, it is reducible; and if P contains *two* factors as in Q_i above, for which $q_i = 0$, P is reducible. For in this case no symbols of the first quadric appear, and we can suppose that $l_i \geqslant m_i$ without loss of generality. Then by convolving the duplicates of $R_{k_i} S_{l_i}$ in the second factor, and repeating the process if necessary, we obtain concomitants of the type $(RU)(RV)$.

Hence we may suppose P to contain not more than one factor lacking symbols A. In other words $q_\nu > 0$; and there are 2ν factors containing A's, while ϖ, the total number of factors in P, is either 2ν or $2\nu+1$. This gives two cases to be considered in turn.

Case (i) $\varpi = 2\nu$.

Case (ii) $\varpi = 2\nu+1$.

Case (i). This follows exactly the method already employed for the invariants*, and shows that P always contains a reducible factor

$$(A_q R_k U_l)(A_q R_k V_l) \quad (q+k+t = n). \tag{6}$$

* *Invariants*, 305.

For we can now write P more specifically as

$$P = (A_{q_1}\ldots)\ldots(A_{q_\nu}R_{k_\nu}S_{l_\nu}U_{t_\nu})(A_{q_\nu}R_{k_\nu}T_{m_\nu}V_{s_\nu}), \tag{7}$$

where $l_\nu \geqslant m_\nu$, and then proceed to convolve the duplicates of $R_{k_\nu}S_{l_\nu}$ in the final factor. Such a process will throw l_ν of the $q_\nu+m_\nu+s_\nu$ symbols among its ATV into factors further to the left; and even if a symbol in A_{q_ν} is so moved, it merely increases a previous suffix q_i $(i<\nu)$ of an A. As in the case of invariants, this leftward movement of a symbols is a reducing process. It replaces P by a series $\Sigma P'$ and reducible terms, where the last two factors of each P' can be written

$$(A_{q_\nu}R_{k_\nu}S_{l_\nu}U_{t_\nu})(A_{q_\nu}R_{k_\nu}S_{l_\nu}T_iV_j). \tag{8}$$

Here T_iV_j denote what remain of the original $m_\nu+s_\nu$ symbols in $T_{m_\nu}V_{s_\nu}$, when S_{l_ν} has ousted l_ν such symbols. If $i=0$, the term is reducible; and, if $i>0$, the process can be repeated by convolving the duplicates of $R_{k_\nu}S_{l_\nu}T_i$ in the first of these two factors. In short, the process of preparing these last two factors of P', for symbols of the second quadric, reduces the form; and this settles case (i).

Case (ii). Here $\varpi = 2\nu+1$, and P can be taken to be

$$(A_{q_1}\ldots)\ldots(A_{q_\nu}\ldots)(A_{q_\nu}R_iS_kU)(R_iT_jV), \tag{9}$$

where R_i denotes all symbols of F' common to the last two factors, and S_k, T_j denote $k+j$ different symbols of F'. Also U and V denote the requisite variables to fill up these bracket factors. If $i+j=n$, V is naturally omitted; and similarly for U.

If k and j are both zero, S and T do not appear, and we pass to the next pair of factors leftwards. If $k \geqslant j$, we convolve the duplicates of R_iS_k in the last factor as before. If $k<j$, we convolve the duplicates of R_iT_j in the last but one factor, and again may drive some of the a symbols in A_{q_ν} leftwards, so raising the values of earlier suffixes q_i. The joint effect of these processes is to express P as a sum $\Sigma P'$ and reducible terms, where

$$P' = (A_{q_1}\ldots)\ldots(A_{q_\nu}\ldots)(A_{q_\nu}R_{i_1}U)(R_{i_1}V) \quad (i_1 \geqslant i). \tag{10}$$

Let the third factor from the right now be written $(A_{q_\nu}R_mU')$. If $m > i_1$, we interchange these adjacent factors, so that

$$P' = (A_{q_1}\ldots)\ldots(A_{q_\nu}R_{i_1}U)(A_{q_\nu}R_mU')(R_{i_1}V). \tag{11}$$

Then, by further repetition of the above process on the last two factors, we ultimately increase R_{i_1}, in the last factor, to at least R_m. The rôles

of suffixes i_1 and m have thus been interchanged, so we can now assume that, for the future, m is not greater than i_1.

As the notation R_i, S_k, T_j is no longer needed for the two final factors, let it be used in a similar way for the next pair of factors leftwards. Thus the typical form is

$$P' = (A_{q_1}\ldots)\ldots(A_{q_{\nu-1}}R_iS_kU')(A_{q_\nu}R_iT_jV')(A_{q_\nu}R_{i_1}U)(R_{i_1}V), \qquad (12)$$

where neither $i+k$ nor $i+j$ exceeds i_1. It is essential to notice in this form that, as all symbols of the second quadric, standing to the right of V', are properly duplicated, the $k+j$ symbols duplicating those of S_k and T_j must necessarily be found, possibly scattered and possibly convolved, somewhere among the earlier factors towards the *left*.

As before, if k and j are both zero, we pass to the next pair of factors leftwards. Also, if $k \geqslant j$, we convolve the duplicates of R_iS_k in the factor immediately to its right, which at present contains R_iT_j. This process may diminish A_{q_ν}, so raising the values of earlier suffixes q_i. But the symbols R_{i_1} on the right are in no way disturbed. If, however, $k<j$, we convolve the duplicates of R_iT_j in the factor immediately on its left. The joint effect of these and all similar processes, on these factors, is to express P' as $\Sigma P''$ and reducible terms, where

$$P'' = (A_{q_1}\ldots)\ldots(A_{q_{\nu-1}}R_{i_2}U'')(A_{q_\nu}R_{i_2}V'')(A_{q_\nu}R_{i_1}U)(R_{i_1}V). \qquad (13)$$

For reasons already stated we may assume that $i_1 \geqslant i_2$.

If $\nu > 1$ we proceed in exactly the same way through pairs of terms leftwards, until all the symbols of F' have been paired off as

$$R_{i_\mu},\ R_{i_\mu}\ \ldots,\ R_{i_2},\ R_{i_2},\ R_{i_1},\ R_{i_1}.$$

This requires 2μ bracket factors, so that the first $\varpi - 2\mu$ factors will contain A_{q_1}, A_{q_1}, ..., and variables only. Hence the form reduces unless $\mu = \nu$, because $\varpi = 2\nu+1$. The only possible irreducible form, for further consideration, is

$$Q = (A_{q_1}U)(A_{q_1}R_{i_\nu}U')(A_{q_2}R_{i_\nu}U'')(A_{q_2}R_{i_{\nu-1}}U''')\ldots(A_{q_\nu}R_{i_1}U^{(\varpi-1)})(R_{i_1}U^{(\varpi)}),$$

where
$$q_1 \geqslant q_2 \geqslant \ldots \geqslant q_\nu > 0,$$
$$i_1 \geqslant i_2 \geqslant \ldots \geqslant i_\nu > 0. \qquad (14)$$

This set of forms Q is now symmetrical in general structure with regard to the symbols of the two quadrics.

Finally the equality signs among the suffixes may be removed.

For let consecutive q's, say q_j and q_{j+1} be equal; and let the four matrices A_{q_j}, etc., to which these equal suffixes belong, be written, still

in their usual order, as

$$A_q,\ A_q,\ A'_q,\ A'_q.$$

Also let the four consecutive factors of Q in which they occur be written

$$(A_q R_{i_{m+1}} V)(A_q R_{i_m} V')(A'_q R_{i_m} V'')(A'_q R_{i_{m-1}} V''').$$

Then by the Sylvester fundamental identity II, which transfers the matrix A'_q, as a whole, from the last of these factors into the second factor, entirely reducible terms are, as before, obtained. Hence each q_j is different. And on account of the symmetry of the result, a like argument applies to each suffix i. Since each q (or i) is different and is less than n, the number of sets q or i is *finite*.

This completes the investigation, and establishes the following theorem.

COMPLETE SYSTEM OF TWO QUADRICS. *All possible irreducible forms belonging to two quadrics a_x^2 and r_x^2 are included among the following and their polars :—*

$$P_{ij} = (A_i R_j U_k)^2$$

$$i, j, k = 0, 1, \ldots;\ i+j+k = n;\ i+j > 0,$$

$$Q = (A_{q_1} U)(A_{q_1} R_{j_1} U')(R_{j_1} A_{q_2} U'') \ldots (A_{q_\nu} R_{j_\nu} U^{(2\nu-1)})(R_{j_\nu} U^{(2\nu)}),$$

$$n > q_1 > q_2 > \ldots > q_\nu > 0,$$

$$0 < j_1 < j_2 < \ldots < j_\nu < n.$$

There is also the absolute concomitant $(U_n) = (u_1 u_2 \ldots u_n)$.

4. *Maximum weight.*

In Q the number of factors depends on ν, and it is greatest when $\nu = n-1$. In this case the suffixes $q_1, q_2, \ldots, q_\nu$ are the integers $1, 2, \ldots n-1$ in descending order, while $j_1, j_2, \ldots, j_\nu$ are the same integers, but in ascending order; so that $q_i + j_i = n$. Hence the second, and even, factors in Q contain no variables u, while the other alternate factors contain each a single u. By using the same variable u in each such factor, we can identify Q with the contravariant of order n, given by the Jacobian*, with regard to the set u, of the n quadratic contravariants.

$$P_i = (A_i R_{n-i-1} u)^2 \quad (0 \leqslant i < n).$$

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* *Trans. Camb. Phil. Soc., loc. cit.*, 238.

For example, if $n = 4$,

$$Q = (aa'a''u)(aa'a''r)(rbb'u)(bb'r'r'')(r'r''b''u)(b''ss's'')(ss's''u),$$

where a, a', a'', b, b', b'' are equivalent symbols for one quadric, and r, r', r'', s, s', s'' for the other.

Such a form (when not identically zero because special quadrics have been chosen) actually resolves into n linear factors, so giving the common self-conjugate simplex for the two quadrics. This can be verified non-symbolically, although it would be interesting also to have a proof by symbolic methods.

In no other case can Q furnish a contravariant, so that any further such forms can only occur among the P's, and then only when $k = 1$. This accounts for the usual n quadratic contravariants.

5. *Enumeration of the complete system.*

It is interesting to give the actual number of forms, apart from polars, classified according to weight. Among those of odd weight they show a remarkable reciprocity. In fact,

The number of forms of weight $2\nu+1$ *is equal to the number of forms of weight* $2n-\nu-1$.

These numbers, for odd weights $w = 1, 3, 5, \ldots$, are as follows :—

$$1, \quad \frac{n(n-1)}{1.2}, \quad \frac{n(n-1)^2(n-2)}{1.2^2.3}, \quad \frac{n(n-1)^2(n-2)^2(n-3)}{1.2^2.3^2.4}, \quad \ldots, \quad 1.$$

The sum of these n terms can be found by the binomial theorem and is $2.(2n-1)!/\{(n-1)!(n+1)!\}$. These give the number of forms Q, as ν takes the values 0, 1, 2, ..., $n-1$ respectively. The single form answering to the zero value is (U_n).

The number of forms P_{ij}, of even weight, is $\frac{1}{2}n(n+3)$, being one less than the triangular number $\frac{1}{2}(n+1)(n+2)$. It enumerates the values of i and j satisfying the condition

$$0 < i+j \leqslant n.$$

Combining these results, the total number of forms is given by

$$N = \tfrac{1}{2}n(n+3) + \frac{2.(2n-1)!}{(n-1)!\,(n+1)!}.$$

In tabular form the numbers can be arranged horizontally according to weight, and vertically according to binary, ternary, ..., n-ary fields;

and we have the following result :—

Weight	$n=2$,	3,	4,	5,	6,	...,	n
2	5	9	14	20	27	...	$\frac{1}{2}n(n+3)$
1	1	1	1	1	1	...	1
3	1	3	6	10	15	...	$\frac{1}{2}n(n-1)$
5		1	6	20	50	...	$\frac{1}{12}n(n-1)^2(n-2)$
7			1	10	50	...	...
9				1	15	...	...
11					1	...	...
Total	7	14	28	62	159	...	N

The total seven for binary quadratics includes the absolute concomitant (uv), besides the usual six forms.

I have not succeeded in finding a general proof of these results beyond the case when the weight is 5, although the entries in the columns as far as $n=6$ have been verified. The problem is one of combinatory analysis, and may be stated quite simply in terms of integer matrices. For example, if $n=5$, $\varpi=7$, it resolves into enumerating the sets

$$\begin{bmatrix} q_1 & q_2 & q_3 \\ j_1 & j_2 & j_3 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}, \quad \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}, \quad \ldots, \quad \begin{bmatrix} 4 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix},$$

subject to the rules $5 > q_1 > q_2 > q_3 > 0$, $0 < j_1 < j_2 < j_3 < 5$, $q_i + j_i \leqslant 5$.

The number of different upper (or lower) rows is obviously ${}_4C_3$: the whole difficulty of the problem lies in the restriction imposed upon the columns.

The reciprocity theorem, enunciated above, can be proved as follows. Consider the matrices

$$\begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 4 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

The first is one of the typical set above ; the second is complementary and contains all integers less than 5, not found in the rows of the first matrix. The third matrix is the difference of the second and of one consisting exclusively of 5's. It can be proved that the third matrix obeys the same ordinal laws as the first : and in this way the reciprocity for weights $2\nu+1$ and $2n-2\nu-1$ is made apparent.

[*Added* 26 *June*, 1929. I have to thank Dr. A. C. Aitken for suggesting the following proof of the main enumeration theorem. It is

illustrated by the above example. Instead of $\begin{bmatrix}3&2&1\\1&2&4\end{bmatrix}$ take $\begin{bmatrix}2&3&4\\1&2&4\end{bmatrix}$ where the upper row consists of the complements of numbers in the original upper row, namely 5—3, 5—2, 5—1. The stipulation now requires each upper index to be at least equal to the one beneath it. Of the $({}_4C_3)^2$ pairs—in general $({}_{n-1}C_\nu)^2$—formed by choosing the ν numbers of each row in all possible ways, we are to reject all cases such as $\begin{bmatrix}1&2&3\\1&2&4\end{bmatrix}$, because the lower index 4 exceeds the upper 3. Consider such a reject. Its first upper index, which is *less* than a lower index, can obviously not be among the lower indices. Let it be taken and put among the lower indices, so that we obtain, *e.g.*, $\begin{bmatrix}1&2&&\\1&2&3&4\end{bmatrix}$. The enumeration of all such modified matrices is immediate: it gives ${}_4C_2 \times {}_4C_4$ or $\binom{n-1}{\nu-1}\binom{n-1}{\nu+1}$ in general. Hence the required number is

$$\binom{n-1}{\nu}\binom{n-1}{\nu} - \binom{n-1}{\nu-1}\binom{n-1}{\nu+1},$$

which reduces to

$$\frac{n(n-1)^2(n-2)^2\ldots(n-\nu-1)^2(n-\nu)}{1\,.\,2^2\,.\,3^2\,.\ldots\,\nu^2\,.\,(\nu+1)}$$

and agrees with what has already been given.]

ON THE DENSITY OF IRREGULAR LINEARLY MEASURABLE PLANE SETS

By G. Walker.

[Received 17 May, 1929.—Read 13 June, 1929.]

1. The upper and lower densities of a linearly measurable plane set of points at any point of the set have been defined by Besicovitch† as follows:—Let E be a measurable set of points, e any point, and $c(e, r)$ a circle with centre e and radius r. Then, if $D^*(e, E)$ and $D_*(e, E)$ be the upper and lower densities respectively,

$$D^*(e, E) = \overline{\lim_{r \to 0}} \frac{L[E \times c(e, r)]}{2r},$$

$$D_*(e, E) = \underline{\lim_{r \to 0}} \frac{L[E \times c(e, r)]}{2r},$$

where $L[E \times c(e, r)]$ is the linear Carathéodory measure‡ of the plane set of points common to E and $c(e, r)$.

If
$$D^*(e, E) = D_*(e, E) = \alpha, \qquad (1)$$

α is called the density of the set at the point e. If $\alpha = 1$ at almost all points of the set, the set is said to be regular; a set which is not regular is said to be irregular.

In his paper Besicovitch has proved the following properties:—

Theorem A§. *At almost all points of a linearly measurable set E,*

$$\tfrac{1}{2} \leqslant D^*(e, E) \leqslant 1, \quad 0 \leqslant D_*(e, E) \leqslant 1.$$

Theorem B‖. *At almost all points of an irregular set the lower density is not arbitrarily near to* 1.

† *Math. Annalen*, 98 (1928), 423.
‡ Carathéodory, *Göttingen Nachrichten*, 1914, 404.
§ *Loc. cit.*, 428, *et seq.*
‖ *Loc. cit.*, 454.

The value of the upper bound of the lower density is given very near 1, and it is suggested that the exact upper bound may be $\frac{1}{2}$. The properties of the lower density proved below show that it is greater than or equal to $\frac{1}{2}$. We prove the following results :—

THEOREM 1. *If α be any number, $0 \leqslant \alpha \leqslant \frac{1}{2}$, then there exist irregular sets of positive measure, whose lower density at almost all points is α.*

THEOREM 2. *If β be any number, $\frac{1}{2} \leqslant \beta \leqslant 1$, then there exist irregular sets of positive measure, whose upper density at almost all points is β.*

The following notation is used throughout. $U(\rho, E)$ denotes a finite or enumerable set of convex areas $U_1(E)$, $U_2(E)$, $U_3(E)$, ..., of diameters $\delta_1, \delta_2, \delta_3, \ldots$, respectively, which possess the following properties :—

(*a*) $\delta_\nu < \rho$ for each ν $(\nu = 1, 2, 3, \ldots)$;

(*b*) every point of E is an interior point of $U_\nu(E)$ for at least one ν.

A typical member of the set of areas is denoted by $U_i(E)$, of diameter $\delta < \rho$. If to each member of a set P there corresponds a number l, then by $\sum_P l$ is meant the sum of all these numbers.

2. THEOREM 1. *There exist linearly measurable sets of positive measure, for which at almost all points*

$$D_*(e, E) = \alpha \quad (0 \leqslant \alpha \leqslant \tfrac{1}{2}).$$

The set given by Gross and discussed by Besicovitch† shows that the theorem is true for $\alpha = 0$. Suppose then that $\frac{1}{2} \geqslant \alpha > 0$. If $\alpha \leqslant \frac{1}{3}$, there exists an integer $m_0 \geqslant 2$ satisfying

$$\frac{1}{2m_0 - 1} \geqslant \alpha > \frac{1}{2m_0 + 1};$$

if $\frac{1}{2} \geqslant \alpha > \frac{1}{3}$, we put $m_0 = 1$.

Let n be a positive integer, and write

$$A = \frac{1}{2\alpha} \sqrt{\{1 - 4\alpha^2 (m_0 - \tfrac{1}{2})^2\}}.$$

† *Loc. cit.*, 435.

Now let P be a rectangle of length l and breadth d. Divide the shorter sides of P into m_0+1 equal parts, and denote by L_1' the set of lines parallel to the longer sides of P obtained by joining corresponding points of division. Next construct a system of parallel lines L_2' at a distance $x = Al/(m_0 n^2)$ from each line of L_1' and on the same side. We denote any line of L_1' by l_1', and the corresponding line of L_2' distant x from it by l_2'. If $m_0 = 1$ we rotate the rectangle formed by the lines l_1', l_2' and the shorter sides of P about its centre through an acute angle $\theta = \sin^{-1}(2n)^{-1}$, and denote l_1' and l_2' in their new positions by l_1'' and l_2'' respectively. The remainder of the construction is to be carried out for $m_0 \geqslant 1$, but, in the case $m_0 = 1$, L_1' and L_2' refer to l_1'' and l_2'', *i.e.* to l_1' and l_2' respectively, in their new positions. Now divide each of the lines of L_1' and L_2' into n^2 equal parts, and enumerate the centres of corresponding sections of l_1' and l_2', beginning from one end of P, corresponding suffixes denoting corresponding points. We thus obtain m_0 pairs of sequences, which we typify as follows :—

$$x_1^1,\ x_2^1,\ x_3^1,\ \ldots,\ x_{n^2}^1, \text{ from } l_1',$$
$$x_1^2,\ x_2^2,\ x_3^2,\ \ldots,\ x_{n^2}^2, \text{ from } l_2',$$

where $x_i^1 x_i^2 = x,\ i = 1, 2, 3, \ldots, n^2$.

Now form from these the sequence

$$x_1^1,\ x_2^2,\ x_3^1,\ x_4^2,\ \ldots,$$

the last member being $x_{n^2}^1$ or $x_{n^2}^2$ according as n is odd or even. With these points as centres construct rectangles of length $l/(m_0 n^2)$, breadth $l/(m_0 n^3)$, with their longer sides parallel† to l_1'. This process we suppose carried out for each of the m_0 pairs of corresponding lines‡. The construction of the $m_0 n^2$ rectangles as above we define to be the operation $\Omega(n)$§. It will be seen below that the choice of x, θ, and n is such that the rectangles so obtained are non-overlapping and completely interior to P.

Now let n_1 be an integer such that

$$n_1 > 4(A+1), \tag{2}$$

† In the case $m_0 = 1$, the rectangles are constructed with their longer sides parallel to l_1' in its rotated position.

‡ The distance x between corresponding lines is chosen so that the distance of a centre of a rectangle in one line from the nearest point of the adjacent one in the corresponding line, when divided into the length of a rectangle, tends to 2α as $n \to \infty$.

§ The object of the rotation in $\Omega(n)$, for $m_0 = 1$, is to ensure that the rectangles obtained by that operation which are nearest to the ends of P are completely interior to P. For $m_0 \geqslant 2$, this rotation is obviously unnecessary.

and write $$n_i = n_1 + i - 1, \quad i = 1, 2, 3, \ldots.$$

Take a rectangle P_0, of length 1 and breadth $\frac{1}{2}$, and perform on it the operation $\Omega(n_1)$. We denote the $m_0 n_1^2$ rectangles, each of length $l_1 = 1/(m_0 n_1^2)$, breadth $d_1 = l_1/n_1$, by P_1, and any member of the system by R_1. On each R_1 of P_1 operate with $\Omega(n_2)$, and denote the resultant system of $m_0^2 n_1^2 n_2^2$ rectangles by $P_2 = \{R_2\}$; in general on each rectangle R_{i-1} of P_{i-1} perform the operation $\Omega(n_i)$, and denote the resultant system of rectangles so obtained by $P_i = \{R_i\}$.

We have the following equations :—

$$\left.\begin{aligned} l_i = l_{i-1}/(m_0 n_i^2), \qquad & d_i = l_i/n_i \\ x_i = A l_i, \qquad & \sin\theta_i = 1/(2n_i) \end{aligned}\right\}. \tag{3}$$

Now for $m_0 = 1$, in order that each R_i may be interior to the R_{i-1} from which it is derived, it is sufficient that

$$d_{i-1} > x_i(1+\cos\theta_i) + l_{i-1}\sin\theta_i + d_i\cos\theta_i, \tag{4}$$

$$l_{i-1} > x_i \sin\theta_i + l_{i-1}\cos\theta_i + d_i \sin\theta_i. \tag{5}$$

From (3) it follows that (4) and (5) will be satisfied provided that

$$\sin\theta_i < \frac{1}{n_i} - \frac{2A+1}{n_i^2}, \tag{4a}$$

$$\tan\tfrac{1}{2}\theta_i > (A+1)/n_i^2. \tag{5a}$$

Since $\sin\theta_i = 1/(2n_i)$, and $\tan\frac{1}{2}\theta \geqslant \frac{1}{2}\sin\theta$, $0 \leqslant \theta \leqslant \frac{1}{2}\pi$, it follows from (2) that (4a) and (5a) are true.

Let E denote the set of points

$$P_0 \times P_1 \times P_2 \times P_3, \ \ldots,$$

where P_i denotes the set of points both interior to and on the boundary of each R_i belonging to P_i ($i = 0, 1, 2, 3, \ldots$), R_0 being identical with P_0.

Since P_i is the sum of a finite number of closed rectangles, it is a closed set. Thus E, being the product of an enumerable number of closed sets, is also closed, and therefore measurable†. Further, each point of E is an interior point of P_i, for each i ($i = 0, 1, 2, 3, \ldots$).

3. We now prove that $LE = 1$.

If t_i denote the length of the diagonal of a rectangle of P_i it follows, since $\sum_{P_i} l_i = 1$, that

$$\sum_{P_i} t_i = \sqrt{\left\{1 + \frac{1}{n_i^2}\right\}}.$$

† Carathéodory, *loc. cit.*, 415.

Hence† $$LE \leqslant 1. \qquad (6)$$

Now, if e be any point of E, e is an interior point of $U_\nu(E)$ for at least one ν. Thus to each point e there corresponds a sequence of rectangles R_p including e, and lying entirely in $U(\rho, E)$, *i.e.* completely interior to some member of the set $U(\rho, E)$. Let $R(e)$ be the first rectangle of the sequence. Then, by Borel's theorem, there exists a finite set of rectangles $R(e)$ which cover all the points of E. If k denote the greatest suffix occurring in this set, then the set P_k of all rectangles R_k possesses the same properties, viz., it covers E and is interior to $U(\rho, E)$. Further, the set P_k forms a non-overlapping set of rectangles. We denote by r_k the set of rectangles R_k which are completely interior to $U_i(E)$. If r_k consists of more than one rectangle R_k, then there exists an integer $k_0 \leqslant k$ such that the members of r_k belong to different rectangles R_{k_0} but to the same rectangle R_{k_0-1}.

Since $\delta < \rho$ and $U_i(E)$ has points in common with at least two rectangles R_{k_0}, for $k_0 < k$, and contains at least one rectangle R_{k_0} interior to it for $k_0 = k$, it follows that $\rho > \frac{1}{2} l_{k_0}$ and $k_0 \to \infty$ as $\rho \to 0$ uniformly for all $U_i(E)$ $(i = 1, 2, 3, \ldots)$.

Suppose that $U_i(E)$ has points in common with t rectangles R_{k_0}. If $k_0 < k$, these t rectangles need not be interior to $U_i(E)$. We prove that

$$\delta > \left\{1 - O\left(\frac{1}{n_{k_0}}\right)\right\} \sum_{r_k} l. \qquad (7)$$

Suppose first that $m_0 = 1$.

If $t = 2$, and $\sum_{r_k} l = \gamma l_{k_0}$ $(0 < \gamma \leqslant 2)$, then

$$\delta > \min\left[l_{k_0} \left\{ \gamma^2 + \left(A - \frac{1}{n_{k_0}}\right)^2 \right\}^{\frac{1}{2}}, \quad (\gamma+1)\, l_{k_0} \right]$$

$$> \sum_{r_k} l. \qquad (8)$$

Suppose then that $t > 2$. We enumerate all the rectangles R_{k_0} in the rectangle R_{k_0-1} under consideration, beginning from one end of the latter. We can clearly arrange that the centres of two successive rectangles in the enumerated sequence shall not lie in the one line, and, further, that the distance of the centres of the rectangles from one end of R_{k_0-1} shall increase steadily as we pass from one member of the

† Carathéodory, *loc. cit.*, 420. With the notation of § 1, $LE = \lim_{\epsilon \to 0} L_\rho$, where L_ρ is the lower bound of the sums $\sum_{U(\rho, E)} \delta$.

sequence to its successor. We proceed on the assumption that $U_i(E)$ has points in common with two rectangles which are at least $t-1$ apart, *i.e.* between which there are $t-2$ rectangles R_{k_0} of the sequence, denoting two such rectangles by R'_{k_0} and R''_{k_0}. In the event of there being no two R_{k_0} possessing this property the inequalities proved below will be true *a fortiori*. Suppose also that $U_i(E)$ has points in common with q' rectangles R_{k_0+1} of R'_{k_0}, and with q'' rectangles R_{k_0+1} of R''_{k_0}. Then

$$\begin{aligned}\delta &> (t-2)\, l_{k_0} + (q'+q''-2)\, l_{k_0+1} \\ &> \left[t-2+\frac{q'+q''}{n^2_{k_0+1}}\right]\left[1-\frac{2}{(t-2)\, n^2_{k_0+1}}\right] l_{k_0}, \quad \text{from (3),} \\ &\geqslant \left[1-O\left(\frac{1}{n_{k_0}}\right)\right]\sum_{r_k} l, \end{aligned} \tag{9}$$

since $$\sum_{r_k} l \leqslant \left[t-2+\frac{q'+q''}{n^2_{k_0+1}}\right] l_{k_0}.$$

Suppose now that $m_0 \geqslant 2$.

If $t = 2$,
$$\begin{aligned}\delta &> (m_0-1)\, l_{k_0} \\ &\geqslant 2 l_{k_0}, \quad \text{for } m_0 > 2, \\ &\geqslant \sum_{r_k} l. \end{aligned} \tag{10}$$

For $m_0 = 2$,
$$\begin{aligned}\delta &> l_{k_0} \\ &\geqslant \sum_{r_k} l, \quad \text{if } \sum_{r_k} l \leqslant l_{k_0}. \end{aligned} \tag{10a}$$

But, if $\sum_{r_k} l > l_{k_0}$, then
$$\begin{aligned}\delta &> 2 l_{k_0}\left(1-\frac{2}{n_{k_0}}\right) \\ &\geqslant \left[1-O\left(\frac{1}{n_{k_0}}\right)\right]\sum_{r_k} l. \end{aligned} \tag{10b}$$

Now, if $m_0 \geqslant 2$, the distance between two rectangles R_{k_0}, whose centres do not lie in corresponding lines, exceeds

$$\frac{d_{k_0-1}}{m_0+1} - x_{k_0} - 2d_{k_0} = \frac{m_0 n^2_{k_0} l_{k_0}}{(m_0+1)\, n_{k_0-1}}\left[1-O\left(\frac{1}{n_{k_0}}\right)\right].$$

Suppose then that $$2 < t \leqslant \frac{m_0 n^2_{k_0}}{(m_0+1)\, n_{k_0-1}}.$$

We have

$$\delta > \min\left[\left\{(t-1)m_0-1\right\} l_{k_0}, \quad \frac{m_0 n^2_{k_0} l_{k_0}}{(m_0+1)\, n_{k_0-1}}\left\{1-O\left(\frac{1}{n_{k_0}}\right)\right\}\right].$$

Thus, either
$$\delta > tl_{k_0}\left(2-\frac{3}{t}\right)$$
$$\geqslant \sum_{r_k} l, \tag{11}$$

or
$$\delta > \left\{1-O\left(\frac{1}{n_{k_0}}\right)\right\}\sum_{r_k} l. \tag{12}$$

Finally, let
$$t > \frac{m_0 n_{k_0}}{(m_0+1)\,n_{k_0-1}}.$$

We may assume that there exists at least one pair of corresponding lines in R_{k_0-1} which contain the centres of at least $[t/m_0]$ rectangles of the t rectangles R_{k_0}, where $[t/m_0]$ denotes the integral part of t/m_0.

Thus
$$\delta > \left\{\left[\frac{t}{m_0}\right]-1\right\} m_0 l_{k_0} - l_{k_0}$$
$$\geqslant \left\{1-\frac{2m_0}{t}\right\} t l_{k_0}.$$

Since $t > \frac{m_0 n_{k_0}}{m_0+1}$ and $\sum_{r_k} l \leqslant t l_{k_0}$, it follows that
$$\delta \geqslant \left\{1-O\left(\frac{1}{n_{k_0}}\right)\right\}\sum_{r_k} l. \tag{13}$$

From (8), (9), (10), (10a), (10b), (11), (12), and (13) we have
$$\delta \geqslant \left\{1-O\left(\frac{1}{n_{k_0}}\right)\right\}\sum_{r_k} l, \quad \text{for } m_0 \geqslant 1.$$

Thus, since $\sum_{P_k} l = 1$ for each k,
$$\sum_{U(\rho,\,E)} \delta > 1-O\left(\frac{1}{n_{k_0}}\right).$$

Hence
$$LE \geqslant 1,$$

and so, from (6),
$$LE = 1.$$

In the same way it may be shown that
$$L[E \times R_k] = l_k.$$

4. We now prove that, at all points e of E,
$$D_*(e, E) \geqslant \alpha.$$

Consider the expression
$$\frac{L[E \times c(e, r)]}{2r} \quad (\tfrac{1}{3} l_i \leqslant r \leqslant \tfrac{1}{3} l_{i-1}),$$

where e is any point of E, and $c(e, r)$ denotes the circle with centre e and radius r. For any given e this ratio is a minimum for that value of r for which $c(e, r)$ just excludes points of $E \times R_i'$, where R_i' is the rectangle adjacent to R_i'' and nearest to e, and R_i'' is the rectangle R_i to which e is interior. Now, considering the possible rectangles $R_{i+1} \subset R_i''$ to which e may belong, it is obvious that the above expression will have its least value when e is interior to an R_{i+1} situated near to the centre of R_i''. For such a value of r, R_i'' is completely interior to $c(e, r)$ and

$$L[E \times c(e, r)] = l_i.$$

Also
$$r = \sqrt{\{x_i^2 + l_i^2(m_0 - \tfrac{1}{2})^2\}} + l_i O\left(\frac{1}{n_i}\right)$$
$$= l_i \left\{\frac{1}{2a} + O\left(\frac{1}{n_i}\right)\right\},$$

since
$$x_i = \frac{l_i}{2a}\sqrt{\{1 - 4a^2(m_0 - \tfrac{1}{2})^2\}}.$$

Thus
$$\frac{L[E \times c(e, r)]}{2r} = a\left\{1 - O\left(\frac{1}{n_i}\right)\right\}$$

and
$$D_*(e, E) \geqslant a. \tag{14}$$

5. Finally we show that, for almost all points e of E, there exist sequences of circles $c(e, r)$ such that

$$\frac{L[E \times c(e, r)]}{2r} = a\left\{1 + O\left(\frac{1}{n_i}\right)\right\} \quad (i \geqslant i_0).$$

Consider the set of rectangles P_i ($i = 0, 1, 2, 3, \ldots$), and operate on them as follows. If $m_0 = 1$, we have a rectangle in each R_i, of the same length as R_i, formed by lines corresponding to l_1' and l_2' and the shorter sides of R_i, which has been rotated about its centre through an angle $\sin^{-1}(2n_{i+1})^{-1}$. From this rotated rectangle we cut off a rectangular strip by lines parallel to its shorter sides, the other boundaries of the strip being the sides of the rotated rectangle, of width l_i/n_{i+1}, the centre of the strip coinciding with the centre of the rotated rectangle. If $m_0 \geqslant 1$, we cut off from each R_i a rectangular strip of width l_i/n_{i+1} by lines parallel to its shorter sides, the centre of the strip coinciding with the centre of R_i. In either case, any rectangle R_{i+1} which has points in common with the strip is completely interior to it or, at most, has points of its boundary in common with the boundary of the strip.

We denote the set of rectangular strips obtained by operating on each R_i of P_i, $m_0 \geqslant 1$, by $p_i = \{s_i\}$. Now take all the rectangles $P_{i+1} \subset P_i - p_i$ and cut off strips of width l_{i+1}/n_{i+2} in the manner described above, and denote the new system of strips by $p_{i+1} = \{s_{i+1}\}$. Proceeding in this way we obtain a system $p_i, p_{i+1}, p_{i+2}, \ldots$ of non-overlapping rectangles.

Next we show that, if η' be an arbitrary positive number, then there exists a positive integer $j(i)$ such that, for $j \geqslant j(i)$,

$$L[E - E \times (p_i + p_{i+1} + p_{i+2} + \ldots + p_{i+j-1})] < \eta'.$$

We have

$$\sum_{P_{i+1} \subset P_i - p_i} l = \left(1 - \frac{1}{n_{i+1}}\right) \sum_{P_i} l$$

$$= \left(1 - \frac{1}{n_{i+1}}\right);$$

$$\sum_{P_{i+2} \subset P_i - p_i - p_{i+1}} l = \left(1 - \frac{1}{n_{i+1}}\right)\left(1 - \frac{1}{n_{i+2}}\right);$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\sum_{P_{i+j} \subset P_i - p_i - p_{i+1} - p_{i+2} - \ldots - p_{i+j-1}} l = \prod_{t=1}^{j-1} \left(1 - \frac{1}{n_{i+t}}\right).$$

Since $\sum_{t=t_0}^{\infty} \frac{1}{n_t}$ diverges,

$$\lim_{j \to \infty} \prod_{t=1}^{j-1} \left(1 - \frac{1}{n_{i+t}}\right) = 0.$$

It follows that there exists a number $j(i)$ such that

$$L[E - E \times (p_i + p_{i+1} + p_{i+2} + \ldots + p_{i+j(i)-1})] < \eta'.$$

Write $$B_i = E - E \times \sum_{t=i}^{i+j(i)-1} p_t, \quad \eta' = \eta/2^{i+1} \quad (i = 0, 1, 2, \ldots).$$

It follows that $$L[B_0 + B_1 + B_2 + \ldots] < \eta.$$

Now consider the set

$$E' = E - B_0 - B_1 - B_2 - \ldots.$$

Corresponding to each point e of E' there exists a sequence of rectangles s_i $(i \geqslant i_0)$ to which e is interior. We choose r to have a value such that $c(e, r)$ just excludes points of $E \times R_i'$, where R_i' is the R_i adjacent to that

one to which s_i belongs, and is nearest to e. Then

$$r = \frac{l_i}{2a}\left\{1-O\left(\frac{1}{n_i}\right)\right\}$$

and

$$\frac{L[E\times c(e,r)]}{2r} = a\left\{1+O\left(\frac{1}{n_i}\right)\right\}. \tag{15}$$

From (14) and (15) it follows that, at almost all points e of E,

$$D_*(e, E) = a.$$

Remark. In §§ 10 and 11 we shall consider the question of possible values of the upper density of irregular sets. By an argument similar to the one employed there, it is not difficult to see that the upper density of the sets which we have been considering is 1 at almost all points.

6. In defining the set for which $D_*(e, E) = \frac{1}{2}$, almost everywhere, we have constructed the second line at a distance $\frac{1}{2}l\sqrt{3}$ from the first, where l is the length of one of the rectangles obtained by the operation. If we denote the distance between the lines by βl and let β have a value less than or equal to $\sqrt{2}$, then the lower density at almost all points increases as β decreases until β equals $\frac{1}{2}\sqrt{3}$. For values of $\beta \leqslant \frac{1}{2}\sqrt{3}$ the lower density is constant and equal to $\frac{1}{2}$ so long as β is not actually zero. In order to see this, it is sufficient to consider points which are near to the ends of R_i, instead of points which are near to its centre. By a method similar to that adopted in §§ 4 and 5 it may be proved that, corresponding to almost all points e of E, there exist sequences of rectangles

$$R_{i_1},\ R_{i_2},\ R_{i_3},\ \ldots,\ R_{i_p},\ \ldots \quad (i_1 < i_2 < i_3, \ldots),$$

such that e is interior to each R_{i_p} $(p = 1, 2, 3, \ldots)$ and situated within a distance l_{i_p}/n_{i_p+1} of its end.

If $\beta = 0$ the lower density becomes 1 and the set regular. *This suggests that the upper bound of the lower density of irregular sets is exactly* $\frac{1}{2}$.

7. We give the following alternative construction for a set in which the lower density at almost all points is $\frac{1}{2}$.

Let $1 > \phi_1 > \phi_2 > \phi_3 > \ldots$ be a sequence of positive numbers tending steadily to zero such that $\sum_{i=1}^{\infty} \sin \phi_i$ diverges, and let $9 < n_1 < n_2 < n_3 \ldots$ be a sequence of positive integers such that $\sum_{i=1}^{\infty} \frac{1}{n_i}$ is divergent. Further

choose n_i and ϕ_i so that $n_i \sin \phi_i$ tends steadily to ∞ as $i \to \infty$, the choice of ϕ_1 and n_1 being such that

$$n_1 \sin \phi_1 > 3.$$

We define the operation $\Omega(n, \phi)$ as follows, P having the same significance as in § 2. Divide the line L joining the midpoints of the shorter sides of P into n equal parts (n denoting a positive integer), and, with the midpoints of the n segments so obtained as centres, construct rectangles with their longer sides making an angle ϕ with L, each of length l/n and breadth l/n^2.

Now, beginning with P_0 as before, perform on it the operation $\Omega(n_1, \phi_1)$, denoting the n_1 rectangles so obtained by $P_1 = \{R_1\}$. On each R_1 of P_1 we operate with $\Omega(n_2, \phi_2)$ and obtain a new system $P_2 = \{R_2\}$ of $n_1 n_2$ rectangles. In general we apply the operation $\Omega(n_i, \phi_i)$ to each R_{i-1} of P_{i-1} and denote the new set by $P_i = \{R_i\}$. n_i and ϕ_i are chosen so that P_i ($i = 1, 2, 3, \ldots$) form a system of non-overlapping rectangles completely interior to P_{i-1}.

Let E denote the set

$$P_0 \times P_1 \times P_2 \times P_3 \times \ldots,$$

where P_i includes the boundaries of each R_i as well as its interior. This set has the following properties :—

(*a*) $LE = 1$;

(*b*) $D_*(e, E) = \frac{1}{2}$, at almost all points;

(*c*) $D^*(e, E) = 1$, almost everywhere.

8. THEOREM 2. *There exist linearly measurable irregular sets of positive measure for which, at almost all points,*

$$D^*(e, E) = \beta, \quad \tfrac{1}{2} \leqslant \beta \leqslant 1.$$

Besicovitch† has proved that $\beta = \frac{1}{2}$ and $\beta = 1$ are the exact bounds of the upper density, and he has given examples in which these values are attained.

We proceed to construct a set for which, at almost all points, $D^*(e, E) = \beta$, the method depending on whether

$$\tfrac{1}{2} < \beta \leqslant 1/\sqrt{2} \quad \text{or} \quad 1/\sqrt{2} \leqslant \beta < 1.$$

Case 1. Let P be an ellipse of eccentricity a, whose major axis is of length $2a$. We divide the angle at its centre into $2(2n+1)$ equal parts

† *Loc. cit.*, 431 *et seq.*

in such a way that the minor axis is one of the dividing lines, and denote the points of intersection of these lines with the ellipse by $P_0, P_1, P_2, \ldots, P_{4n+1}$. At P_i $(i=0, 1, 2, 3, \ldots, 4n+1)$ construct ellipses of eccentricity a, with major axes of length $a/(2n+1)$, which touch P internally at P_i at one end of their major axes. By means of this operation, which we denote by $\phi(n)$, we construct a system of $2(2n+1)$ non-overlapping ellipses, having the same eccentricity as the original, each member of the system touching it internally, two of the points of contact being the ends of the minor axis of the original ellipse.

Now take an ellipse Γ_0, of eccentricity $a=\sqrt{(4\beta^2-1)}/(2\beta)$, whose major axis is two units in length, and perform on it the operation $\phi(1)$. We obtain a system Γ_1 of $2(2+1)$ non-overlapping ellipses, any one of which we denote by γ_1. On each γ_1 of Γ_1 we perform the operation $\phi(2)$, obtaining a system Γ_2 of $2^2(2+1)(4+1)$ ellipses, any member of which we denote by γ_2. In general the system of ellipses obtained by the operation $\phi(n)$ on each γ_{n-1} of Γ_{n-1} is denoted by $\Gamma_n = \{\gamma_n\}$.

Clearly Γ_n denotes a non-overlapping set, and Γ_{n+2} is interior to Γ_n for each n $(n=0, 1, 2, 3, \ldots)$. Further, if $2a_n$ and $2b_n$ are the lengths of the major and minor axes respectively of γ_n, we have

$$\left.\begin{aligned} a_n &= \frac{1}{2^n.3.5.7.\ldots.(2n+1)}, \\ b_n &= a_n/(2\beta), \quad \sum_{\Gamma_n} 2a = 2. \end{aligned}\right\} \tag{16}$$

Let $E=\Gamma_0\times\Gamma_1\times\Gamma_2\times\Gamma_3\times\ldots$, where Γ_n denotes not only the points interior to, but also on the boundary of each γ_n. Clearly the set E is closed and therefore measurable.

9. We now prove that $LE=2$.

Since Γ_n is a set of convex areas covering all points of E, it follows, from (16), that

$$LE \leqslant 2. \tag{17}$$

Corresponding to each point e of E there exists a sequence of ellipses containing that point and interior to $U(\rho, E)$. By *interior to* $U(\rho, E)$ we mean interior to at least one of the convex areas of which $U(\rho, E)$ is composed. It follows from Borel's theorem that we can find a finite set of ellipses covering all the points of E. If n be the greatest suffix which occurs in this set, then Γ_n possesses the same properties, viz., it is

interior† to $U(\rho, E)$, covers all the points of E, and the ellipses forming it are non-overlapping.

Let R_n be the set of ellipses of Γ_n which are entirely included in $U_i(E)$. If the number of γ_n belonging to R_n is greater than one, then there exists an integer $k \leqslant n$ such that the members of R_n belong to the same γ_{k-1} but to different γ_k.

Since $a_k < \rho$ it follows that $k \to \infty$ as $\rho \to 0$, uniformly for all $U_i(E)$.

If $k = n$, it is easily seen that

$$\delta \geqslant \sum_{R_n} 2a. \tag{18}$$

We suppose then that $k < n$.

The distance between two consecutive ellipses γ_k of γ_{k-1}, taken in order around the boundary of γ_{k-1}, is not, in general, constant as we proceed along the boundary, but increases as we pass from one end of the minor axis until one of the ends of the major axis is reached, when it begins to decrease. Thus the distribution of the γ_k is most dense around the ends of the minor axis of γ_{k-1}. In the first instance, then, in finding the value which δ must exceed, we assume that the γ_k which have points in common with $U_i(E)$ are distributed about one of the ends of the minor axis of γ_{k-1} until the distance between the points where the two most distant ellipses touch γ_{k-1} becomes equal to the distance of these points from the other end of the minor axis of γ_{k-1}. When we come to the case in which the number of γ_k is in excess of that required for this limiting position, we must assume them distributed around this end of the minor axis, until the distribution becomes symmetrical with respect to the axes of γ_{k-1}.

Let t be the number of ellipses γ_k which have points in common with $U_i(E)$. We first consider $t = 2$.

If O be the centre of γ_{k-1}, B and B' the extremities of its minor axis, P any point on its boundary, and θ the acute angle which OP makes with BOB', then

$$OP = a_{k-1}(1-\alpha^2)^{\frac{1}{2}}(1-\alpha^2 \sin^2\theta)^{-\frac{1}{2}}.$$

Denote by $\gamma_{k,1}$ the ellipse γ_k which touches γ_{k-1} at B, and by $\gamma_{k,2}$ one of the ellipses γ_k adjacent to it. Let P' be the point where the latter touches γ_{k-1}, N the foot of the perpendicular from P' on to BOB', P'' the point where the normal at P' meets $\gamma_{k,2}$ again, and N' the foot of the perpendicular from P'' on to $P'N$. Then the least distance between $\gamma_{k,1}$ and $\gamma_{k,2}$ exceeds $NN' - 2b_k$.

† By the set Γ_n being interior to $U(\rho, E)$ we mean that each member of this finite set is interior to $U(\rho, E)$ in the sense defined above.

Thus

$$\delta > a_{k-1}(1-\alpha^2)^{\frac{1}{2}}\left(1-\alpha^2\sin^2\frac{\pi}{2k+1}\right)^{-\frac{1}{2}}\sin\frac{\pi}{2k+1}-2b_k-2a_k\sin\frac{\pi}{2k+1}. \tag{19}$$

Since $a_k=\dfrac{a_{k-1}}{2(2k+1)}$ and $\theta-\dfrac{\theta^3}{6}<\sin\theta<\theta, \quad 0<\theta\leqslant\frac{1}{2}\pi,$ (20)

$$\delta > 2a_k\left[(1-\alpha^2)^{\frac{1}{2}}\left(\pi-1-\frac{\pi^3}{6(2k+1)^2}\right)-\frac{4}{2k+1}\right]$$

$$> 3a_k\left[1-\frac{5}{2k+1}\right] \quad \text{for } k\geqslant 3,$$

since $\sqrt{(1-\alpha^2)}\geqslant 1/\sqrt{2}$.

Thus, if $\sum\limits_{R_n}2a\leqslant 3a_k$,

$$\delta > \left(1-\frac{5}{2k+1}\right)\sum_{R_n}2a. \tag{21}$$

But if $$3a_k<\sum_{R_n}2a\leqslant 4a_k,$$

we may omit the second term in (19), and (21) is true in this case also.

It is now necessary to find the maximum value of t in order that we may assume that the γ_k are grouped around B only. We denote by ψ the acute angle between BOB' and the radius vector drawn from O to a point on the boundary of γ_{k-1}, where an ellipse γ_k touches γ_{k-1}. Then for the above assumption to be satisfied it is easily verified that ψ must satisfy the relation

$$\tan\psi\leqslant\frac{2\sqrt{3}}{\alpha^2+2}.$$

We may clearly suppose that t is odd, so that there are the same number of γ_k on either side of B. If t_0 be the greatest possible value of t corresponding to the k considered, then

$$\frac{t_0}{2(2k+1)}\leqslant\frac{1}{\pi}\tan^{-1}\frac{2\sqrt{3}}{\alpha^2+2}+\frac{1}{2(2k+1)},$$

acute angles only being considered. We have, since $1/\sqrt{2}\geqslant\alpha>0$,

$$\tfrac{3}{10}<\frac{1}{\pi}\tan^{-1}\frac{2\sqrt{3}}{\alpha^2+2}<\tfrac{1}{3}. \tag{22}$$

Consider now $2<t\leqslant 5<t_0$.

Write $t=r+s+1$, where $r=s$ if t be odd, and $r=s-1$ if t be even.

Then

$$\delta > a_{k-1}(1-\alpha^2)^{\frac{1}{2}}\left[\sin\frac{r\pi}{2k+1}+\sin\frac{s\pi}{2k+1}\right] - 2a_k\left[\sin\frac{r\pi}{2k+1}+\sin\frac{s\pi}{2k+1}\right]-2b_k$$

$$> 2a_k\left[\frac{t\pi-\pi-1}{\sqrt{2}}-\frac{t\pi-\pi}{2k+1}-\frac{6(t-1)^3}{(2k+1)^2}\right]$$

from (20), since $\frac{(t-1)\pi}{2k+1} < \frac{1}{2}\pi$ and $(1-\alpha^2)^{\frac{1}{2}} \geqslant \frac{1}{\sqrt{2}}$.

Thus
$$\delta > 2ta_k\left[\frac{t\pi-\pi-1}{t\sqrt{2}}-\frac{5}{2k+1}\right] \quad \text{for } k \geqslant 48,$$

$$> \left[1-\frac{5}{2k+1}\right]\sum_{R_n} 2a, \tag{23}$$

since $\sum_{R_n} 2a \leqslant 2ta_k$.

Next consider $6 \leqslant t \leqslant t_0$.

Then

$$\delta > 2a_k\left[\left\{(1-\alpha^2)^{\frac{1}{2}}(2k+1)-1\right\}\left\{\sin\frac{r\pi}{2k+1}+\sin\frac{s\pi}{2k+1}\right\}-(1-\alpha^2)^{\frac{1}{2}}\right]$$

$$\geqslant 2a_k\left[(1-\alpha^2)^{\frac{1}{2}}(2t-3)-\frac{2t-2}{2k+1}\right],$$

since $\sin\theta \geqslant 2\theta/\pi$, $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

Hence
$$\delta > 2a_k t\left[1-\frac{2}{2k+1}\right], \quad \text{for } t \geqslant 6,$$

$$\geqslant \left[1-\frac{2}{2k+1}\right]\sum_{R_n} 2a. \tag{24}$$

We finally consider $t_0 < t \leqslant k_0$, where we have written, for brevity, $k_0 = 2(2k+1)$.

Since
$$\frac{t_0+1}{k_0}\pi > \tan^{-1}\frac{2\sqrt{3}}{\alpha^2+2} > \tfrac{3}{10}\pi$$

from (22), we have
$$\frac{t_0}{k_0} > 0{\cdot}29$$

provided that $k_0 > 100$.

Further

$$\left(1-\alpha^2\sin^2\frac{t\pi}{2k_0}\right)^{-\frac{1}{2}}-\left(1-\alpha^2\sin^2\frac{(t-5)\,\pi}{2k_0}\right)^{-\frac{1}{2}}$$

$$\leqslant \frac{\alpha^2}{2(1-\alpha^2)^{\frac{3}{2}}}\left(\sin^2\frac{t\pi}{2k_0}-\sin^2\frac{(t-5)\,\pi}{2k_0}\right)$$

$$= \frac{\alpha^2}{2(1-\alpha^2)^{\frac{3}{2}}}\sin\frac{(2t-5)\,\pi}{2k_0}\sin\frac{5\pi}{2k_0}$$

$$< \frac{5}{k_0}(1-\alpha^2)^{-\frac{1}{2}}, \qquad (25)$$

since $$\frac{\alpha^2}{1-\alpha^2}\leqslant 1.$$

Thus $$\delta \geqslant 2a_k\left[k_0(1-\alpha^2)^{\frac{1}{2}}\left(1-\alpha^2\sin^2\frac{(t-5)\,\pi}{2k_0}\right)^{-\frac{1}{2}}-2\right]$$

$$> 2a_k t\left[k_0 t^{-1}(1-\alpha^2)^{\frac{1}{2}}\left(1-\alpha^2\sin^2\frac{t\pi}{2k_0}\right)^{-\frac{1}{2}}-\frac{5}{t_0}\right], \quad \text{from (25)}.$$

Since $\frac{t}{k_0}\leqslant 1$ and $\frac{\alpha^2}{1-\alpha^2}\leqslant 1$, it follows that

$$k_0 t^{-1}(1-\alpha^2)^{\frac{1}{2}}\left(1-\alpha^2\sin^2\frac{t\pi}{2k_0}\right)^{-\frac{1}{2}}\geqslant 1.$$

Hence, since $t_0 > 0{\cdot}29\,k_0$, we have

$$\delta > 2a_k t\left[1-\frac{20}{k_0}\right]$$

$$\geqslant \left[1-\frac{20}{k_0}\right]\sum_{R_n} 2a. \qquad (26)$$

From (18), (21), (23), (24), and (26), it follows that, in all cases,

$$\delta > \left[1-\frac{20}{k_0}\right]\sum_{R_n} 2a.$$

Thus $$\sum_{U(\rho,\,E)}\delta > \left(1-\frac{10}{2k+1}\right)\sum_{\Gamma_n} 2a$$

$$= 2\left(1-\frac{10}{2k+1}\right)$$

and $$LE \geqslant 2.$$

From (17) it follows that $LE = 2$.

In the same way we see that

$$L[E \times \gamma_n] = 2a_n.$$

10. We now prove that $D^*(e, E) \leqslant \beta$.

Corresponding to any point e of E there exists a sequence $\{\gamma'_n\}$, $n = 1, 2, 3, \ldots$, of ellipses to which it is interior. Let $c(e, r)$ be a circle with centre e and radius r, where $2a_n \leqslant r \leqslant 2a_{n-1}$. $c(e, r)$ has points in common with only one γ_{n-1}, viz. γ'_{n-1}. Then the expression

$$\frac{L[E \times c(e, r)]}{2r}$$

has its greatest value for the least value of r for which $c(e, r) \supset E \times \gamma'_{n-1}$. By considering the different positions in γ'_{n-1} which γ'_n may occupy for different points e we see that $L[E \times c(e, r)]$ remains constant and equal to $2a_{n-1}$, whereas r decreases from $2a_{n-1}\{1-O(1/n)\}$ to $2b_{n-1}\{1-O(1/n)\}$, the value of r being least when γ'_n is one of the ellipses which touch γ'_{n-1} at the end of the minor axis of γ'_{n-1}. Hence, for all positions of e,

$$\frac{L[E \times c(e, r)]}{2r} \leqslant \frac{2a_{n-1}}{4b_{n-1}}\left\{1+O\left(\frac{1}{n}\right)\right\}.$$

From (16) it follows that

$$D^*(e, E) \leqslant \beta.$$

11. Finally we show that, for almost all points e of E, there exist sequences of circles $c(e, r)$ such that, for $2a_n \leqslant r \leqslant 2a_{n-1}$,

$$\frac{L[E \times c(e, r)]}{2r} = \beta\left\{1-O\left(\frac{1}{n}\right)\right\} \quad (n \geqslant n_0).$$

Denote by γ_n^* a particular ellipse of Γ_n, and by γ'_i $(i > n)$ the ellipses γ_i which touch γ_{i-1} at the ends of the minor axis of the latter. We have

$$\sum_{\gamma_{n+1} \subset \gamma_n^* - \gamma'_{n+1}} 2a = 2a_n\left(1-\frac{1}{2n+3}\right),$$

$$\sum_{\gamma_{n+2} \subset \gamma_n^* - \gamma'_{n+1} - \gamma_{n+2}} 2a = 2a_n\left(1-\frac{1}{2n+3}\right)\left(1-\frac{1}{2n+5}\right),$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\sum_{\gamma_{n+j} \subset \gamma_n^* - \gamma'_{n+1} - \gamma'_{n+2} - \ldots - \gamma'_{n+j}} 2a = 2a_n \prod_{i=1}^{j}\left(1-\frac{1}{2n+2i+1}\right).$$

Since $\sum_{i=1}^{\infty} \frac{1}{2n+2i+1}$ diverges, we can find $j_0(n)$ such that, for $j \geqslant j_0(n)$,

$$\sum_{\gamma_{n+j} \subset \gamma_n^* - \gamma'_{n+1} - \gamma'_{n+2} - \cdots - \gamma'_{n+j}} 2a < 2a_n \eta', \tag{28}$$

where η' is an arbitrary positive number.

Write $$E'_{n,t} = E \times \gamma'_{n+t},$$

$$E_{n,t} = \sum_{\Gamma_n} E'_{n,t}.$$

We have, from (28),

$$L[E - E_{n,1} - E_{n,2} - \ldots - E_{n,j_0(n)}] < \eta'.$$

Write $$B_n = E - \sum_{t=1}^{j_0(n)} E_{n,t} \quad (n = 1, 2, 3, \ldots),$$

$$E' = E - B_1 - B_2 - B_3, \ldots.$$

Then, if we put $\eta' = \eta/2^n$, we have

$$L[E - E'] < \eta.$$

If e be any point of E', then e belongs to $E_{n,t}$, for some t, $1 \leqslant t \leqslant j_0(n)$ $(n = 1, 2, 3, \ldots)$. Thus there exists a sequence of ellipses $\{\gamma_{n_r}\}$ corresponding to each point e of E' such that e is interior to each member of the sequence and γ_{n_r} touches γ_{n_r-1} at the end of the minor axis of the latter. Now take the least value of r for which $c(e, r) \supset \gamma_{n_r-1}$. Clearly $c(e, r)$ has points in common with only one γ_{n_r-1}. Hence

$$\frac{L[E \times c(e, r)]}{2r} = \frac{2a_{n_r-1}}{4b_{n_r-1}} \left\{1 + O\left(\frac{1}{n_r}\right)\right\},$$

from which it follows that $D^*(e, E) \geqslant \beta$.

Then, from (27), it follows that we may delete the sign of inequality. This proves the theorem for $\frac{1}{2} < \beta \leqslant 1/\sqrt{2}$.

12. *Case* II. Suppose now that $1/\sqrt{2} \leqslant \beta < 1$.

We replace $\phi(n)$ by the operator $\psi(n)$ defined as follows. Divide the major axis of P into n equal parts, where n is an odd integer, and through the midpoints of the sections so obtained draw lines parallel to the minor axis, intersecting the ellipse P in $2n$ points. With these points as points of contact, construct, as in § 8, ellipses which touch P

internally, of eccentricity equal to that of P and with major axes of length a/n. Let n_1 be an odd integer satisfying $n_1 > 1/(1-\beta^2)$, Γ_0 an ellipse of eccentricity β, whose major axis is of length two units. Perform the operation $\psi(n_1)$ on Γ_0, and denote the system of $2n_1$ ellipses so obtained by $\Gamma_1 = \{\gamma_1\}$. On each γ_1 of Γ_1 we operate with $\psi(n_1+2)$ and denote the new system by $\Gamma_2 = \{\gamma_2\}$, and in general on each γ_{i-1} of Γ_{i-1} we perform the operation $\psi[n_1+2(i-1)]$, denoting the set of new ellipses by $\Gamma_i = \{\gamma_i\}$. The construction is such that there always exist two γ_i which touch the γ_{i-1} to which they are interior at the ends of the minor axis of the latter.

We write $E = \Gamma_0 \times \Gamma_1 \times \Gamma_2 \times \Gamma_3 \times \ldots$. Then, by methods similar to those adopted above, it may be shown that

(a) the set E is closed and therefore measurable,

(b) $LE = 2$,

(c) $D^*(e, E) = \beta$, almost everywhere.

ON QUADRIC REPRESENTATIONS OF THE LINES OF FOUR-DIMENSIONAL SPACE

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[Received 15 May, 1929.—Read 13 June, 1929.]

1. *Introduction.*

In the study of line-geometry in space of n dimensions S_n, a considerable simplification is usually introduced by a suitable birational representation of the lines of S_n by the points of some locus Σ. One such representation is well known*, namely that which represents the lines of S_n on the points of a locus M_{2n-2} of order $(2n-2)_n \div (n-1)$ immersed in a $[(n+1)_2-1]$, and this representation has the advantage of being without any exceptional elements whatever. It has also the disadvantage, however, for $n>3$, that the locus M has a high order and is not otherwise well known. Of other representations the most useful are those in which Σ is a flat space, and, even though these do introduce exceptional elements, they will clearly be very necessary for $n>3$.

We are concerned in this paper with birational representations of the ∞^6 lines of a [4] S_4 by means of the ∞^6 points of a [6] S_6; and we set out, in particular, to establish a method of obtaining representations of this kind in which the linear complexes of S_4 are represented in S_6 by quadrics. Two such quadric representations have already been obtained by Kantor† by a different method; but the first of these is only a particular case of a more general representation obtained here, and the second also arises naturally from our point of view. We discuss each case in some detail, and obtain the systems of complexes in S_4 which correspond to primes of S_6; and we also give quadric representations on

* See Segre, *Encyk. der Math. Wiss.*, III C.7, 964.

† *Journal für Math.*, 118 (1897), 105, 106. The representations there obtained are for the lines of $[n]$.

S_5 and S_4 respectively of the linear complex and the linear congruence in S_4, and a representation of the general quadratic complex in S_4 on a cubic primal in S_6. The results which we obtain are all capable of generalization to space of higher dimensions, and we keep to four dimensions chiefly in order to fix the ideas, but also because of the special interest of this case.

2. *The representation on* M_6^5. It is worth while at the outset to point out the connection of representations of our type with the well-established representation* of the lines of S_4 on the points of a quintic locus M_6^5 in [9]. This representation is set up by taking the ten coordinates of a line in S_4 to be proportional to those of a point in a [9], S_9 say; and it is found then that the ∞^6 points of S_9 so arising form a quintic locus M_6^5 whose prime sections represent the linear complexes of S_4. Any birational representation of the lines of S_4 on the points of a *flat* space S_6 implies then a birational representation of the points of M_6^5 on those of S_6; and, in the latter representation, prime sections of M_6^5 will be represented in S_6 by the primals of S_6 which correspond to linear complexes of S_4. In particular, therefore, any *line-point quadric representation of* S_4 *on* S_6 *is equivalent to a quadric representation of* M_6^5 *on* S_6, *and conversely.*

3. The method by which we propose to obtain quadric line-point representations of S_4 on S_6 can be summarized briefly as follows. We take a [4], S_4' say, and relate it homographically to the given [4] S_4, the two being supposed immersed in space of nine dimensions, S_9. We make correspond then to any line l of S_4 the solid, $[l, l']$ say, containing l and the corresponding line l' of S_4'; and this preliminary correspondence between lines of S_4 and solids $[l, l']$ of S_9 we call the *intermediate correspondence*. We then obtain a correspondence between lines of S_4 and points of S_6 by taking a section of the figure by a [6] S_6, and making correspond to any line of S_4 the point in which S_6 is met by the solid $[l, l']$. This correspondence we shall show to be birational and such that the linear complexes in S_4 are represented in S_6 by quadrics; there are various cases to discuss according as the given [6] is of general or special position. We begin now by examining the intermediate correspondence as defined above.

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* See Stephanos, *Comptes rendus*, 93 (1881), 578 and 633; also Palatini, *Atti. Ist. Veneto*, 59 (1899–1900), 861; and Segre, *loc. cit.*, note 605. The locus M_6^5 is implied in Stephanos' paper, though it is not explicitly referred to.

(We shall use dashed letters in this paper to denote space elements of S_4'—the corresponding undashed letters being always understood to refer to the corresponding elements in S_4.)

4. *The intermediate correspondence.* In this correspondence between lines l of S_4 and solids $[l, l']$ of S_9, the fact of fundamental importance is that *one and only one of the solids* $[l, l']$ *passes through any given point of* S_9, or, in other words, the solids $[l, l']$ just fill S_9. For let the given point be O. There passes, then, through O a unique transversal, POQ' say, of S_4 and S_4', meeting these latter in P and Q' respectively; and any solid $[l, l']$ through O must clearly contain this transversal. Such a solid, however, since it contains the points P, Q', must contain also the corresponding points P' and Q; and it is therefore uniquely determined by the four points P, Q', P', Q.

This consideration shows that the suggested line-point representation of S_4 on S_6 is birational. For any one of the solids $[l, l']$ meets S_6 in one point, and only one of them passes through a given point of S_6.

Consider now a linear complex of lines k in S_4; and let the locus of solids $[l, l']$ corresponding to lines of k be denoted by K, where K is clearly of dimension eight, *i.e.* a primal. The complex K, like any other linear complex, generates a self-correlation of the containing space S_4, the solid Π corresponding in this correlation to any point P of S_4 being that containing all the lines of the complex through P*. Let this self-correlation be denoted by the symbol $\{P, \Pi\}$, and let the image self-correlation of S_4' be denoted by $\{P', \Pi'\}$. These two together determine, then, a correlation, $\{P, \Pi'\}$ say, between S_4 and S_4', in which the solid of S_4' which corresponds to a point P of S_4 is that which corresponds to P' in $\{P', \Pi'\}$. Now it is easy to see that the locus K must contain every line joining P to a point of Π', *i.e.* it contains the solid $[P, \Pi']$. It is generated, therefore, by the ∞^4 such solids joining corresponding elements of two correlated [4]'s; and since any locus so generated is known to be a quadric, K must be a quadric†. To any linear complex in S_4, then, there corresponds in S_9 a quadric containing S_4 and S_4'.

We note now, however, that the correlation between S_4 and S_4' is not quite general, but is subject, in fact, to the condition that the solid Π' corresponding to P is constrained always to pass through P'. The quadrics K, then, are not the general quadrics through S_4 and S_4', but are restricted to contain every line PP' joining corresponding points of

* See Segre, *Encyk. der Math. Wiss.*, III C.7, 871.

† See Segre, "Studio sulle quadriche . . .", *Mem. Acc. Torino* (2), 36 (1884), 42.

S_4 and S_4'. Now these lines PP', being the joins of corresponding points of two homographic [4]'s, are known to generate a quintic locus* M_5^5, and this contains, besides S_4 and S_4', ∞^1 other [4]'s, R say, all met homographically by the lines PP'. *To linear complexes of S_4 there correspond therefore in S_9 quadrics through a rational quintic fivefold locus M_5^5, locus of ∞^1 [4]'s.*

Conversely, it is easy to see that any quadric through M_5^5 determines a correlation of the right kind between S_4 and S_4' and corresponds therefore to a linear complex in S_4.

5. *The line-point representations of S_4 on S_6.* Consider now the section of the figure by a [6], S_6 say, and the birational line-point representation of S_4 on S_6 thereby determined. To linear complexes of S_4 there will correspond in S_6 the sections by S_6 of quadrics through M_5^5, and these will all have in common the section by S_6 of M_5^5. This section will be a quintic surface in general, but it may also be a threefold locus. For, if S_6 be chosen so as to contain two corresponding planes in S_4 and S_4' respectively, it will contain every line PP' joining corresponding points of these planes, and all such lines will form a cubic locus V_3^3, containing ∞^1 planes, and common to S_6 and M_5^5.

It might be thought, then, that S_6 could be taken to contain a solid or a [4] of M_5^5; but it is found that in such a case the representation we are seeking breaks down completely. For each of the solids $[l, l']$ meets not only S_4 and S_4' but every generating [4], R, of M_5^5 in a line, so that it meets every solid of such a space in a point; and if, then, S_6 were taken to contain a solid Π of M_5^5, it would be met by every solid $[l, l']$ in a point of Π.

There arise therefore only two distinct line-point representations of S_4 on S_6, in which the linear complexes of S_4 are represented in S_6 respectively by

(i) quadrics through a quintic scroll F_2^5 (the section of M_5^5 by a [6]),

(ii) quadrics through a V_3^3 of planes.

The latter of these is much the simpler, and we deal with it first.

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* If the coordinates of a point P in S_4 be $(x_0, \ldots, x_4)$, and those of the corresponding point P' in S_4' be $(x_0', \ldots, x_4')$, then the locus of the line PP' will clearly be given by the equations

$$\frac{x_0}{x_0'} = \frac{x_1}{x_1'} = \ldots = \frac{x_4}{x_4'},$$

which indicate its properties as stated.

6. *The representation by quadrics through a V_3^3 of planes.*

We consider then the representation which is by means of quadrics in S_6 through a V_3^3 of planes. The system of these quadrics we call Φ. The representation is equivalent, as we have shown, to some representation of M_6^5 on S_6 (§ 2), in which the prime sections of M_6^5 are represented by quadrics of Φ. This latter representation we shall now show to be obtainable by direct projection.

In the first place V_3^3 lies in a [5] S_5, and quadrics of S_5 which contain it are known to form a net, any two of them meeting residually in a solid Π. There are thus ∞^2 of these solids Π, one through any arbitrary point of S_5, and they are clearly such that any Φ-quadric which is made to contain a point of Π contains the whole solid. There will correspond to them therefore only points of M_6^5, and it is easy to see then that to the whole of S_5 there will correspond only a plane α of M_6^5. There will correspond to the same solids, moreover, only lines of S_4, and it is likewise easy to see that these lines lie in a plane σ, so that the plane α corresponds on M_6^5 to the plane σ of S_4. Prime sections of M_6^5, then, which contain α are represented by primes of S_6, for each of the latter makes up with S_5 a quadric of Φ; and this means that the representation in question is obtainable by direct projection of M_6^5 from the plane α. Thus *M_6^5 projects birationally on to S_6 from any plane of itself which represents a plane of lines of S_4, and its prime sections project into quadrics through a V_3^3 of planes.*

It appears incidentally, moreover, that primes of S_6 correspond to linear complexes of S_4 which contain the fixed plane σ. It follows, therefore, that *linear complexes in S_4 which contain a fixed plane σ form a homaloidal system, i.e. any six of them have only one variable line in common.*

We can now derive properties of the line-point representation of S_4 on S_6 most easily from known properties of M_6^5 by projection.

In particular, to the planes of S_4 there correspond on M_6^5 ∞^6 planes which we may call α-planes, and these project from α into the ∞^6 planes of S_6, each of which contains a directrix line of V_3^3.

To the ∞^4 point-stars of lines of S_4, moreover, there correspond ∞^4 solids Π of M_6^5; and these, in general, project into solids each containing a plane of V, but ∞^2 of them meet α in a line and project into the directrix lines of V. The solids Π contain in all ∞^7 planes, all of which lie on M_6^5 and are quite distinct from the α-planes; we shall refer to them therefore as π-planes, and there correspond to them in S_6 planes which meet a plane of V_3^3 in a line. Incidentally, each π-plane corresponds in

S_4 to the section of a point-star of lines by a solid through its vertex, and we shall call any such section a *point-star solid* of S_4.

To the ∞^4 solids of S_4, finally, there correspond ∞^4 fourfold quadrics Ω on M_6^5 and these project into [4]'s of S_6, each of which meets V_3^3 in a quadric surface. Of the quadrics Ω, however, ∞^1 contain the plane a, and these project into the planes of V.

We shall show now how the present representation leads immediately to striking quadric representations of the general linear complex and the general linear congruence on S_5 and S_4 respectively.

6.1. *Quadric representation on S_5 of a linear complex of S_4.*

Let L be a given linear complex of S_4, and let it be represented on M_6^5 by the section, M_5^5 say, of M_6^5 by a prime S. Let L, in addition, contain the fundamental plane σ of S_4 (this involves no loss of generality), so that S contains the plane a, and M_5^5 projects from a on to a prime, Σ say, of S_6. The lines of L, then, are represented on the points of Σ, and the sections of L by the other line-spaces* of S_4 are represented on Σ by the sections of the quadrics Φ. These sections are quadrics in Σ whose base is the section of V_3^3 by Σ, *i.e.* a cubic scroll Γ. Thus *the general linear complex in S_4 can be represented on a* [5] Σ, *so that its sections by the line-spaces of S_4 are represented in Σ by quadrics through a cubic scroll.*

In this representation the ∞^3 planes of Σ which pass through the unique directrix line λ of Γ are projections of a-planes which lie on M_5^5, and they represent therefore the ∞^3 planes of S_4 contained in L.

The ∞^4 planes which each contain a generator of Γ are projections likewise of π-planes on M_5^5, and they correspond accordingly to ∞^4 point-star solids of L.

The directrix line λ, moreover, is the projection of a solid Π lying on M_5^5, and it corresponds in consequence to the *unique complete point-star of lines contained in L.* This illustrates the value of the representational method, which here, and in the following case also, sets in prominence the chief special feature of the represented variety.

6.2. *Quadric representation on* [4] *of a linear congruence in S_4.*

Consider now a linear congruence K defined as the system of lines common to two linear complexes in S_4. K is known to contain just one plane†, which we take to be the fundamental plane σ, and it will be repre-

* By line-spaces of S_4 we mean linear complexes of lines in S_4.

† See Castelnuovo, *Atti. Ist. Veneto* (7), 2 (1891), 855–901; the remark in question is on p. 865.

sented then on M_6^5 by a secundum section, M_4^5 say, which contains the plane a. M_4^5 projects, then, from a on to a [4], Λ say, of S_6, and in the resulting representation of K on Λ the sections of K by the line-spaces of S_4 will be represented in Λ by sections of the quadrics Φ. These sections will form the system of quadrics through the cubic curve C^3, in which V_3^3 is met by Λ. Thus *a general linear congruence of lines common to two linear complexes in S_4 can be represented on a* [4] Λ, *so that its sections by the linear spaces of S_4 are represented in Λ by quadrics through a rational cubic curve.*

In this representation the points of C^3 are projections of ∞^2 π-planes on M_4^5, and their neighbourhoods represent, therefore, ∞^1 point-star solids on K. It can easily be shown, moreover, that the vertices of these latter lie on a conic in σ and that their solids all contain* σ.

We remark, in conclusion, that the solid which contains C^3 lies in S_5, and it corresponds therefore to the plane a on M_5^5, and also to the unique plane σ of K in S_4. This is another example of the power of the representational method in pointing out the special elements of its subject.

6.3. *Representations of the quadratic complex of S_4.*

We now make our last use of the representation under consideration to find out what kind of primal corresponds in S_6 to a quadratic complex, K say, of S_4; and, in particular, denoting the primal in question by K', we seek to know the order of K' and the multiplicity on it of V_3^3.

The complex K, being quadratic, is represented on M_6^5 by the intersection, $[M, Q]$ say, of M_6^5 with a general quadric Q of S_9, and K' then is just the projection of $[M, Q]$ from the plane a. Now any general line of S_6 meets every Φ-quadric twice and meets S_5 once, so that there corresponds to it on M_6^5 a conic which meets a; while if the line meet V_3^3, there corresponds to it similarly a line not meeting a. The order of K', then, the number of its intersections with a line, is equal to the number of intersections of $[M, Q]$ with a conic of M, which is four; and the multiplicity of V_3^3 on K' is equal to $4-x$, where x is the number of free intersections of K' with a line of S_6 which meets V_3^3, and is therefore equal to the number of intersections of $[M, Q]$ with a line of M, which is two. Thus *to quadratic complexes of S_4 there correspond in S_6 quartic primals having V_3^3 as double locus.*

We can get, however, a still simpler representation of the general quadratic complex. For, if we take K to contain the fundamental plane σ of S_4, we lose nothing in generality, since the general quadratic com-

* Cf. Castelnuovo, *loc. cit.*, 864, 865.

plex is known to contain a finite number of planes, while at the same time the primal K' has to break up into S_5 and a cubic primal containing V_3^3 simply. Thus *the lines of a general quadratic complex K in S_4 can be represented in S_6 on the points of a cubic primal K' which contains a V_3^3 of planes; sections of K by linear complexes are represented by sections of K' by quadrics through V_3^3, and sections by quadratic complexes by sections of K' by quartic primals having V_3^3 as double locus.*

We return now to the other line-point representation of S_4 on S_6, which is by means of quadrics through a quintic surface F_2^5, the section of M_5^5 by S_6 (cf. § 5).

7. *The representation by quadrics through a quintic scroll F_2^5.*

We assume now that S_6 meets M_5^5 only in a surface, which will be of order five, F_2^5. F_2^5 then will be ruled, for S_6 meets every generating [4] R of M_5^5 in a line; and since the surface lies in addition in a [6], it must be a normal rational quintic scroll or some degeneration of such a surface. Of non-degenerate surfaces of this type there are known to be just two, of which one has a directrix conic, while the other has a directrix line*; but the former of these is, in fact, the more general, the latter being actually derivable from it by a special limiting process in which the directrix conic breaks up into a directrix line and a generator. Now the surface F_2^5 will not *in general* contain a directrix line; for any such line would also be a directrix line PP' of M_5^5, and S_6, since it meets S_4 and S_4' in lines only, will not in general contain any line PP'. Thus *in the most general case F_2^5 is the most general rational normal quintic scroll.*

The linear complexes in S_4, then, are represented in S_6 by quadrics through F_2^5, and we shall again denote the system of these quadrics by Φ.

7.1. *Special forms of F_2^5.* Before discussing the general case in detail, as we shall do later, it is of interest to note the various special cases obtained by taking S_6 in certain special positions, yet such as always to meet M_5^5 only in a surface. Among these we shall see that already obtained by Kantor.

We may specialize, in particular, in either of two ways :—

(*a*) By making S_6 contain one or more of the directrices PP' of M_5^5;

(*b*) By making S_6 contain one or more planes of M_5^5; and a specialization of either kind may or may not involve one of the other kind. As

* Cf. Bertini, *Introduzione alla geometria proiettiva degli iperspazi* (1923), 361.

regards (*a*) there are only two possibilities; for, if S_6 contains two directrices, PP' and QQ' say, the lines PQ and $P'Q'$ will be corresponding lines in S_4 and S_4' respectively, and every line joining a point of PQ to the corresponding point of $P'Q'$ will be a directrix line of M, all such lines forming a directrix quadric surface of M; while, if S_6 were taken to contain three directrices of general position, it would meet M_5^5 in a cubic threefold V We now give the five special cases that arise under (*a*) and (*b*).

(i) Let S_6 contain a plane π of S_4. Its residual intersection with M_5^5 will be a quartic scroll F_2^4. Now, if a rational quintic curve break up into a rational quartic and a line, the line must meet the quartic in one point. F_2^4 must therefore meet π in a line. Thus F_2^5 *breaks up into a plane and a rational quartic scroll* F_2^4 *meeting the plane in a line.*

(ii) Let S_6 contain a directrix PP'. F_2^5 will now have PP' as directrix line, and it will therefore be a rational normal quintic scroll of the second kind. Thus in this case F_2^5 *is a rational normal quintic scroll with a directrix line.*

(iii) Let S_6 contain a directrix PP' and a plane π of S_4. The point P, then, will lie in π, for otherwise S_6 would contain a solid of S_4, which is forbidden by hypothesis. As in (i), F_2^5 *breaks up into the plane* π *and a quartic scroll* F_2^4 *which has* PP' *as directrix line and which meets* π *in a line through* P.

(iv) Let S_6 contain a plane π of S_4 and a plane σ' of S_4'. It will also contain then a directrix PP' for which the point P is the intersection of the planes π, σ in S_4, and P' is the corresponding point in S_4'. F_2^5 *breaks up into the planes* π *and* σ' *and a cubic scroll* Γ *meeting each of* π *and* σ' *in a line and having* PP' *as directrix.* This is the case noted by Kantor.

(v) Let S_6 contain a directrix quadric surface Q of M_5^5 meeting S_4 and S_4' in corresponding lines l, l'. One system of generators of Q will consist of directrices of M_5^5 joining corresponding points of l and l'; and, since the lines in which S_6 meets the spaces R of M_5^5 must meet those directrices, those lines must form the other system of generators of Q. The residual cubic intersection of S_6 with M_5^5 must consist therefore of a triad of planes, each of which lies in a space R and meets Q in a generator of the second system. Thus F_2^5 *breaks up into a quadric surface* Q *and three planes meeting* Q *in generators of the same system.* If we had taken S_6 to contain three planes of M_5^5 we should have arrived at the same result.

7.2. *The general case—the reverse correspondence.*

In resuming our discussion of the most general case, the first and most natural inquiry is as to what system of line complexes in S_4 corresponds to the primes of S_6. Denoting the system in question, then, by Ψ, all we know about Ψ to begin with is that it is

(a) linear and of freedom six,

(b) homaloidal (in a sense already explained).

The first step towards obtaining further information is to find out what corresponds in S_6 to a plane pencil of lines λ in S_4. We consider first, then, the solids $[l, l']$ which correspond in S_9 to lines of λ; and these define themselves as the solids which join corresponding lines of λ and of the corresponding pencil λ' in S_4'. The solids in question generate therefore a quadric line-cone, and this is met by S_6 in a conic, which thus corresponds in S_6 to the pencil λ. Now this conic will meet any quadric of Φ altogether in four points; but, since a linear complex and a plane pencil in S_4 have in general only one line in common, only one of the four is free, and the remaining three must be at base points of Φ. *To any plane pencil of lines in S_4, therefore, there corresponds in S_6 a trisecant conic of F_2^5.* Now the order of a line complex in S_4 is, by definition, the number of lines which it has in common with a general plane pencil. For a complex of Ψ then, this will be two; for any prime of S_6 is met by a trisecant conic of F_2^5 in two points. Thus *the complexes of Ψ are quadratic.*

7.3. *The base congruence of Ψ.* We want next to find the system of lines common to all the complexes Ψ, *i.e.* the base congruence of Ψ. Any chord of F_2^5 corresponds to one such line; for, since it imposes only one condition on quadrics of Φ to contain it, every point of it must correspond to the same line l of S_4; and, since the chord meets every prime of S_6, l must belong to every complex Ψ. We have thus to find the system of lines in S_4 which correspond to chords of F_2^5. These latter clearly form an ∞^4 system and fill the ∞^2 solids of S_6 which contain the cubic curves of F^*.

We remark now that any chord T_1T_2 of F_2^5 lies in just one solid $[l, l']$, namely that which contains the two directrices p_1, p_2 of M_5^5 which pass through its end points T_1, T_2. The line of S_4 which corresponds to the chord will clearly be just that in which the solid in ques-

* F_2^5 can be represented on a plane by means of cubic curves which have a fixed node; the lines of the plane therefore represent ∞^2 rational cubic curves of F_2^5.

tion meets S_4, and this again is just that which joins the points P_1, P_2 in which p_1, p_2 meet S_4. All we have to do then is to find the locus of the points P_1, P_2, in which S_4 is met by directrices of M_5^5 which meet F_2^5; for this locus will be a surface Γ any chord of which will correspond in S_4 to some chord of F_2^5.

Consider then the directrices in question. Any one of them, since it meets F_2^5, lies in a secundum* through S_6; and, conversely, any directrix which lies in a secundum through S_6 clearly meets F_2^5. Consider any one of these secunda, S_7 say. It will meet S_4 in a plane, α say, passing through the line λ in which S_4 is met by S_6, and it will meet S_4' similarly in a plane β' through the line μ' in which S_4' is met by S_6. Let the lines of S_4' and S_4 which correspond to λ and μ' be λ' and μ respectively, and let the planes corresponding similarly to α and β' be α' and β. The only directrix of M_5^5, then, which lies in S_7, is the line joining the point of intersection P of α and β to the point of intersection P' of α' and β'. Now as the secundum S_7 varies about S_6 as base, the plane α describes a net of planes about the line λ as axis, while β describes a similar net about μ as axis, and there is clearly a (1, 1) homographic correspondence between the planes of the two nets. The point of intersection P of corresponding planes describes therefore a cubic scroll containing λ and μ, and this is the required surface Γ. In conclusion, then, we have established a (1, 1) relation, $\{T, P\}$ say, between the points T of F_2^5 and the points P of a cubic scroll Γ in S_4, such that to any chord T_1T_2 of F_2^5 there corresponds in S_4 the related chord P_1P_2 of Γ, and chords of Γ therefore form the base congruence of Ψ. Hence *to the system of primes of S_6 there corresponds in S_4 the system Ψ of quadratic complexes which contain the chords of a cubic scroll* Γ.

We shall often refer in the sequel to the relation $\{T, P\}$ between points of F and Γ.

The remaining details of the correspondence are now easily filled in.

In particular, *to any point-star of lines of S_4 there corresponds in S_6 an M_3^4 through F_2^5.* It is easily seen, in fact, that the locus of the solids $[l, l']$ in S_9, which correspond to the lines through a point P of S_4, is just the M_6^4 which projects M_5^5 from PP', and this M_6^4 is clearly met by S_6 in an M_3^4 as stated. To any point-star, moreover, whose vertex P lies on Γ there will correspond the M_3^4 which projects F_2^5 from the point P of F *related to* T (in the relation $\{T, P\}$).

Neighbourhoods of points of F_2^5. This last statement suggests an

* A secundum is a space whose dimension is one less than that of a prime.

inquiry into the homologues in S_4 of the neighbourhoods of the points T of F_2^5. To any point of the neighbourhood of such a point T there corresponds, as usual, a line of S_4; but to all those points of the neighbourhood which lie in a solid through the tangent plane to F at T there corresponds the *same* line, for they all impose one and the same condition on quadrics of Φ to contain them. Also, since any such solid meets F in one point not at T, it contains one generator of the cone M_3^4 which projects F from T. To the whole neighbourhood of T, therefore, there corresponds the point-star of S_4 whose vertex is the point P of Γ related to T. Thus, *in general, to neighbourhoods of points T or F_2^5 there correspond the point-stars of S_4 whose vertices are the related points P of Γ.*

Consider next the planes of S_4 and their representation in S_6. Every plane of S_4 contains ∞^2 lines and ∞^2 line-pencils, and to the latter there must correspond ∞^2 trisecant conics of F_2^5. Obviously, then, *the homologues in S_6 of the planes of S_4 are the trisecant planes of F_2^5; and, in particular, to any plane of S_4 whose points of intersection with Γ are P_1, P_2, P_3 there corresponds the plane of S_6 joining the three related points T_1, T_2, T_3 of F_2^5.* It follows, then, that any two points of S_6 represent intersecting lines of S_4 if, and only if, they lie in the same trisecant plane of F.

Consider, finally, the solids of S_4. Any one of these, Π say, contains ∞^4 lines to which there correspond in S_9 ∞^4 solids $[l, l']$ just filling the $[7]$ S_7 which joins Π to its homologue Π' in S_4'. S_7 clearly contains a V_4^4 of M_5^5, the locus of lines joining corresponding points of Π and Π', and it meets S_6 therefore in a secundum containing a rational normal C^4 of F. Thus *to solids of S_4 there correspond in S_6 secunda which each meet F_2^5 in a rational normal C^4.*

The complexes Ψ. We can now demonstrate simply the very special nature of the complexes Ψ. Any one of these, Ψ_0 say, corresponds to some prime Π of S_6 which meets F_2^5 in a quintic curve C^5, and to neighbourhoods of points of C^5 there correspond point-stars of S_4 whose vertices lie on the curve of Γ, C' say, related to C^5. Ψ_0, then, must contain each of these point-stars and so must have C' as singular curve. Moreover, we have shown that to any plane containing three points of F there corresponds the plane of S_4 which contains the three related points of Γ, so that to any trisecant plane of C^5 there corresponds a trisecant plane of C'. Trisecant planes of C^5, however, just fill the whole prime Π, there being just one of them through any point of Π^*, and

* Cf. G. Marletta, *Rend. di Palermo*, 19 (1905), 101.

trisecant planes of C' must therefore generate the whole complex Ψ_0. The order of the curve C', finally, is the number of its intersections with a solid, and is therefore equal to the number of intersections of C^5 with a secundum of S_6 which meets F in a rational C^4; and, since Π meets the C^4 in four points which must lie on C^5, the number in question must be four. Thus C' is a rational quartic on Γ and belongs to an ∞^6 system of such curves related to the ∞^6 prime sections of Γ. It is easy, moreover, to verify otherwise that Γ does contain an ∞^6 system of quartic curves, and that this system is complete. In conclusion, then, *the quadratic line-complexes in S_4 which contain the chords of a cubic scroll Γ form a homaloidal ∞^6 system Ψ, and each one of them consists of all the trisecant planes of a rational quartic curve on Γ.* The latter part of this result can also be deduced from the known fact that every quadratic complex in ordinary space which contains the chords of a rational cubic curve is tetrahedral, and this also shows incidentally that *each of the complexes Ψ is cut by any solid in a tetrahedral complex.*

8. *Two homaloidal systems of quadrics in S_6.* We may mention in conclusion two interesting homaloidal systems of quadrics in S_6 suggested by the preceding work. These both arise from the interpretation in S_6 of the fact, already noted, that linear complexes in S_4 which contain a given plane form a homaloidal system (cf. § 6).

In the first representation (§ 6) any plane of S_4 is represented in S_6 by a plane through a directrix line of V_3^3, and we get the theorem that *quadrics in S_6 which contain a V_3^3 of planes and a plane meeting V_3^3 in a directrix line form a homaloidal system.* This fact was noticed by Kantor (*loc. cit.*).

In the second representation (§ 7), however, any plane of S_4 is represented in S_6 by a trisecant plane of F, and we now find that *quadrics in S_6 which contain a rational normal quintic scroll F_2^5 and a trisecant plane of F_2^5 form a homaloidal system.*

THE LOCUS REPRESENTING THE LINES OF FOUR-DIMENSIONAL SPACE AND ITS APPLICATION TO LINEAR COMPLEXES IN FOUR DIMENSIONS

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[Received 25 May, 1929.—Read 13 June, 1929.]

Introduction.

The study of the theory of line systems in ordinary space is greatly clarified and simplified when the lines are regarded as represented by the points of a quadric in five dimensions. The theory of the linear and quadratic complexes in particular arises quite naturally from this point of view. The object of this paper is to study in detail the locus which similarly represents the lines of four-dimensional space, and to show how, from a knowledge of the properties of this locus and its sections, the theory of systems of linear complexes in four dimensions can be developed in a very simple manner.

The linear complex in four dimensions has been studied at length in an algebraical manner by Castelnuovo*, and the first section of this paper is devoted to a brief summary of his results for general cases, obtained here, however, by methods which are more synthetic than his. The results of this section are consequently not all new, but they do not appear to be very well known in this country.

In the second section of the paper the relations between the co-ordinates of a line in [4] are made the basis of a general study of the locus by points of which the lines are represented; in particular, it is shown how the locus can be generated by solids, and the nature of the flat spaces which lie on it is studied.

In the third section of the paper the various sections of the locus by linear spaces are examined in detail. Some of these loci have been

* *Atti del R. Ist. Veneto* (7), 2 (1891), 855–901.

studied by Scorza* from the point of view of loci whose curvilinear sections are elliptic, and a few of our theorems are proved in his paper by considerations different from those employed here. The methods here used are purely synthetic, and the properties of the locus are obtained directly, apart from any consideration of the corresponding theorems of [4]. It is believed that no account of the matter has been published as yet which is similarly self-contained.

The representation of the locus by the points of [6] forms the subject of the fourth section; this is obtained by projecting the locus from its own space. An indication is given of an extension to the representation of the lines of $[n]$.

The last section deals with certain enumerative results which are important in the study (from our point of view) of the quadratic complex of lines.

Certain fundamental theorems regarding the existence of algebraic complexes and the equations by which they are represented were proved by Severi in a paper† to which reference will be made when necessary.

The following notation will be adopted :—

$[n]$ will denote a space of n dimensions.

V_m^k will denote a manifold of dimension m and order k.

C_p^n will denote a curve of order n and genus p.

A "prime" is a linear space of one less dimension than the fundamental space in which we are working.

A "primal" is a manifold of one less dimension than the fundamental space, and is represented by a single equation in the coordinates.

V will denote the variety representing lines of [4].

$a, a_1, \ldots$ will denote solids lying on V.

$\alpha, \alpha_1, \ldots, \pi, \pi_1, \ldots$ will denote planes (of two kinds) lying on V.

1. *The linear complex and systems of linear complexes in* [4]‡.

1.1. *The null system and linear complex in* [4].

The method by which line coordinates are defined in [3] extends at

* *Annali di Matematica* (3), 15 (1908), 217.

† *Ibid.*, 24 (1915), 89–120.

‡ See Castelnuovo, *loc. cit.*, for a (fuller) algebraic treatment.

once to space of any number of dimensions. If $X = (x_0, x_1, \ldots, x_n)$ and $Y = (y_0, y_1, \ldots, y_n)$ be two points of $[n]$, the ratios of the quantities $p_{ik} = x_i y_k - x_k y_i$ $(i \neq k;\ i, k = 0, 1, \ldots, n)$ are unaltered by the substitution for x_i, y_i of the corresponding coordinates of any other two points of the line XY. Consequently p_{ik} may be taken to be line-coördinates in $[n]$. When, in particular, $n = 4$ we have ten linearly independent quantities $p_{01}, \ldots, p_{34}$; the relations connecting these are discussed below (2.1).

Using ξ to denote coordinates of primes and x to denote coordinates of points, a correlation of [4] can be expressed, with the usual summation convention, by the equations

$$\xi_i = a_{ik} x_k \quad (i = 0, 1, \ldots, 4),$$

which determine, for each point X, a prime Ξ. This correlation will be a *null system* if, for all points X, the prime Ξ which corresponds to X passes through X. This requires

$$\xi_i x_i = 0 \quad \text{or} \quad a_{ik} x_i x_k = 0,$$

which, being an identity, requires $a_{ii} = 0$, $a_{ik} + a_{ki} = 0$, for all i, k.

The determinant $|a_{ik}|$ then is skew symmetrical and has five rows, consequently it vanishes. The five expressions $a_{ik} x_k$ are thus linearly connected by a relation of the form $y_i a_{ik} x_k = 0$ where the numbers y_i depend only on a_{ik}. This shows that, for all values of x_i, the prime Ξ passes through the point $(y_0, y_1, \ldots, y_4)$, which we may call the "*centre*" of the null system*. Hence we have the result

1.11. *In any null system in* [4] *there is one point (in general) through which all the primes of the system pass.*

In this null system the point $X' = (x'_0, x'_1, \ldots, x'_n)$ is said to be *conjugate* to X if it lies in the prime corresponding to X. This requires $\xi_i x'_i = 0$ or $a_{ik} x_k x'_i = 0$. Since $a_{ik} = -a_{ki}$, this may be written in the form $a_{ik} x'_k x_i = 0$, showing that X is likewise conjugate to X'. The centre O of the null system is conjugate to every point X. We may define the aggregate of lines joining pairs of conjugate points in the null system as the lines of a *linear complex*. Through each point of [4] pass ∞^2 lines of the complex all lying in the prime corresponding to the point; one line of the complex lies in any general flat pencil. The coordinates of the line XX' above are connected by the relation $a_{ik} p_{ik} = 0$ (supposing that $i < k$, so that p_{ik} and p_{ki} are combined in one term), and

* This property is clearly valid in space of any even number of dimensions.

conversely it is clear that any linear relation between the line coordinates defines a null system. Since O is conjugate to every point of space we have

1.12. *The lines of a linear complex through an arbitrary point of* [4] *lie in a prime; there is one point O, however, such that every line through O belongs to the complex. O may be called the centre of the complex.*

It may happen that the a_{ik} are so chosen that the four-rowed minors of the determinant $|a_{ik}|$ all vanish; since $|a_{ik}|$ is skew symmetrical it follows that the three-rowed minors vanish as well. Instead of a singular point O for the null system we have a singular plane π, every point of which is conjugate to every point of space. The number of such special null systems (and corresponding linear complexes) is ∞^6 as opposed to ∞^9 in the general case. We conclude

1.13. *The special linear complex in* [4] *consists of all lines which intersect a given plane.*

1.2. *Systems of linear complexes.*

Consider the ∞^4 lines which are common to two linear complexes, and therefore to the pencil (∞^1) determined by these two*.

1.211. *In an arbitrary plane of* [4] *will lie one of these,* joining the vertices of the two flat pencils of lines of the two complexes which lie in the plane.

1.212. *Through a general point will pass ∞^1 lines forming a flat pencil* lying in the plane of intersection of the "polar" solids of the point with regard to the two complexes, meaning by the polar solid of a point the solid which corresponds to the point in the corresponding null system. The lines of the system which intersect a given line l are ∞^2 and lie in ∞^1 planes through the various points of l. If l' be a second line the lines of the system meeting l and l' lie in the solid (ll') and belong to the linear congruence of lines cut in this solid by the two complexes. Two lines of this congruence meet l and l'. Consequently

1.213. *The lines of the system which meet l form a quadric point-cone whose vertex is the unique point P such that the plane through P*

* Algebraically, if $a_{ik}p_{ik} = 0$, $b_{ik}p_{ik} = 0$ are the complexes, we consider the pencil $(\lambda a_{ik} + \mu b_{ik})p_{ik} = 0$.

containing the lines of the system which pass through P is the plane (Pl). Castelnuovo does not give this result.

The pencil of linear complexes containing the system of lines will be such that one of the complexes contains an arbitrary line of space. If X, Y are the centres of two of the complexes the line XY will lie in both and hence belong to the system. Supposing the centres of the complexes to be distinct, consider the locus of these points. It is a curve such that the cone of lines which projects it from one of its points lies in the polar solid of that point with respect to one (and therefore all) of the complexes whose centre does not lie at the point. The curve thus lies in ∞^1 solids, and is thus a plane curve. Let π be the plane of this curve. An arbitrary solid meets the system in a pencil of complexes of which two are special; and since the only sections of a linear complex in [4] which are special arise from solids through the centre of this complex it follows that two points of the locus of centres must lie on an arbitrary solid; these are the intersections of π with the directrices of the congruence in which the solid cuts the system of lines. Hence

1.22. *The locus of centres of the complexes of an ∞^1 system is a conic* (Castelnuovo, *loc. cit.*).

Leaving aside particular cases, we next consider the lines (∞^3) common to three (and hence to ∞^2) linear complexes. This system we shall call a *congruence*. We have clearly

1.231. *Through an arbitrary point passes one line of the congruence.*

Let l be any line and π a plane not meeting l. The ∞^1 lines of the congruence which meet l form a ruled surface whose order is the number of its generators which meet π. Consider the congruence as being determined by the intersection of an ∞^4 system S of the type just described with a linear complex C. The lines of S through any point P of l lie in a plane which meets π in a point Q; the lines of C through Q lie in a solid meeting l in a point P'. Conversely the lines of C through P' lie in a solid meeting π in a line l', and the lines of S which meet l' generate a quadric point-cone (1.213) meeting l in two points P. Thus there is a (2, 1) correspondence on l between the points P and P', and for three points of l the point P' will coincide with P. When this happens the line PQ belongs both to C and to S, and hence to the congruence which is their common part. Consequently

1.232. *The lines of a linear congruence which meet a given line generate a ruled cubic surface having the line as directrix.*

If two lines of the congruence pass through a point it is evident that all the lines of the pencil determined by them also belong to the congruence. Such a point is called a *singular point* of the congruence.

Since a complex of the ∞^2 system can be drawn to contain any two arbitrary lines it follows, by taking these to pass through a singular point, that

1.233. *The singular points of the congruence are the centres of the complexes which contain it.*

If the line l of (1.232) belongs to the congruence the cubic ruled surface must degenerate into three planes passing through l, each plane containing a pencil of lines of the congruence. The vertices of these pencils lie on l and are singular points. Any two complexes of the system determine a pencil, and the locus of the centres of the complexes of the pencil is a conic (1.22). The singular surface thus contains ∞^2 conics, of which any pair meet in a point, consequently it is the projection of a Veronese surface*. Hence (Castelnuovo)

1.24. *The lines of a linear congruence are the trisecants of a* V_2^4 *which is the locus of centres of the* ∞^2 *complexes containing the congruence.*

The ∞^3 system of linear complexes dependent linearly on four arbitrary ones will, in general, contain a finite number of complexes which are special (1.13). Since a linear complex is determined by nine lines, such a system will be determined by six. But six lines in [4] are met by five (associated) planes†. Thus the system contains five special complexes, and may be determined by any four among these. Since the lines meeting five associated planes generate a cubic primal‡ with ten nodes we have (Castelnuovo)

1.25. *The* ∞^2 *lines common to four linear complexes generate a cubic primal with ten nodes.* The points of the primal are the centres of the complexes.

The following results follow at once.

1.26. *The lines common to five linear complexes generate a quintic ruled surface (elliptic); the* ∞^1 *planes of the special complexes of the system meet this in cubic curves.*

1.27. *Six linear complexes have in common five associated lines.* Seven or more linear complexes have in general no common part.

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* Darboux, *Bull. Sci. Math.* (2), 4 (1880), 348.

† Segre, *Rend. di Palermo*, 2 (1888), 45-52.

‡ Segre, *Atti Acc. Torino*, 22 (1887), 791.

2. *The representation of lines of* [4] *by the points of a locus in* [9].

2.1. *The relations between the ten coordinates of a line.*

It may be verified immediately that the coordinates p_{ik}, as defined in (1.1), satisfy the following five identities :—

$$(2.11)\qquad \begin{cases} (0) \equiv p_{12}p_{34}+p_{13}p_{42}+p_{14}p_{23} = 0, \\ (1) \equiv p_{02}p_{34}+p_{03}p_{42}+p_{04}p_{23} = 0, \\ (2) \equiv p_{01}p_{34}+p_{03}p_{41}+p_{04}p_{13} = 0, \\ (3) \equiv p_{01}p_{24}+p_{02}p_{41}+p_{04}p_{12} = 0, \\ (4) \equiv p_{01}p_{23}+p_{02}p_{31}+p_{03}p_{12} = 0. \end{cases}$$

If, then, we take the ten coordinates p_{ik} as homogeneous coordinates in [9], these five identities will represent five loci in [9], and the lines of [4] will be in exact (1, 1) correspondence with the locus which is common to all the five. It is our present purpose to discuss in detail the properties of this locus.

For convenience we shall replace the coordinates in [9] by $(\xi, \eta, \zeta, \xi', \eta', \zeta', \vartheta, \phi, \psi, \tau)$ defined by

$$\xi = p_{03}, \quad \eta = p_{13}, \quad \zeta = p_{23}, \quad \xi' = p_{04}, \quad \eta' = p_{14}, \quad \zeta' = p_{24}, \quad \vartheta = p_{12},$$

$$\phi = p_{20}, \quad \psi = p_{01}, \quad \tau = p_{34}.$$

The equations (2.11) then become

$$(2.12)\qquad \begin{cases} \tau\vartheta = \eta\zeta'-\eta'\zeta, \\ \tau\phi = \zeta\xi'-\zeta'\xi, \qquad \vartheta\xi+\phi\eta+\psi\zeta = 0, \\ \tau\psi = \xi\eta'-\xi'\eta, \quad \vartheta\xi'+\phi\eta'+\psi\zeta' = 0. \end{cases}$$

2.2. *The locus representing the lines.*

Consider now the five equations (2.12). Each of them represents a quadric in [9] which, since its equation involves only six of the ten coordinates, is a cone with a [3] as its vertex or, as we shall say, a [3]-cone. The five equations are clearly linearly independent, since no product of two of the coordinates occurs in more than one term. Consequently they define a linear system of ∞^4 quadrics, all of which contain the variety V, which is their common part. These are, in fact, the only quadrics* which do contain V. We now prove three results.

2.21. *V is a V_6^5 and all the quadrics through V are* [3]*-cones.*

* D'Ovidio, *Atti Acc. Torino*, 12 (1877), 334-49.

2.22. *The vertices of these cones lie on V and generate it.*

2.23. *Any two of the* [3] *meet in exactly one point.*

In fact, the V_6^3, which is the intersection of the first three cones, splits up into the V_6^3 whose equations are $\tau = 0$, $\xi/\xi' = \eta/\eta' = \zeta/\zeta'$, and a residual V_6^5. If we multiply the first three equations of (2.12) by ξ, η, ζ, and add, we see that this residual V_6^5 must lie on the locus $\vartheta\xi+\phi\eta+\psi\zeta = 0$, which is the fourth cone. Similarly, it lies on the fifth. This V_6^5 is thus to be identified with the locus V common to all the five quadrics.

Any quadric belonging to the net defined by the first three cones of (2.12) has an equation of the form

$$Q \equiv \tau(a\vartheta+b\phi+c\psi)-a(\eta\zeta'-\eta'\zeta)-b(\zeta\xi'-\zeta'\xi)-c(\xi\eta'-\xi'\eta) = 0,$$

and we have the identity

$$aQ \equiv a\tau(a\vartheta+b\phi+c\psi)+(c\xi-a\zeta)(b\xi'-a\eta')-(c\xi'-a\zeta')(b\xi-a\eta).$$

If we now put

$$\Theta = a\vartheta+b\phi+c\psi, \quad \mathrm{T} = a\tau, \quad \mathrm{H} = a\eta-b\xi, \quad \mathrm{H}' = a\eta'-b\xi',$$
$$\mathrm{Z} = a\zeta-c\xi, \quad \mathrm{Z}' = a\zeta'-c\xi',$$

the equation of Q is

$$\mathrm{T}\Theta = \mathrm{H}\mathrm{Z}'-\mathrm{H}'\mathrm{Z},$$

while the remaining four equations of (2.12) become

$$\mathrm{T}\phi = \mathrm{Z}\xi'-\mathrm{Z}'\xi, \qquad \Theta\xi+\phi\mathrm{H}+\psi\mathrm{Z} = 0,$$
$$\mathrm{T}\psi = \xi\mathrm{H}'-\xi'\mathrm{H}, \quad \Theta\xi'+\phi\mathrm{H}'+\psi\mathrm{Z}' = 0.$$

This shows first that Q is also a [3]-cone, and secondly that we may replace the first equation by any linear combination of the first three without disturbing the form of the equations of V. Since the equations (2.11) are symmetrical it follows that this process may be repeated until we can identify Q with an arbitrary quadric of the system. These are thus all [3]-cones, and (2.21) is proved. It follows further that the ∞^4 cones form a completely symmetrical system, and hence, to prove (2.22) and (2.23), it is sufficient to consider particular cases. But the vertex of the cone $\tau\vartheta = \eta\zeta'-\eta'\zeta$ is the solid $\eta = \zeta = \eta' = \zeta' = \vartheta = \tau = 0$ which satisfies the other equations, and which meets the solid $\xi = \zeta = \xi' = \zeta' = \phi = \tau = 0$ in the point Ψ. Thus (2.22) and (2.23) hold good for these two cones, and consequently hold in general.

2.3. *The system of solids of V.*

2.311. It is important to note that *there can be no other solids on V than the ones obtained above.* For let σ be any solid of V which is not the vertex of a cone. At any point P of V there is a tangent [6] to V whose intersection with V contains all the linear spaces of V passing through P and, since V is a complete intersection of quadrics, this intersection contains no points of V which do not lie in such a linear space. Since V is a V_6^5 and contains ∞^4 solids, vertices of cones containing it, ∞^1 of these pass through P and lie in the tangent [6] to V there. If P lies in σ, these ∞^1 solids cannot all meet σ in planes, otherwise two of them would meet in points other than P, which, by (2.23), is impossible. Hence there is a point Q of σ not lying in any of these solids. The tangent [6] to V at P and Q both contain σ; if they have no other common part, the intersections of the solids through P with the solids through Q must all lie in σ; these solids must therefore meet σ in planes, which, as we have seen, is impossible. Hence the two [6] have further common points and lie in a prime. Let R be another point of σ which does not lie in any of the ∞^1 solids through P or Q, which are vertices of the cones of the system. The above argument shows that the tangent [6] at P, R, or at Q, R lie in a prime. We thus get ∞^3 [6], tangent to V at the points of σ, such that any two of them lie in a prime. Since these clearly have no common point outside σ they must all lie in one prime. (If all the [6] had in common the point O not lying in σ there would be ∞^2—at least—solids of V through O; the symmetry of the equations of V shows this to be impossible*.) This prime will then contain ∞^4 solids of V, ∞^1 at each point of σ; consequently it contains V or some portion of it. But V is clearly irreducible, and it cannot lie in a prime owing to the linear independence of the equations which represent it. Consequently the solid σ cannot exist.

The following important corollaries follow at once:—

2.312. *V contains no* [4] *or space of higher dimension*; otherwise it would contain [3] which would not be vertices of cones containing V.

2.32. *The intersection of V with its tangent* [6] *at P consists exclusively of the ∞^1 solids of V which pass through P*; for, if Q were a point of the section not lying in a solid through P, any solid of V through Q, since it meets all the solids through P but does not pass through P, must meet the tangent [6] at P in a plane α, and the solid $(P\alpha)$ passing

* Such a point O would be multiple on V and it has been shown by Severi (*loc. cit.*) that V contains no such points.

through Q would lie on V, contrary to the supposition that there is no such solid.

2.33. *The intersection of the tangent* [6] *at* P *with any solid not passing through* P *is a line*, for it contains ∞^1 points, one in each solid through P, and, by (2.32), cannot be a plane curve other than a line.

2.34. *Through every line of* V *passes one, and only one, solid of* V, lying in the tangent [6] to V at every point of the line.

2.4. *Generation of* V *by means of solids.*

We shall now show how, given six solids in [9], of which any two meet in a point, a system of ∞^4 solids is determined which have the properties mentioned above, and which are in (1, 1) correspondence with the points of [4].

The most direct proof is by showing how the solids define a system of quadric cones with equations of the form (2.11). The freedom of a [3]-cone in [9] with an assigned vertex is the same as the freedom of a general quadric in [5], that is twenty. One such cone will pass through twenty assigned points outside the vertex. For a cone to contain a solid meeting the vertex in one point it is enough to make it contain six further points of the solid. Consequently, exactly one cone can be drawn to have one of the six given solids as vertex and contain the other five. Denoting the six solids by 0, 1, ..., 5, and the intersection of the solids i, j by X_{ij}, where we suppose that $X_{ij}+X_{ji}=0$, the fifteen points so obtained are subject to six syzygies, which we may take in the form $\Sigma X_{ij}=0$ $(i=0, 1, \ldots, 5)$, of which five determine the sixth. Taking as fundamental simplex that formed by the five solids 0, 1, ..., 4, it is easy to see that the equation of the cone whose vertex is 0 and which contains the other five solids is precisely $p_{12}p_{34}+p_{13}p_{42}+p_{14}p_{23}=0$, which is the first equation of (2.11) Similarly the other four equations could be reproduced. Hence

2.41. *Six solids in* [9], *meeting in pairs, define* V.

We next show how to construct the solids by incidences. Since all the ∞^4 solids of V meet 0 it will be sufficient to construct the ∞^1 solids of V through any point of 0 (2.3). This we are enabled to do by using the result of (2.33), that the pencil of solids through any point of V is met by any other solid in the points of a line.

Denote by l_i the line (X_{0i}, X_{1i}). The ∞^1 solids through X_{01} meet the solid i $(i=2, 3, 4, 5)$ in points of a line, of which X_{0i}, X_{1i} are two; the line is consequently l_i. The ∞^1 solids of V through X_{01} are then those which meet the four lines l_i; one such solid passes through an arbitrary point P of l_2, being the intersection, in the [6] determined by the solids 0, 1, of the

three primes (X_{01}, P, l_4, l_5); (X_{01}, P, l_5, l_3); (X_{01}, P, l_3, l_4). We can thus get ∞^1 solids through X_{01} or any other point X_{ij}.

If R be a point on the line $(X_{01}X_{02})$ we can, by what has been said, construct one solid of the system through X_{12} which passes through R. This, with the solid 0, gives two solids through R from which, as above, the ∞^1 may be constructed. This extends the construction to any point on the line (X_{ij}, X_{ik}).

If Q is any point in the plane (X_{01}, X_{02}, X_{03}), and R the projection of Q from X_{03} on the line (X_{01}, X_{02}), we can construct the solids through R; Q is then in the same position as was R above, and the solids through Q can be constructed in an analogous manner.

Finally, if S be any point of the solid 0, and Q its projection from X_{04} on the plane (X_{01}, X_{02}, X_{03}), the solids through Q may be constructed as above, and hence those through S, considered as a point of (X_{04}, Q). Thus

2.42. *The ∞^4 solids of V may be constructed by direct incidence methods given any six of them.*

We next exhibit the ∞^4 solids in a form which shows their relation to the points of [4].

Of the ∞^1 solids through X_{01} one passes through the point $\alpha X_{20}+\beta X_{21}$ of the line l_2. The points where it meets $l_i\,(i=3, 4, 5)$ may be denoted by $a_i X_{i0}+b_i X_{i1}$. If these three points lie in a solid together with X_{01} and $\alpha X_{20}+\beta X_{21}$ we must be able to find values of λ, λ_i which satisfy

$$\lambda X_{01}+(\alpha X_{20}+\beta X_{21})+\sum_{i=3}^{5}\lambda_i(a_i X_{i0}+b_i X_{i1}) = 0,$$

taking into account the identities $\sum_{j\neq i} X_{ij} = 0$ $(i = 0, 1, \ldots, 5)$. This leads to

$$\lambda = \beta-\alpha, \quad \lambda_i = 1, \quad a_i = \alpha, \quad b_i = \beta.$$

Thus the five points

$$\left\{\begin{array}{l} X_{01}, \\ \alpha X_{20}+\beta X_{21}, \\ \alpha X_{30}+\beta X_{31}, \\ \alpha X_{40}+\beta X_{41}, \\ \alpha X_{50}+\beta X_{51}, \end{array}\right. \tag{2.431}$$

lie in a solid for all values of α/β, and the ∞^1 solids through X_{01} are obtained by varying this ratio.

It follows that the ∞^1 solids through the point $\alpha X_{20}+\beta X_{21}$ of the line l_2 contain the points

$$(2.432)\qquad \begin{cases} \beta X_{01}+\gamma X_{02}, \\ \alpha X_{10}+\gamma X_{12}, \\ \alpha X_{20}+\beta X_{21}, \\ \alpha X_{30}+\beta X_{31}+\gamma X_{32}, \\ \alpha X_{40}+\beta X_{41}+\gamma X_{42}, \\ \alpha X_{50}+\beta X_{51}+\gamma X_{52}. \end{cases}$$

As $\alpha:\beta:\gamma$ vary, this gives the ∞^2 solids which meet the line (X_{01}, X_{02}) [or (X_{10}, X_{12}) or (X_{20}, X_{21})].

Similarly the ∞^3 solids meeting the plane (X_{01}, X_{02}, X_{03}) are determined by the points

$$(2.433)\qquad \begin{cases} \beta X_{01}+\gamma X_{02}+\delta X_{03}, \\ \alpha X_{10}+\gamma X_{12}+\delta X_{13}, \\ \alpha X_{20}+\beta X_{21}+\delta X_{23}, \\ \alpha X_{30}+\beta X_{31}+\gamma X_{32}, \\ \alpha X_{40}+\beta X_{41}+\gamma X_{42}+\delta X_{43}, \\ \alpha X_{50}+\beta X_{51}+\gamma X_{52}+\delta X_{53}, \end{cases}$$

and finally the whole system of solids may be expressed in the form

$$(2.434)\qquad \begin{cases} \beta X_{01}+\gamma X_{02}+\delta X_{03}+\epsilon X_{04}, \\ \alpha X_{10}+\gamma X_{12}+\delta X_{13}+\epsilon X_{14}, \\ \alpha X_{20}+\beta X_{21}+\delta X_{23}+\epsilon X_{24}, \\ \alpha X_{30}+\beta X_{31}+\gamma X_{32}+\epsilon X_{34}, \\ \alpha X_{40}+\beta X_{41}+\gamma X_{42}+\delta X_{43}, \\ \alpha X_{50}+\beta X_{51}+\gamma X_{52}+\delta X_{53}+\epsilon X_{54}, \end{cases}$$

giving the six points in which the solid meets the six fundamental solids. By subtracting from the i-th expression $\xi\sum_j X_{ij}$ (which is zero on account of the syzygies), replacing $\alpha-\zeta, \ldots, \epsilon-\zeta$ by $\alpha, \ldots, \epsilon$, and choosing ζ so that $\alpha+\beta+\gamma+\delta+\epsilon+\zeta=0$, the expressions (2.434) take the more

symmetrical form

$$(2.44)\qquad \begin{cases} P_0 \equiv \quad \beta X_{01}+\gamma X_{02}+\delta X_{03}+\epsilon X_{04}+\zeta X_{05}, \\ P_1 \equiv \alpha X_{10} \quad +\gamma X_{12}+\delta X_{13}+\epsilon X_{14}+\zeta X_{15}, \\ P_2 \equiv \alpha X_{20}+\beta X_{21} \quad +\delta X_{23}+\epsilon X_{24}+\zeta X_{25}, \\ P_3 \equiv \alpha X_{30}+\beta X_{31}+\gamma X_{32} \quad +\epsilon X_{34}+\zeta X_{35}, \\ P_4 \equiv \alpha X_{40}+\beta X_{41}+\gamma X_{42}+\delta X_{43} \quad +\zeta X_{45}, \\ P_5 \equiv \alpha X_{50}+\beta X_{51}+\gamma X_{52}+\delta X_{53}+\epsilon X_{54}, \end{cases}$$

where $P_0+\ldots+P_5=\alpha P_0+\ldots+\zeta P_5=0$, so that only four of the points P_i are independent.

The form of this result shows that the solids are in $(1,1)$ correspondence with the points of a $[4]$ in which $(\alpha, \beta, \gamma, \delta, \epsilon, \zeta)$, subject to $\alpha+\beta+\gamma+\delta+\epsilon+\zeta=0$, are taken as coordinates. If $(P_0', \ldots, P_5')$ denote the same expressions with $\alpha', \beta', \ldots, \zeta'$ substituted for $\alpha, \beta, \ldots, \zeta$, the relation $\alpha' P_0+\ldots+\zeta' P_5+\alpha P_0'+\ldots+\zeta P_5'=0$ shows that the two corresponding solids of V meet in a point, thus verifying (2.23). Thus

2.45. *In the representation on V the ∞^4 solids of V correspond to the ∞^4 points of $[4]$; the point of intersection of two solids of V being the map of the line joining the corresponding points.*

2.5. *Interpretation of the preceding results.*

We now consider briefly the relation of the preceding results to the geometry of $[4]$. If two lines XY, XZ of $[4]$ have the point X in common, the four-rowed determinants in the matrix

$$(2.51)\qquad \begin{Vmatrix} x_0, & x_1, & x_2, & x_3, & x_4 \\ y_0, & y_1, & y_2, & y_3, & y_4 \\ z_0, & z_1, & z_2, & z_3, & z_4 \\ x_0, & x_1, & x_2, & x_3, & x_4 \end{Vmatrix}$$

having two identical rows vanish. Taking, for example, the determinant containing the last four columns, and using the Laplace development in terms of the first two rows, we obtain between the coordinates p_{ik}, p_{ik}' of the two lines the relation

$$(2.52)\qquad p_{12}p_{34}'+p_{12}'p_{34}+p_{13}p_{42}'+p_{13}'p_{42}+p_{14}p_{23}'+p_{14}'p_{23}=0,$$

showing that the points on V corresponding to the lines are conjugate with regard to the cone $p_{12}p_{34}+p_{13}p_{42}+p_{14}p_{23}=0$. Similarly, they are

conjugate with regard to the other cones of the system. The join of the points thus lies on all the cones, and consequently on V_4.

The points of the join of these points represent in [4] lines of the pencil whose vertex is X, and whose plane is XYZ.

The solids correspond, as we have seen, to the stars of lines passing through the various points of [4]; a plane lying in such a solid corresponds to the lines through the corresponding point of [4], which themselves lie in a solid. We proceed next to identify the loci of V which represent lines lying in the planes and solids of [4].

2.6. *The two systems of planes lying on V.*

The ∞^7 planes which lie in the solids of V will be called the planes of the *first* system on V. There is, however, a second family of planes, ∞^6 in aggregate, which lie on V; we shall call these the planes of the *second* system. We shall use the letters α, π to denote planes of the two systems.

Let O be a point of V, and a_1, a_2, a_3 three solids through O. Let l be any line of a_1 through O. The [4] (la_2), (la_3), lying in the tangent [6] to V at O, meet in a plane π through l which intersects a_2, a_3 in lines through O. This plane contains three lines of V and lies entirely on it. Since every line of V passing through O lies in one of the solids through O (2.34), it follows that this plane π meets in lines each of the solids through O. All the planes of this kind through O lie in the tangent [6] at O; one such clearly passes through every line through O in the solid a_1. These planes π correspond to the lines which lie in a plane of [4]; two planes of [4] meeting in a line are represented by two planes π which intersect. We clearly have

2.61. *Through every line of V passes one plane of the second system; through every point pass ∞^2. A solid of V, which meets a plane of this system at all, meets it in a line. Two planes of this system may or may not intersect.*

2.7. *The V_4^2 of V representing lines lying in solids of* [4].

That the lines of a solid will be represented on V by the points of a V_4^2 lying in a [5] follows at once from the fact that the coordinates established for a line of the solid are the ordinary Plücker line coordinates We here give a direct geometrical proof of the existence of such quadrics on V, and show how one of them is determined by any two points of V.

First we prove

2.71. *The planes of the second system which meet in lines through O a plane α of the first system generate a quadric point-cone in* [4], *whose*

vertex is O, which is met by each solid through O in a plane of the first system.

Let a_1, a_2, a_3, ... be solids through O, and α_1 a plane of a_1 through O. The two [5] (a_1a_2), (a_1a_3), lying in the tangent [6] at O, have in common a [4] through α_1 which meets a_2, a_3 in planes α_2, α_3. If l is a line through O in α_1 the plane of the second system of V through l is the intersection of the [4] (la_2), (la_3) which lie respectively in the [5] (a_1a_2), (a_1a_3). Consequently the plane lies in the [4] in which these [5] intersect, and meets the solids a_2, a_3 (2.61) in lines of α_2, α_3. Thus, for each plane α_1 of a_1 passing through O we get a [4] containing α_1 and meeting all the solids through O in planes α_2, α_3, ..., and which further contains the ∞^1 planes of the second system through the lines of the pencil $(O\alpha_1)$, these meeting the solids a_2, a_3, ... in lines of the pencils $(O\alpha_2)$, $(O\alpha_3)$, ... The two systems of planes thus generate a quadric point-cone whose vertex is O, and one such cone is determined by each plane α_1 of the first system passing through O. This proves (2.71).

Now let P be any point, other than O, of such a cone, and α, π the planes of V lying on the cone, of the first and second systems respectively, which pass through OP. We can construct, as in (2.71), a point-cone whose vertex is P, which contains α; the point O and the plane π necessarily lie on this cone.

The [4] containing this point-cone thus meets the [4] of the cone whose vertex is O in the solid defined by the planes α, π. The two [4] thus lie in a [5]. We now prove

2.72. *This* [5] *is independent of the position of P on the point-cone whose vertex is O and meets V in a V_4^2.*

This result is clearly true if the point P be always taken to lie in α, for, if P, Q be two points of α, the two [5] determined by the pairs of cones whose vertices are O, P, and O, Q respectively have in common the [4] of the cone whose vertex is O and the plane of the second system through PQ; this does not lie in the [4] unless O, P, Q are collinear If O, P, Q are collinear, and R be a point of α not on OPQ, then the [5] determined, as above, by O, P, and O, Q coincide with that determined by O, R, and thus with each other. We may thus suppose P to be any point in α which we may choose at pleasure.

Let S be a point of the cone whose vertex is O which does not lie in α, and let π' be the plane of the second system through OS; this will lie on the cone and meet α in a line through O on which, as we have seen, we may suppose P to lie.

Let π, π_1 be two planes of the second system, through P, S respec-

tively, lying on the cones whose vertices are P, S, which respectively lie in a [5] with the cone whose vertex is O. The planes of the second system of V, which lie on the cones whose vertices are O, P, meet α in lines, and hence any one of the planes through O meets each of those through P. A similar result holds if we replace P by S. Now let π_0 be a fixed plane of the second system through O, meeting π, π_1 in L, L_1 respectively. The line LL_1 lies in π_0, and therefore on V; the lines PL, SL_1 lie in the planes of the first system of the cone, whose vertex is O, which pass through OP, OS; these lines are skew, since the planes of the first system containing them do not meet except at O, and π_0 can be chosen so that PL, SL_1 do not pass through O. Further, the line PS lies in π'. The solids of V through LL_1, PS are distinct, since the [4] containing the cone whose vertex is O does not contain a solid of V, lying, as it does, in the tangent [6] at O to V, and meeting all the solids through O in planes. Thus the solids through LL_1, PS meet in a point X. The lines XP, XL, PL all lie on V, and hence so does the plane XPL; it is of the second system, meeting in lines the solids through LL_1, PS. Similarly XSL_1 is a plane of the second system*. These planes do not lie in the [4] of the cone whose vertex is O, since this [4] contains no planes of V which do not pass through O. Thus X does not lie in this [4]. The two [5], determined respectively by O, P and O, S, contain the point X in addition to the [4] of the cone whose vertex is O, and consequently coincide. This [5] meets V in a locus, at every point of which is a [4] cutting the locus in a point-cone; it thus meets V in a V_4^2 of which the [4] are the tangent [4]†.

2.73. *One such* [5] *is, in fact, determined by any two points* A, B *of* V, *which do not lie on a line of* V. This accords with the fact that two skew lines determine a solid.

For the tangent [6] to V at A, B meet V in the ∞^1 solids through A, B respectively. Each of the solids through A meets the tangent [6] at B in a line (2.33), and *vice versa*; the two systems (∞^1) of lines so obtained clearly generate an ordinary quadric (V_2^2). The generators of this quadric join to A, B by planes of V. Let P be a point of the quadric, and l, m the generators through P. The four planes (Al), (Am), (Bl), (Bm) lie on V, and if (Al) is of the first system, so is (Bm), while (Am), (Bl) belong to the second system. The planes joining A to the generators of the quadric generate a point-cone in [4]; the point-cone

* In fact, XPL is the plane π and XSL_1 is π_1.

† Since the point-cones are complete intersections of V with their containing space the same is true of V_4^2.

constructed as in (2.71), with its vertex at P to contain the planes (Al), (Am), determines with the former cone a [5], independent of the position of P, cutting V in a V_4^2 (2.72). But the cone whose vertex is P must contain the plane (Bl) of the second system on V which passes through l: it therefore contains B. The [5] which we have constructed thus contains the [4] joining B to the quadric V_2^2. Thus A, B determine the [5] uniquely.

Two V_4^2, lying on V, must clearly have common points, in general ∞^2 in number. We may regard these V_4^2 as being determined by pairs of points A, B; A, C; A being common to both. Let a be a solid through A. The tangent [4] to the quadrics at A meet the quadrics in point-cones, which intersect a in two planes α, α'; these have a line l in common. The plane of the second system through l is then common to both quadrics (α, α' cannot coincide, since one of the point-cones determines a unique V_4^2). The two quadrics can have no further common part, for if X were another point common to both, since two V_4^2 pass through XA, the line XA must lie on V; in that case it must lie in the plane just found. Hence

2.74. *Two V_4^2 of V meet in a plane of the second system*; and, since two planes of the same system on a V_4^2 meet in a point,

2.75. *Three V_4^2 of V meet in a point.* If, then, from three points A, B, C of V we construct the V_4^2 defined by (B, C), (C, A), (A, B), these meet in a point D common to the three tangent [6] to V at A, B, C; and the lines DA, DB, DC lie on V. In [4] the last two theorems correspond to the facts that two solids meet in a plane and three meet in a line, while three arbitrary lines have a common transversal. A corollary of (2.75) is

2.76. *On V every V_4^2 meets every plane π*, as may be seen at once by drawing two V_4^2 through π to lie on V. This corresponds to the fact that a plane meets a solid in [4] in a line.

3. *The linear sections of V and the line systems which they represent.*

3.1. *Prime sections of V; the two species.*

To the points of the section of V by one or more primes will correspond the aggregate of lines common to one or more linear complexes. In this section and the following ones we obtain the principal properties of these sections.

3.11. Consider the ∞^2 solids of V which pass one through each line (2.34) of a plane π of the second system of V. *These solids lie in a prime.*

To see this we observe first that the intersection of any two of the solids lies in π, and is the intersection of the lines in which the solids meet π. Let P be any point of π, and a a solid of V which does not meet π. The ∞^1 solids through P meet π in lines through P and a in points of a line l (2.33). From a second point P' of π a line l' arises which meets l in the point where the solid through PP' meets a. By varying P in π, we get in a ∞^2 lines l of which any two meet. These cannot all pass through a point, since such a point would be common to all the ∞^2 solids, which have no common points except in π (2.23). The lines l thus all lie in a plane u. If b is a second solid not meeting π, a plane v is similarly determined in b. Any one of the ∞^2 solids thus meets π in a line, and u, v in points X, Y. If U be the point of intersection of a and b the plane XUY lies on V, and is of the second system; since this plane goes through U, it must meet a, b in lines through U (2.61). Since U can be taken arbitrarily on V, we can suppose the tangent [6] at U not to meet π; the solids a, b then fulfil the necessary conditions. No lines through U meet π and lie on V, so that XY cannot intersect π. The three planes u, v, π are independent, and hence determine a prime; this contains the ∞^2 solids meeting π in lines, since it meets such a solid in a line of π and two further independent points in u and v.

The linear complex represented by such a prime is special, as defined in (1.13). An immediate corollary of this result is

3.12. *Any ruled surface in* [4] *is represented on* V *by a curve of the same order.*

3.13. *The* ∞^2 *solids of* (3.11) *form the complete intersection of* V *with the prime* (Ω).

Let P be a point of V lying in Ω. If all the solids through P lay in Ω the [6] containing them would, of necessity, meet π in a point Q at least, and the solid through PQ would belong to the system. If this is not the case, some solid a through P meets Ω in a plane α. The ∞^2 solids intersecting π meet a (which is supposed not to meet π, not being one of the system) in the points of a plane α', which clearly coincides with α; one of these solids thus passes through P. The result stated follows.

3.14. *If* Ω *is any prime, the section of* V *by* Ω *contains at least one plane of the second system through every point**; for, if one of the solids

* In general only one; for if there are two there must be ∞^1 generating a point-cone and the prime would cut the tangent [6] at the point in a [5] whose intersection with V would be composite or else lie entirely in [4]; this will not happen in general.

through P meets Ω in a plane a, and we consider the quadric point-cone whose vertex is P and which contains a (2.71), this meets Ω again in a plane π of the second system passing through P, or else lies entirely in Ω.

Now let π_1, π_2 be two non-intersecting planes of the second system, and Ω_1, Ω_2 the primes containing the solids which meet them respectively in lines. Let π_1 meet Ω_2 in l_2, and π_2 meet Ω_1 in l_1. Since, by (3.13), the solids in Ω_1 form the complete intersection of Ω_1 and V, one solid in Ω_1 passes through an arbitrary point P of l_1. This solid therefore meets π_2 in a line through P (2.61) and lies in Ω_2, and, since it lies also in Ω_1, it meets π_1 in a line, which is consequently the line l_2. The line in π_2 is similarly l_1. Since in any prime section it is possible to find two planes π which are skew (the section of V being a V_5^5), we have

3.15. *Every prime section of V contains at least one solid of V.*

3.16. *Any two non-intersecting planes of the second system of V have a solid of V meeting both in lines.*

The second result corresponds to the theorem that two planes of [4] have a common point, while the first indicates that every linear complex possesses at least one singular point (*cf.* 1.12).

Now, taking a prime Ω to contain two solids of V, we show that it must contain ∞^2. (Clearly the number of such primes is less than or equal to ∞^8, so that the primes are not general.) Let a, b be the solids meeting in O. Then Ω clearly contains the tangent [6] to V at O and the ∞^1 solids lying therein. Let P, Q be two other points of the section of V by Ω not lying in the tangent [6] at O, and such that PQ does not lie on V^*. The tangent [6] at O, P, Q to V meet in a point T (2.75), and TP, TQ, TO lie on V. The point T lies in the tangent [6] to V at O, and hence in Ω; TP, TQ thus lie in Ω. The plane of the second system through TO lies in the tangent [6] at O to V, and hence in Ω. Consequently the tangent [6] to V at T, meeting Ω in the plane of the second system and the solid through TO, together with the lines TP, TQ, lies entirely in Ω. If X is any point of the plane π of the second system through TO and a_1, a_2 the solids of V through TX, OX, both these lie in Ω (being solids through T, O); thus the tangent [6] at X lies in Ω also. Hence we have

3.17. *Any prime containing two solids of V contains ∞^2 meeting a*

* This supposition requires that residual intersection is not a single flat space; this is true since V contains no [5] (2.312).

certain plane π in lines. There are thus ∞^6 such primes. Consequently it follows from (3.15) that

3.18. *The prime sections of V form two classes, those containing one solid and those containing ∞^2. The latter touch V at all points of a plane π.* We shall call them *tangent primes.*

3.2. *Further properties of the prime section of V.*

We will consider first the tangent primes to V.

3.21. *Any tangent prime to V contains ∞^1 [5] meeting V in V_4^2* (2.7).

Suppose the prime Ω to touch V along π (3.18). Since the prime contains all the solids of V meeting π, and the tangent [6] at all points of π, it contains all the point-cones whose vertices lie in π and which contain π; taking two of these cones, it follows from (2.72) that such a [5] as is required can be constructed. The ∞^1 such [5] arise from the ∞^1 point-cones which can be drawn through any point of π to contain π; these correspond to the ∞^1 planes of the first system passing through any line of π.

3.22. Conversely *every prime through a V_4^2 is a tangent prime.*

For let a be a solid of the prime; by (3.15) there is at least one such. This meets the [5] containing V_4^2 in a point P of V (at least), which thus lies on V_4^2. Denote this V_4^2 by ω. The ∞^1 planes of ω of the second system through P meet a in lines through P (2.61), consequently a meets ω in a plane α of the first system passing through P. The solid a and the [5] containing ω hence lie in a [6]. Let Q be a point of the prime section not lying in the [6]. The tangent [6] to V at Q meets a in a line l (2.33), and l meets a in a point R. The tangent [6] to V at R contains (i) the tangent [4] at R to ω, (ii) the solid a meeting the former in α, (iii) the line RQ which does not lie in the [6] (a, ω). These three determine the tangent [6] at R to V, which hence lies in the prime. Hence, by (3.17), the prime is a tangent prime.

3.231. *Any plane π_1 of the second system, lying in the tangent prime to V along π, meets π.* For let P be a point of π_1; the solid of the ∞^2 system through P meets π_1 in a line p. If Q be a point of π_1 not lying on p, the solid through Q meets π_1 in a line q. If R is the point where p meets q, two solids of the system pass through R, which can only happen if R is in π. Thus π_1 meets π in R.

3.232. *All the V_4^2 in a tangent prime contain the plane of contact π;* since π meets all the planes, ∞^3 in number, of the second system which lie on V_4^2.

In terms of the special linear complex these results give the following :—

3 . 24. *Every linear complex containing all the lines of a solid is special ; such complexes contain* ∞^1 *solids all passing through the singular plane of the complex.*

Next consider the section of V by a general prime. This contains just one solid of V, a say (3 . 15), (3 . 17). We prove that :—

3 . 25. *Any plane of the second system of* V *lying in this prime meets* a *in a line.* For suppose that π were a plane of the second system lying in the prime Ω and not meeting a. The ∞^1 solids through any point O of π meet a in the points of a line l (2 . 33) ; if O' is a second point of π, the line l' constructed in the same way for O' meets l in the point of intersection of a with the solid of V through the line OO'. We get, by varying O, ∞^2 lines in a, all meeting one another. If these all passed through a point P, the solid $P\pi$ would lie on Ω, and π would not be a plane of the second system. Consequently the lines lie in a plane α of a. There is then a correlation between the planes α, π, such that the planes joining lines of α to corresponding points of π lie on V. Since α, π do not meet by hypothesis, these planes generate a V_4^2. Ω thus contains a V_4^2 of V and is a tangent prime.

It is important to notice that V is a self-dual construct*, and that the system of ∞^6 tangent primes to V has properties strictly analogous to the system of points. Since each tangent prime corresponds to a plane of the second system (along which it touches V), we have a reciprocity between the points of V and the planes of this system, which is clearly the representation on V of the duality of line and plane in [4]. The correspondence can thus be expressed either in terms of spaces of [9] or in terms of elements of V. We can then construct the following table :—

3 . 26.

Point of [9]	Prime of [9]
Point lying on V	Tangent prime to V or plane π of contact
Line of [9]	[7] of [9]
Line lying on V	[7] which is the base of ∞^1 tangent primes
Solid lying on V	[5] containing a V_4^2 of V
Point of intersection of two such solids	Prime containing two V_4^2 or plane π common to both
∞^1 solids through this point	∞^1 V_4^2 lying in this prime and passing through π
$[x]$ meeting V in r points ($x = 1$, $r = 1, 2$; $x = 2$, $r = 1, 2, 3$).	$[8-x]$ lying in r tangent primes.

* A remark due to Mr. P. Du Val.

3.3. *Sections of V by* [7].

We now consider the sections of V by [7]; these will be V_4^5. Each [7]-section will be dual to a line (3.26), and the various [7]-sections possible will correspond to the various relations in which a line can stand in regard to V. Since V is the intersection of a number of quadrics, every trisecant line of V lies on V entirely. We can thus distinguish five cases of [7]-sections of V corresponding to lines as follows :—

(i)	General line not meeting V	General [7] not lying in any tangent prime
(ii)	Line meeting V in one point	[7] lying in one tangent prime
(iii)	Chord of V	[7] lying in two tangent primes
(iv)	Tangent line to V	[7] lying in two consecutive tangent primes
(v)	Line lying on V	[7] lying in a pencil (∞^2) of tangent primes

Take, first, the general case (i), in which none of the primes of the pencil of which the [7] is the base touches V. Let Ω_1, Ω_2 be two of these primes, and a_1, a_2 the solids of V lying in them (3.18); these solids meet in a point O_3 which is common to the two planes a_1, a_2 in which the solids meet the [7]. Let Ω_3 be a third prime through the [7], and let a_3, the solid in Ω_3, meet a_1, a_2 in O_2, O_1. The plane $O_1O_2O_3$ lies in the [7] and is a plane of V, since it contains the three lines O_2O_3, etc., lying in the solids a_1, etc., of V. This plane is thus of the second system on V, meeting a_1, a_2, a_3 in lines. There can be no other plane of the second system in the [7], for, if π were such a plane, the prime Ω_1 through π not being special, π must meet a_1 in a line (3.25); similarly π meets a_2, a_3 in lines, and coincides with $O_1O_2O_3$. Thus the plane $O_1O_2O_3$, or π, is symmetrical with regard to the primes through the [7] and the planes of the first system lying in the [7], which are sections of solids in these primes. [There can be no other planes of the first system in the [7], since, if α were such a one, the prime containing the [7] and the solid through α not being special, α must meet π (3.25).] These planes meet π in lines, and envelop a curve which is, in fact, a conic. For, if three of the lines l_1, l_2, l_3 passed through some point O of the plane, and a_1', a_2', a_3' were the solids of V through l_1, l_2, l_3, the [7] must cut these solids in planes a_1', a_2', a_3' through O. These do not lie in a [4], since, if they did, the ∞^1 planes of the second system through O* lying in this [4] would lie on V_4^5 (which we have seen not to be the case), and since none of the primes through the [7] is special, the tangent [6] to V at O does not lie in the [7]. It

* Clearly if a [4] contains three planes α of V through O it must contain a point-cone (2.71).

must thus meet the [7] in a [5] through a_1', a_2', a_3'; this [5] will then meet all the ∞^1 solids through O in planes. Since each such solid determines a prime through the [7], all the ∞^1 solids in the primes through the [7] pass through O; this is impossible, as three of them meet π in the sides of a triangle $O_1O_2O_3$. Thus the class of the envelope in $\pi < 3$, and it clearly is at least 2, since two of its lines go through O_1, O_2, O_3. Hence we have the theorem

3.31. *The general section of V by a* [7] *contains one plane π of the second system, and ∞^1 planes α of the first system meeting π in the tangents of a conic* [*cf.* (1.22)].

3.32. If one of the primes Ω_1 through the [7] touches V along a plane π_1, the line l_1, in which π_1 meets the [7], is a *double line* of V_4^5. If two of the primes are tangent to V we get two double lines; the system of lines in [4] which corresponds to this V_4^5 consists of the lines which meet two planes. The solid containing the two nodal lines lies on V, and is the unique solid of V meeting the two planes of contact of the primes (3.16). If the two primes are consecutive the two nodal lines approach coincidence, and we get in the limit a line of *binodes*. [Cases (ii), (iii), (iv) above.] The plane π through this line is a plane of ordinary nodes of the V_4^5, being the double plane of the tangent prime defining V_4^5.

3.33. It is true conversely that *every* [7] *containing a solid of V lies in two tangent primes to V*. For the dual of such a [7] is a line lying in a [5] meeting V in a V_4^2. This line meets V_4^2 in two points, distinct or coincident, and the corresponding V_4^5 has two distinct or coincident tangent primes through it, and hence contains two nodal lines or a line of binodes which is their limit.

3.34. Since every trisecant line of V lies entirely on V, it follows that, *if three tangent primes to V pass through a given* [7], *every prime through the* [7] *touches V*. It is clear that the pencil of primes determined by the tangent primes along two planes π_1, π_2 of V will contain other tangent primes if, and only if, π_1 and π_2 meet; it will then contain the ∞^1 tangent primes along the planes of the quadric point-cone determined in the [4] (π_1, π_2) by these two planes (2.71), and these will all contain the tangent [6] to V at the point of intersection of π_1 and π_2. Consequently

3.35. *A* [7] *lying in three tangent primes to V contains the tangent* [6] *to V at a point of V.*

3.36. *The residual intersection of such a* [7] *with* V *is a* V_4^2, for if Q is a point of the residual intersection, and P the point at which the tangent [6] to V lies in the [7], the V_4^2 determined (2.73) by P and Q meets the tangent [6] to V at P in a point-cone of [4] and contains Q; consequently it lies in the [7]. That there can be only one such follows from the fact that two V_4^2, since they meet in a plane, determine a prime (2.74).

3.37. Conversely, *any* [7] *through a* V_4^2 *meets* V *again in a tangent* [6] *section.* For if X, Y are two points of the residual intersection, the V_4^2 determined by X, Y meets the given V_4^2 in a plane π of the second system. The ∞^1 solids through X cut this quadric (X, Y) in its tangent [4] at X, this [4] meets π in a line l, and the plane (Xl) lies on V and is of the first system. Similarly, there is a line l' of π such that (Yl') is a plane of the first system. The tangent [6] at (l, l') then contains the tangent [4] to V_4^2 at the point and the lines joining it to X, Y; it thus lies in the [7]. Every prime through this [7], containing the ∞^1 solids through (l, l'), is a tangent prime (3.17).

The line system represented by such a [7] section consists of the lines meeting two planes which intersect in a line; these form two systems: (*a*) all the lines in the solid determined by the planes and (*b*) all the lines meeting their line of section. The former correspond to the points of V_4^2 and the latter to the residual V_4^3.

We deduce from the last three sections the following:—

3.38. *The* ∞^1 *solids through a point of* V *form a* V_4^3.

3.39. *The section of* V *by a* [7] *of type* (v) *above consists of a* V_4^3 *and a* V_4^2; *the vertex of* V_4^3 *lies on* V_4^2.

3.4. *Sections of* V *by a* [6].

The various [6] sections of V will correspond to the various planes which can be drawn in [9]. If a [6] lies in a tangent prime to V, the corresponding plane will meet V, and the section (V_3^5) by the [6] will have a double point where the plane of contact of the tangent prime meets the [6].

No plane can contain as many as four isolated points of V. For if A, B, C, D are four coplanar points of V, the tangent [6] to V at A, B, C meet in a point O (2.75), and the tangent [6] to V at O contains the plane ABC (passing necessarily through these points), and hence the line OD. The solid joining O to this plane thus meets the V_4^3 which is the section of V by the tangent [6] at O (3.38) in four lines only; this is impossible, since the variety is of the third order. It is also clear that

a plane containing a line of V cannot contain two further points of V without containing their join.

We can consequently enumerate sixteen distinct [6] sections of V, which are enumerated here and discussed below. Eight only of these are uncomposite and of dimension three.

Plane.	[6]-section.
(i) Plane not meeting V	V_3^5
(ii) ,, meeting V in one point	$V_3^5(C_2)$
(iii) ,, ,, ,, two distinct points	$V_3^5(2C_2)$
(iv) ,, ,, ,, two coincident	$V_3^5(B_3 +$ line of $C_2)$ B_3 lying on the line
(v) ,, ,, ,, three points distinct	$V_3^5(3C_2)$
(vi) ,, ,, ,, three points, two coincident	$V_3^5(C_2 + B_3 +$ line of $C_2)$ B_3 lying on the line
(vii) ,, ,, ,, three points, all three coincident	$V_3^5(B_4 +$ line of $B_3)$ B_4 lying on the line
(viii) ,, ,, ,, in a point and lying in the tangent [6] there	V_3^5(plane of $C_2 +$ conic of B_3), the conic lying in the plane
(ix) ,, ,, ,, a line	$V_3^3 + V_3^2$
(x) ,, ,, ,, a line and a point	$V_3^3 + V_3^2(C_2)$ (*i.e.* quadric cone)
(xi) ,, ,, ,, a line touching V at a point of it	$V_3^3 +$ point-cone V_3^2 through one of its planes
(xii) ,, ,, ,, a conic	$[3] + V_3^4$ of planes π
(xiii) ,, ,, ,, two intersecting lines	$[3] +$ two V_3^2 of planes π
(xiv) ,, ,, ,, two coincident lines	$[3] +$ two coincident V_3^2
(xv) Plane α of first system on V	$V_4^1 + [3]$
(xvi) Plane π of second system on V	V_4^3 (tangent [6])

Explanation. In (i)–(viii) the letters C_2, B_3, B_4 denote the singularities of V_3^5, denoting respectively the ordinary conical node, the binode, and the special binode whose edge lies on the locus*.

The degenerations of (ix)–(xv) are readily obtained by considering the dual representation, and noticing that a conic lying in a plane that does not lie on V lies on a V_4^2. (For if A, B, C, D are four points on the conic the V_4^2 determined by A, B; C, D if distinct meet in a plane π of V, the intersection of their containing spaces; these spaces, however, both go through the point $[(AB), (CD)]$, which does not lie on V. The V_4^2 thus coincide.)

The interesting and important cases, to which most attention will be given in the sequel, are (i), (v), and (xvi).

* I am indebted to Mr. P. Du Val for the first seven of these cases.

3.41. Consider first (xvi), the section of V by its tangent [6]; this consists of ∞^1 solids through a point P; we have defined these in (2.4) as the solids through P meeting four lines. *This configuration of solids is the projection from P of the configuration in* [5] *of the ∞^1 planes which meet four lines; these are known to form a V_3^3* containing ∞^2 lines meeting all the planes; the join of these lines to P gives the ∞^2 planes of the second system passing through P. We have thus a verification of (3.38).

The section of V by a general [6] will be a V_3^5. If P is any point of this, the tangent [6] at P to V will be met by the [6] considered in a solid which meets the V_4^3 of solids through P in three lines through P. Consequently three lines of V_3^5 pass through each point of the locus, and V_3^5 is generated by ∞^2 lines. The ∞^1 lines of V_3^5 meeting a given line l generate a ruled surface having l as double line, since two of the lines pass through each point of l. The tangent prime to V along the plane π of the second system through l meets the [6] in a [5] which contains all these lines. An arbitrary [4] of the [6] meets V in a quintic curve, and this prime section in five points. Hence five generators of the ruled surface meet an arbitrary [4] of the [6]. Consequently

3.421. *V_3^5 is generated by ∞^2 lines, three of these passing through each of its points. Along each line of V_3^5 there is a* [5] *touching V_3^5 along the line and meeting it in a rational ruled quintic surface with the line as a double directrix.* (Scorza, *loc. cit.*, p. 228.)

The singularities of (iv) and (vi) above arise by cutting the V_4^5 with a double plane and a binodal line in it by a prime, general or tangent to V. [See (3.32).] The case (vii) arises as the intersection of three "consecutive" tangent primes to V.

Since the plane of the second system, which lies in a general [7] section, and the ∞^1 planes of the first system meeting π in the tangents to a conic (3.31) are cut by a general [6] in a line and the lines of V_3^5 meeting it, it follows that

3.422. *The ∞^1 planes a generate a V_3^5 having π as a double plane.* This is case (viii) above. Since the envelope in π is a conic, it follows that

3.423. *There are two points on every line of V_3^5 for which two of the three generators coincide.* In the [7], any [6] through π meets the V_3^5 of planes in π, counted twice, and a residual V_2^3 which must necessarily

consist of three planes a, since the only intersection of the [6] with an arbitrary one of the planes a is the line in which a meets π. The three vertices of the triangle formed by the lines of intersection of a and π are double points of the section. Hence

3.43. *Any* [6] *containing a plane* π *of* V *meets* V *in a* V_3^5 *containing three other planes* a *meeting* π *in lines, and the vertices of the triangle formed by these lines are double points of* V_3^5.

3.441. *Any* [6] *section through a plane* a *of the first system has two nodes in* a, since the [7] which contains the [6] and the solid of V through a has two nodal lines (3.33).

3.442. *Conversely, since a binodal* V_3^5 *lies in two tangent primes, it contains the plane* a *in which the solid, meeting in lines the two planes of contact, meets the* [6] *containing* V_3^5 (3.16), (3.32).

Since, as we have seen, no plane can contain four isolated points of V, the greatest number of isolated double points which V_3^5 can have is three. Such a V_3^5 will be the intersection of three tangent primes. We consider its properties in detail; it represents the lines meeting three planes of [4].

Let Ω_1, Ω_2, Ω_3 be three primes touching V along π_1, π_2, π_3. The [6] which we consider is the intersection of Ω_1, Ω_2, Ω_3. Let P_1 be the point of intersection of π_1, Ω_2, Ω_3, and let P_2, P_3 be similar points. Each point P_i ($i = 1, 2, 3$) lies in all the primes Ω_i and on V; it thus lies on V_3^5. The line P_1P_2, in general, will not pass through P_3. There is a solid a_3 of V meeting π_1, π_2 in lines (3.16), and the line (a_3, π_1) is the same as the line (Ω_2, π_1) (3.16), and contains P_1. Similarly (a_3, π_2) contains P_2. P_1P_2 thus lies in a_3 and on V_3^5. The plane $P_1P_2P_3$ contains three lines of V and lies entirely thereon, it is of the second system, meeting a_3 in a line. The solid a_3 meets Ω_3 in a plane α_3 which, lying in Ω_1, Ω_2, Ω_3, lies in the [6] considered. The plane α_3 thus lies on V_3^5 and contains P_1, P_2. Consequently the V_3^5 is the same as that considered in (3.43), and we have

3.451. *The intersection of three tangent primes to* V *is a* [6] *meeting* V *in a trinodal* V_3^5 *containing one plane* π *and three planes* a *meeting* π *in the sides of the nodal triangle.*

The ∞^4 system of [3]-cones which define V is met by an arbitrary [6] in an ∞^4 system of point-cones through V_3^5, the ∞^2 lines of V_3^5 are vertices of the line-cones of the system, three of these lines passing through an arbitrary point of V_3^5. If we consider the V_3^5 discussed in

the last section, and take any line l of it, the line-cone projecting V_3^5 from l contains the three planes a_1, a_2, a_3 meeting in pairs in points; this is impossible, unless l meets one of the planes a.

3.452. *The lines of V_3^5 in this case form three distinct systems, the lines of one system meet one of the planes a, and one line of each system passes through an arbitrary point of V_3^5.* Let l be a line of the first system meeting a_1 in O. Consider the lines of the second system through the points of l; these meet a_2 and lie in the $[4]$ (l, a_2). (In general, l will not meet a_2.) The section of V_3^5 by the prime (a_1, a_2, l) consists of the three planes a_1, a_2, π, together with a residual quadric surface, which must consist of the lines of the second system meeting l as one set of generators and lines of the first system meeting all these as the other set. Hence we have

3.46. *The generators of one system (A) meeting a given line of another system (B) on this V_3^5 lie on a quadric surface, the other system of generators of this surface consisting of lines of system (B).*

3.471. *Through a general point O of a_1 passes only one line l of the first system on V_3^5;* for if there were two, l, l' say, the plane a_1 and the lines l, l' would lie in the tangent solid to V_3^5 at O; the plane (l, l') would thus meet a_1 in a line and lie on V_3^5, which is impossible, since V_3^5 is not a locus of ∞^1 planes. The ruled quintic surface (3.421) formed by lines of V_3^5 meeting l in this case consists of two quadric surfaces passing through l (3.46) and the plane a_1. The generator, passing through O, of the regulus formed by the lines of the second system on V_3^5 which meet l is clearly the line OP_3, P_3 denoting, as before, the double point of V_3^5 in which a_1 and a_2 meet. All the lines of the first system on this quadric thus meet OP_3. There is similarly a line P_3O' in a_2 which is a generator of the quadric of the same system as l.

3.472. *Thus the quadrics whose reguli consist of lines of two systems on V_3^5 determine, in the corresponding planes a_i, a_j, a relation between the two pencils lying in these planes and having as common vertex the point P_k where the planes meet. The planes joining corresponding lines in the planes generate a quadric point-cone whose vertex is P_k which is the tangent cone to V_3^5 at P_k.*

3.473. *If p_1 is a line of a_1, the tangent prime to V_3^5 along p_1* (3.421) *meets V_3^5 in the plane a_1, counted twice, and a ruled cubic surface having p_1 as directrix.*

3.48. The correspondence between the points of this V_3^5 and the lines meeting three planes β_1, β_2, β_3 in [4] may be studied. If Q_1, Q_2, Q_3 are the points of intersection of these planes in pairs, and the plane $Q_1Q_2Q_3$ be denoted by χ, we can easily verify that the correspondence is as follows :—

[4]	V_3^5
Line meeting β_1, β_2, β_3	Point of V_3^5
Line of χ	Point of π
Line through Q_i in solid $(Q_i\beta_i)$	Point of α_i
Line Q_iQ_j	Point P_k
Flat pencil in χ whose vertex is Q_i	Line P_jP_k
Flat pencil joining line of system to Q_i	Line of V_3^5 through point meeting α_i; and so on

We conclude with a mention of the cases in which V_3^5 contains two planes α_1, α_2 of the first system. There are three cases.

3.491. If α_1, α_2 do not meet, the intersection P of the solids α_1, α_2 containing α_1, α_2, does not lie in the [6]; but the ∞^1 solids of V through P meet the [6] in planes α, forming a V_3^3 lying in the prime determined by α_1, α_2 (3.41). The residual intersection of the [6] with V is then a V_3^2; if Q is a point of it, the tangent [6] to V at P, Q meet in a solid containing a quadric surface of V (2.73) meeting the [6] which we are considering in a conic. The lines of V_3^2 through Q thus form an ordinary quadric cone, and hence (in general) the V_3^2 which is the locus of Q is a general quadric. This is case (ix) above; (x) and (xi) can arise as special cases.

3.492. If α_1, α_2 meet in a point P, the [4] (α_1, α_2) lies entirely in the [6], and meets the section of V by the tangent [6] at P in the planes α_1, α_2, and a third plane π of the second system. Hence (3.43) we have the trinodal V_3^5 studied above.

3.493. If α_1, α_2 belong to the same solid α, the residual V_3^4 is a locus of planes of the second system, since, if P is a point of it, the ∞^1 solids through P meet α in the points of a line l, and the plane (Pl) is of the second system on V. This is case (xii), of which (xiii), (xiv) are special cases.

3.5. *Sections of V by spaces of dimension five and less.*

3.51. The general [5]-section of V is a quintic surface which is the complete intersection of five linearly independent quadrics, the residual

intersection of three of these being a cubic scroll (2.2). It is thus a *del Pezzo surface*. This contains ten lines, each meeting three others. This surface represents the lines common to ∞^3 linear complexes in [4]. The number of tangent primes to V which pass through the [5] is equal [dually (3.26)] to the number of points of V lying in a [3]; this is five. This verifies the result (1.25) that the lines of [4] are the transversals of five associated planes.

3.52. *Any* [5]*-section of* V *which contains a plane may always be regarded as a section of a trinodal* V_3^5 (3.451). For if the plane is of the second system any [6] through the [5] meets V in a trinodal V^5 (3.43); if the plane is of the first system it is sufficient to draw the [6] through the [5] and the plane of the second system through one of the lines of the plane.

Considering first the section of the trinodal V_3^5 by a [5] through the plane π of the second system, let l_1, l_2, l_3 be the lines in which π is met by the three planes a_1, a_2, a_3 of the first system of V_3^5, and let these meet in pairs in the three points P_1, P_2, P_3, which are double for V_3^5. Let A, B be any two points of the section not lying in π or on a line of V. The unique V_4^2 through A and B (2.73) meets π (2.76), and meets the [5] in a conic through A, B, meeting π. Hence the residual intersection of the [5] with V_3^5 is a surface such that through any two points of it pass a conic, lying on it entirely, and all the conics meet π. Thus

3.53. *The residual section of* V *by a* [5] *through a plane* π *is a Veronese quartic surface meeting* π *in a conic.*

If the plane through which the [5] be drawn be a_1, the [5] will, in general, meet π only in l_1. If A is any point of the residual intersection, one line of V_3^5 passes through A and meets a_1 (3.452); this line will lie in the [5]. The residual surface is therefore ruled. If l is a line of a_1, the lines of the first system on V_3^5 which meet l form a ruled cubic with l as directrix (3.473), which is met by the [5] in l and two generators. Thus

3.541. *The residual intersection of* V *with a* [5] *through a plane* a_1 *is a ruled quartic meeting* a_1 *in a conic.*

3.542. *If the* [5] *contain* π *as well as* a_1, *this conic becomes a line pair, and the quartic breaks up into* π *and the ruled cubic* (3.473) *formed by the generators of the first system of* V_3^5 *meeting the second line of the pair.* [*The first line is* (π, a_1).]

In the general case of the [5] through a_1, the [5] meets a_2, a_3 in lines through P_3, P_2. Hence

3.543. *The conic in which the ruled quartic meets V_3^5 passes through P_2 and P_3.* For the other sections of V we clearly have, by (3.51), (3.53), and (3.541),

3.551. *The general section of V by a [4] is a C_1^5.*

3.552. *If the [4] contain a line the residual intersection is a C_0^4 having the line as a chord.*

3.553. *If the [4] contain a plane π the residual intersection is a conic meeting π in a point.*

3.554. *If the [4] contain a plane a the residual intersection consists of two lines meeting a.*

3.56. *An arbitrary [3] meets V in five points.*

4. *The representation of V on a linear space.*

4.1. *Direct representation of V and its sections by projection.*

The general [4]-section of V is not a rational curve (3.551); but the V_2^5, which is its [5]-section, is known to be representable on a plane, so that its prime sections correspond to plane cubics having four fixed base-points. This representation is obtainable by projecting V_2^5 from any one of the ∞^1 conics which it contains.

Consider now the projection of V_3^5 from a conic of itself on to [3]. The general prime section of V_3^5, being a quintic surface and meeting the conic in two points, will project into a cubic surface. An elliptic quintic curve, the section of V_3^5 by a general [4], will not, in general, meet the conic, and will thus project into a C_1^5 in [3], which is the variable part of the intersection of the two cubic surfaces which correspond to the two primes which determine the [4]. The system of cubic surfaces in [3] have thus in common a fixed C_0^4. Hence

4.11. *The general V_3^5 can be represented in [3] by means of cubic surfaces through a rational quartic curve.* From this point of view the locus was considered by Scorza (*loc. cit.*).

Now consider the trinodal V_3^5 and consider the effect of projecting it from the plane of the second system π, which it contains. The del Pezzo surface, which is the prime section of V_3^5, is now projected from a line of itself, and becomes a quadric; the three planes of the first

system meeting π in lines give rise, in the [3] on to which V_3^5 is projected, to three points which are common to all the quadrics of the system. Thus

4.12. *The trinodal V_3^5 can be represented by means of quadrics through three points.* The lines of V_3^5 are represented, in the general case, by chords of the rational quartic, and, in the case of the trinodal variety, by lines through the fundamental points.

Since the sections of V by spaces of seven or more dimensions contain necessarily a plane π of the second system, we can get a representation of these varieties by projecting from such a plane. We thus find the following results :—

4.131. *V_4^5, the general [7]-section of V, is representable on [4] by quadrics through a C_0^3.*

4.132. *V_5^5, the prime section, is representable on [5] by quadrics through a cubic scroll.*

4.133. *V is representable on [6] by means of quadrics passing through a V_3^3 of planes lying in a prime.*

We do not get a representation of V by projecting from a plane of the first system, since the solid joining such a plane to a point P of V contains necessarily a line through P meeting the plane in its point of intersection with the tangent [6] at P. [The existence of this point follows from (2.33).]

The representation of V given in (4.133) gives a convenient parametric representation of V. For the general quadric of [6] through the V_3^3 whose equations are

$$u = 0, \quad \frac{x}{x'} = \frac{y}{y'} = \frac{z}{z'} \tag{4.141}$$

is of the form

$$u(\theta u + \alpha x + \beta y + \gamma z + \alpha' x' + \beta' y' + \gamma' z') + A(yz' - y'z) + B(zx' - z'x) + C(xy' - x'y) = 0.$$

Defining homogeneous coordinates in [9] by

$$\frac{\tau}{u^2} = \frac{\xi}{xu} = \frac{\eta}{yu} = \frac{\zeta}{zu} = \frac{\xi'}{x'u} = \frac{\eta'}{y'u} = \frac{\zeta'}{z'u} = \frac{\varsigma}{yz' - y'z} = \frac{\phi}{zx' - z'x} = \frac{\psi}{xy' - x'y}, \tag{4.142}$$

we have conversely

$$(4.143) \qquad \frac{u}{\tau} = \frac{x}{\xi} = \frac{y}{\eta} = \frac{z}{\zeta} = \frac{x'}{\xi'} = \frac{y'}{\eta'} = \frac{z'}{\zeta'}$$

and the five relations

$$\vartheta\tau = \eta\zeta' - \eta'\zeta,$$
$$\phi\tau = \xi\xi' - \zeta'\xi, \quad \vartheta\xi + \phi\eta + \psi\zeta = \vartheta\xi' + \phi\eta' + \psi\zeta' = 0,$$
$$\psi\tau = \xi\eta' - \xi'\eta,$$

which are the equations (2.12) of V.

This form is essentially that given by Severi (*loc. cit.*), with changes in the signs of certain coordinates.

The details of the representation (4.133) are easily worked out directly. The following results are easy to verify :—

4.151. *The ∞^4 solids of V are represented by the solids through the planes of V_3^3.*

4.152. *The ∞^6 planes π of V are represented by planes through the generating lines of V_3^3.*

4.153. *The lines of V_3^3 arise from solids of V meeting the plane of projection in lines.*

4.154. *The planes of V_3^3 arise from V_4^2 on V which contain π.*

4.155. *The lines in these planes arise from planes of the second system of V meeting π.*

4.156. *An arbitrary V_4^2 is represented by a $[4]$ meeting V_3^3 in a quadric surface*, since it meets the plane of projection in a point (2.76), and meets any V_4^2 of V which passes through π in a plane meeting π (2.74); consequently it projects into a $[4]$, meeting each plane of V_3^3 in a line (4.155).

The representation of the section of V by a tangent prime is obtained by projection from a plane of the second system meeting the plane of contact. It is easily found that

4.16. *The section (V_5^5) of V by a tangent prime is represented on $[5]$ by means of quadrics passing through a quadric surface and a plane containing one of its generators.*

It is found similarly that

4.17. *The V_4^5 in which V is met by two tangent primes is represented on $[4]$ by quadrics through three lines, of which one meets the other two.*

4.2. *Representation of V from considerations of line geometry.*

In a recent paper in these *Proceedings**, Mr. J. G. Semple has obtained the representation of V given above and several others by considering the representation of lines of $[4]$, using a method slightly more general than one developed by Kantor†. The other transformations obtained there are more complicated than the one already given, and we shall not have occasion to refer further to them.

4.3. *An extension to line-geometry in $[n]$.*

The equations (4.141)–(4.143) admit of obvious generalization to n dimensions. The variety V, representing the lines of $[n]$, is of dimension $2(n-1)$ and order $\frac{1}{n-1}\binom{2n-2}{n-2}$ ‡; it lies in $[\frac{1}{2}(n-1)(n+2)]$. It is generated by ∞^n $[n-1]$, of which any two meet, these corresponding to the points of $[n]$. If we consider the points

$$X_{rs} \quad (r \neq s, \ r, s, = 0, 1, \ldots, n+1),$$

where

$$X_{rs}+X_{sr} = 0,$$

$$\sum_j X_{rj} = 0 \quad (r = 0, 1, \ldots, n+1),$$

we have enough points to define the space, and the $n+2$ points

$$P_i = \sum_j \theta_j X_{ij} \quad (i = 0, 1, \ldots, n+1), \tag{4.311}$$

where the θ_j are $n+2$ homogeneous parameters whose sum vanishes, satisfy $\sum_i P \equiv 0$ and $\sum_i \theta_i P_i \equiv 0$ and lie in an $[n-1]$, which is one of the $[n-1]$ generating the variety. Two such $[n-1]$, given by "θ", "θ'", where "θ" denotes the set $(\theta_0 \ldots \theta_{n+1})$, etc., meet, since we have

$$\sum_i (\theta_i P'_i + \theta'_i P_i) = 0. \tag{4.312}$$

If, in the $[\frac{1}{2}(n-1)(n+2)]$, we take $\frac{1}{2}(n-1)(n+2)+1$ coordinates denoted by

$$x, \ y_1, \ y_2, \ \ldots, \ y_{n-1}, \ z_1, \ z_2, \ \ldots, \ z_{n-1}, \ u_{pq} \quad (1 \leqslant p < q \leqslant n-1),$$

* *Supra*, 500–512.

† *Journal für Math.*, 118 (1897), 105 *seq.*

‡ Schubert, *Mittheil. der Math. Ges. Hamburg*, 1 (1884), 87; Meyer, *Math. Annalen*, 21 (1883), 125.

and put

$$(4.321)\qquad x=\theta^2,\quad y_r=\theta\phi_r,\quad z_r=\theta\psi_r,\quad u_{pq}=\phi_p\psi_q-\phi_q\psi_p$$
$$(r=1,\ \ldots,\ n-1;\ \ 1\leqslant p<q\leqslant n-1)$$

(where, if $p>q$, $u_{pq}=-u_{qp}$), we have $\binom{n+1}{4}$ quadratic relations

$$(4.322)\qquad \begin{cases} xu_{pq}=y_pz_q-y_qz_p, \\ y_iu_{pq}=y_pu_{iq}-y_qu_{ip} & (i\neq p,\ q), \\ z_ju_{pq}=z_pu_{jq}-z_qu_{jp} & (j\neq p,\ q), \\ u_{pq}u_{rs}=u_{pr}u_{qs}-u_{ps}u_{qr} & (p,\ q,\ r,\ s \text{ all different}), \end{cases}$$

which, on putting

$$x=p_{01},\quad y_i=p_{i1},\quad z_i=p_{0i},\quad u_{pq}=p_{pq},$$

become the Plücker equations

$$(4.323)\qquad p_{\alpha\beta}p_{\gamma\delta}+p_{\alpha\gamma}p_{\delta\beta}+p_{\alpha\delta}p_{\beta\gamma}=0,$$

so that the parametric equations (4.321) represent lines in $[n]$. The equations between the variety and the $[2n-2]$, on which it is represented and in which θ, ϕ_i, ψ_i are coordinates, are consequently the following :—

$$(4.324)\qquad \frac{x}{\theta^2}=\frac{y_i}{\theta\phi_i}=\frac{z_i}{\theta\psi_i}=\frac{u_{pq}}{\phi_p\psi_q-\phi_q\psi_p},$$

$$(4.325)\qquad \frac{\theta}{x}=\frac{\phi_i}{y_i}=\frac{\psi_i}{z_i}\quad (i=1,\ \ldots,\ n-1),$$

together with the equations (4.322) of the variety. This is an obvious generalization of (4.142), (4.143).

By taking in $[2n-2]$ coordinates θ, θ', ϕ_i, ψ_i $(i=1,\ \ldots,\ n-1)$ where $\sum_{i=1}^{n-1}(\phi_i+\psi^i)=0$, we can get another representation in the form

$$(4.326)\qquad \frac{x}{\theta\theta'}=\frac{y_i}{\theta\phi_i}=\frac{z_i}{\theta'\psi_i}=\frac{u_{pq}}{\phi_p\psi_q-\phi_q\psi_p},$$

$$(4.327)\qquad \frac{\phi_i}{y_iZ}=\frac{-\psi_i}{z_iY}=\frac{-\theta}{xY}=\frac{\theta'}{xZ}\quad (i=1,\ \ldots,\ n-1),$$

where

$$y=\sum_{i=1}^{n-1}y_i,\qquad Z=\sum_{i=1}^{n-1}z_i,$$

together with the relations (4.322).

When $n=4$, this gives another of the representations obtained by Semple (*loc. cit. ante*).

5. *V as a locus of linear spaces.*

5.1. *Fundamental properties of an algebraic system of linear spaces*.*

Hitherto we have considered V as a locus of ∞^6 points, of order five, whose points fall into lines, planes, etc., in various ways. To get a more comprehensive view of the variety it is necessary to regard it also as a system of linear spaces (lines, planes, etc.) without relation to the points that lie on it.

In this section we show how a set of numbers can be defined which are projective invariants of a manifold of linear spaces, in much the same way as the "order" of a locus of points is an invariant of the locus, and which play a fundamental rôle in enumerative geometry.

Consider, in $[n]$, a manifold of $[k]$, $(k<n)$. We consider the effect of imposing on this manifold conditions of a certain type defined as follows. We require those $[k]$ of the manifold which meet in a point a prescribed space $[a_0]$, meet in a line a prescribed $[a_1]$ which contains $[a_0]$, &c., and which finally lie in a given $[a_k]$ which contains $[a_{k-1}]$. This will be denoted symbolically by the condition $(a_0, a_1, \ldots, a_k)$; clearly $0 \leqslant a_0 < a_1 \ldots < a_k \leqslant n$. If the manifold of $[k]$ consist of ∞^s spaces, those which satisfy $(a_0, a_1, \ldots, a_k)$ will be ∞^{s-t}, where it is easily verified that

$$t = n(k+1) - \tfrac{1}{2}k(k+1) - \sum_0^k a_i,$$

or

$$\sum_0^k a_i = \tfrac{1}{2}(k+1)(2n-k) - t.$$

The spaces $[a_i]$ are naturally supposed to be in general position. When $t=s$, the spaces satisfying $(a_0, \ldots, a_k)$ are finite in number, and this number is clearly a projective invariant of the manifold. We then have

$$\sum_0^k a_i = \tfrac{1}{2}(k+1)(2n-k) - s. \tag{5.11}$$

We shall denote this number of spaces by $[a_0, a_1, \ldots, a_k]$, and call it a *characteristic* of the manifold. The aggregate of sets of numbers $a_0, \ldots, a_k$, satisfying (5.11) and the inequalities $0 \leqslant a_0 < a_1 \ldots < a_k \leqslant n$, define a set of characteristic numbers for the manifold.

Two simple examples may be given. In $[r]$ a V_k^n, considered as a locus of points, meets an $[r-k]$ in a finite number n of points; it has

* This section is mainly explanatory of (5.2) and contains no new results.

thus one characteristic (its order) in our notation denoted by $[r-k]=n$. Secondly, consider a congruence of lines in ordinary space (∞^2). The conditions of the type (a_0, a_1), which are of power two, are (0, 3) and (1, 2), so that the congruence has two characteristics [0, 3] and [1, 2]; it is easily seen that these are the numbers usually called the order and class of the congruence.

The fundamental theorem regarding these characteristics is due to Schubert*, and may be stated as follows :—

5.12. *If two manifolds of* $[k]$ *have complementary dimensions in* $[n]$, *the number of* $[k]$ *common to them when finite (as is generally the case) is given by*

$$\Sigma[a_0, a_1, \ldots, a_k][n-a_k, n-a_{k-1}, \ldots, n-a_0]',$$

where these are characteristics of the first and second manifolds respectively, and summation extends over all relevant $a_0, \ldots, a_k$ (*i.e.* those for which both characteristics are defined). This includes, for example, Halpen's theorem that the number of lines common to two congruences in [3] is $[0, 3][0, 3]'+[1, 2][1, 2]'$, or the product of the orders added to the product of the classes.

When the dimensions of the systems are not complementary, the characteristics of the system common to both must be obtained indirectly. If $[a_0, \ldots, a_k]$ is one of the characteristics of the system common to A and B, it will usually be possible, from the knowledge of the properties of A, to deduce those of the system $(A, [a_0, \ldots, a_k])$, and, by combining these with the characteristics of B and applying (5.12) we obtain the result sought.

5.2. *The characteristics for the generating spaces of* V.

We will have to calculate four sets of numbers for V, since V can be generated by lines, planes (in two ways), or solids. Consider first the ∞^8 lines of V; the conditions of power 8 of the form (a_0, a_1) are (0, 9); (1, 8); (2, 7); (3, 6); (4, 5).

Now V being only a V_6, an arbitrary plane does not meet V at all. Hence $[2, 7]=[1, 8]=[0, 9]=0$. A [6] meets V in a V_3^5; a [3] of the [6] meets V_3^5 in five points; through each of which pass three lines of V_3^5. Thus $[3, 6]=15$. Since the V_2^5, which is the general section V by a [5], contains ten lines, $[4, 5]=10$. *Thus, for the lines of* V,

$$(5.21)\quad [0, 9]=0,\quad [1, 8]=0,\quad [2, 7]=0,\quad [3, 6]=15,\quad [4, 5]=10.$$

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* *Mitteil. der Math. Ges. Hamburg*, 1 (1886), 134; see also Severi, *Rend. Lincei* (5) 9 $(1900)_2$, 321.

For the ∞^7 planes of the first system on V we shall require $[0, 8, 9]$, $[1, 7, 9]$, $[2, 6, 9]$, $[3, 5, 9]$, $[2, 7, 8]$, $[3, 6, 8]$, $[4, 5, 8]$, $[4, 6, 7]$, these coming from the conditions of power seven. As before, those involving 0, 1, 2 vanish. $[3, 5, 9]$ is the number of solids of V meeting a solid and containing a line of the V_2^5 in which V is met by an arbitrary $[5]$ through the solid. Since in $[5]$ a solid and line do not in general meet, $[3, 5, 9] = 0$.

For $[3, 6, 8]$ it is enough to observe that through each of the fifteen lines (5 . 21) satisfying (3, 6) passes a solid of V meeting the $[8]$ in a plane of the first system. Thus $[3, 6, 8] = 15$, and similarly $[4, 5, 8] = 10$.

Finally $[4, 5, 7]$ is the order of the locus of planes of the first system lying in a $[7]$; this is five (3 . 422). *Thus, for the planes α of V,*

$$(5 . 22) \quad \begin{cases} [0, 8, 9] = 0, & [1, 7, 9] = 0, & [2, 6, 9] = 0, & [3, 5, 9] = 0, \\ [2, 7, 8] = 0, & [3, 6, 8] = 15, & [4, 5, 8] = 10, & [4, 6, 7] = 5. \end{cases}$$

For the planes of the second system we require $[1, 8, 9]$, $[2, 7, 9]$, $[3, 6, 9]$, $[4, 5, 9]$, $[3, 7, 8]$, $[4, 6, 8]$, $[5, 6, 7]$ arising from conditions of power six. We have $[1, 8, 9] = [2, 7, 9] = 0$. From the characteristics (5 . 21) of the lines we have, since each line determines a plane π, $[3, 6, 9] = 15$, $[4, 5, 9] = 10$. Since through each point of a general prime section of V passes one plane π lying in the prime (3 . 14), $[3, 7, 8] = 5$ and, by (3 . 31), $[5, 6, 7] = 1$.

To get $[4, 6, 8]$, we observe that the $[8]$ contains a solid of V meeting the $[6]$ in a line l. The lines of V lying in the $[6]$ and meeting l form a ruled quintic surface (3 . 421), five of them thus meet a $[4]$. The plane π through one of these lines meets the solid in a line, since it meets l; thus it lies in the $[8]$. Hence $[4, 6, 8] = 5$. *Thus, for the planes π,*

$$(5 . 23) \quad \begin{cases} [1, 8, 9] = 0, & [2, 7, 9] = 0, & [3, 6, 9] = 15, & [4, 5, 9] = 10, \\ & [3, 7, 8] = 5, & [4, 6, 8] = 5, & [5, 6, 7] = 1. \end{cases}$$

From (5 . 22) and the fact that each prime section of V contains, in general, one solid of V, we find, for the characteristics of the solids of V,

$$(5 . 24) \quad \begin{cases} [2, 7, 8, 9] = 0, & [3, 6, 8, 9] = 15, & [4, 5, 8, 9] = 10, \\ & [4, 6, 7, 9] = 5, & [5, 6, 7, 8] = 1. \end{cases}$$

The principal application of these numbers is to the section of V by a quadric primal and the quadratic complex in $[4]$; this, however, is beyond the scope of this paper.

THE LONDON MATHEMATICAL SOCIETY.

BALANCE SHEET, 31st October, 1929.

Liabilities.	£	s.	d.	£	s.	d.
Lieut.-Col. Campbell's Fund				499	7	5
Sir A. B. Kempe's Fund				200	0	0
Donations				363	16	0
Life Compositions Fund				3232	12	0
Reserve Fund in 1928	546	15	6			
Depreciation of Investments	112	12	1			
Reserve Fund				434	3	5
Sundry Creditors				24	2	0
Revenue Account, balance in hand				253	19	1
				£5007	19	11

Assets.	£	s.	d.
London, Midland & Scottish Railway £385 4 per Cent. Guaranteed Stock at 74 ...	284	18	0
India Stock £700. 7*s.* 1*d.* 3 per Cent. at 58...	406	4	0
Funding Loan £3000 4 per Cent. at 87 ...	2610	0	0
Conversion Loan £1700 3½ per Cent. at 76...	1292	0	0
The above-mentioned stocks are valued at the present market prices.			
Cash at Bank	412	17	2
Cash in hand	2	0	9
	£5007	19	11

REVENUE ACCOUNT for the year 1st November, 1928, to 31st October, 1929.

	Year 1927-28.							Year 1927-28.					
	£	s.	d.	£	s.	d.		£	s.	d.	£	s.	d.
Balance from last year	38	9	10	*539*	*8*	*2*	Printing and Circulating *Proceedings* ...	1060	17	0	*1305*	*9*	*0*
Income Tax returned (3 years)	98	9	8	—			Ditto *Journal*	427	5	6	*426*	*15*	*0*
Dividends on—							General Printing	94	12	3	*125*	*11*	*3*
Lord Rayleigh's Fund	38	16	6	*38*	*13*	*0*	Purchases and Binding for Library ...	18	14	0	*24*	*1*	*5*
Lieut.-Col. A. Cunningham's Fund ...	36	18	4	—			Expenses of Meetings	30	0	0	*30*	*0*	*0*
The Society's Investments	188	16	4	*205*	*1*	*8*	Postages and Sundries	9	16	11	*9*	*13*	*4*
Entrance Fees	35	14	0	*31*	*10*	*0*	Balance in hand on 31st October, 1929 ...	253	19	1	*38*	*9*	*10*
Annual Subscriptions	495	2	2	*509*	*8*	*0*							
Sales of *Proceedings*	399	5	0	*369*	*12*	*3*							
Ditto *Journal*	78	7	11	*79*	*1*	*9*							
Advertisements revenue	35	5	0	*37*	*5*	*0*							
Grant from Royal Society	300	0	0	—									
Mr. H. T. J. Norton's Donation	150	0	0	*150*	*0*	*0*							
	£1895	4	9	*1959*	*19*	*10*		£1895	4	9	*1959*	*19*	*10*

REVENUE ACCOUNT of the DE MORGAN MEDAL FUND.

	£	s.	d.		£	s.	d.
Balance in hand 31st October, 1928 ...	5	9	6	Cash at Bank	9	1	6
Dividends received	2	8	0				
Income Tax returned (2 years)	1	4	0				
	£9	1	6		£9	1	6

REVENUE ACCOUNT OF LIEUT.-COL. A. J. C. CUNNINGHAM'S TRUST FUND

For Factorisation of Numbers.

	£	s.	d.		£	s.	d.
Dividends on £1538. 3*s.* 1*d.* Local Loans Stock	36	18	4	To the Society's Revenue Account	36	18	4
Income Tax returned	4	12	2	Balance in hand on 31st October, 1929 ...	4	12	2
	£41	10	6		£41	10	6

STATEMENT OF TRUST FUNDS, 31st October, 1929.

Lord Rayleigh's Fund consists of the following Investments:—

Great Indian Peninsula Railway £50. 8*s.* 8*d.* Annuity, Class B.
£229. 13*s.* 4*d.* 4 per Cent. Funding Stock.

This Fund is held in trust to apply the income as part of the general income of the Society.

The *De Morgan Medal Fund* consists of Great Western Railway £60 5 per Cent. Preference Stock.

Lieut.-Col. Allan Cunningham's Trust Fund for Factorisation of Numbers consists of £1538. 3*s.* 1*d.* Local Loans 3 per cent. Stock. The income of this Fund is being applied towards cost of printing papers on the Theory of Numbers.

14th November, 1929.

Signed on behalf of the Council:—

C. T. WHITTAKER, *President.*

A. E. WESTERN, *Treasurer.*

I report to the Members that I have obtained all the information and explanation I required as Auditor, and that I have examined the above Accounts with the books, and that, to the best of my information and of the explanations given to me, such Accounts are properly drawn up so as to exhibit a true and correct view of the state of the affairs of the Society as shown by the books of the Society.

25th November, 1929.

A. E. JOLLIFFE, *Auditor.*

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CORRIGENDA.

Page 163, line 3, *for* $1{\cdot}175\pi$ *read* $1{\cdot}2\pi$. R. G. C.

Page 213, line 6, col. 3, *for* $\kappa_1\kappa_2$ *read* $\kappa_1\kappa_2^3$.

Page 231, line 3, *for* κ^4 *read* κ_2^4.

Page 233, line 3, *for* 15548, 7774 and 15548 *read* 15552, 7776 and 15552.

Page 234, last line, *for* $k_4^{-\frac{3}{2}}$ *read* $k_2^{-\frac{3}{2}}$. R. A. F.

Page 240, line 15, in the integral

for $(\frac{1}{2}+, 0+, \frac{1}{2}-, 0-)$ *read* $\left(\frac{1}{z}+, 0+, \frac{1}{z}-, 0-\right)$.

Page 375, line 8, in the expression for $-i\pi e^{m\pi i}P_n^m(\cos\theta)$,

interchange $\cos(\theta+0i)$ *with* $\cos(\theta-0i)$.

Page 375 line 10, *for* $\frac{1}{2}\pi P_n^m(\cos\theta)$ *read* $-\frac{1}{2}\pi P_n^m(\cos\theta)$.

Page 376, at the foot of the page, in the last two formulae

for 2^{2m} *read* $\dfrac{\Pi(n+m}{\Pi(n-m)}$. E. W. H.

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PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY.

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Issued August 22nd, 1929.

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THE LONDON MATHEMATICAL SOCIETY is instituted for the promotion and extension of Mathematical Knowledge.

It was founded in 1865, and incorporated under Section 23 of the Companies Act 1867 in 1894.

Every Candidate for Membership must be proposed and recommended, according to a form which the Secretaries will supply, by not less than three Members, of whom one at least, except in special cases to be submitted for the decision of the Council, must certify his personal knowledge of the Candidate.

This form is read at one of the Ordinary or Annual General Meetings, and the Candidate is balloted for at the next ensuing meeting, provided that seven Members are present thereat.

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The Council consists of sixteen Members, including the Officers, and is chosen from among the Ordinary Members of the Society at the Annual General Meeting in November.

The Officers are a President, Vice-Presidents, a Treasurer, a Librarian, and Secretaries.

The Ordinary Meetings of the Society are now held at Burlington House, W.1, and begin at 5 o'clock in the evening. The Meetings for the Session 1929–30 are on November 14 and December 12, 1929, and January 16, February 6, March 13, April 24, May 15, and June 19, 1930. Cards giving the Programmes of the meetings can be had by Members a few days beforehand on application to C. F. Hodgson & Son, Ltd., 2 Newton Street, Kingsway, London, W.C.2.

At these meetings papers are read and communications made: upon each paper or communication the Chairman invites discussion.

The Council alone decides whether any paper proposed for reading shall or shall not be read.

After a paper has been presented to the Society, it is referred by the Council to one or more Members, who report to the Council on its fitness for publication in the *Proceedings* or the *Journal*. After hearing the reports, the Council decides by ballot whether it shall be printed or not.

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